

Principal Typings Demystified

*what they are,
why you want them,
and why your type system doesn't have them*

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A Perspective Change via New Notation

Consider the λ -term $P = (\lambda yx.x)$.

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We say the typing $t_1 = (\emptyset \vdash \beta \rightarrow \alpha \rightarrow \alpha)$ can be *assigned* to P .

More Typings for the Example (1)

(Memory from previous overheads)

$$P = (\lambda yx.x) \quad t_1 = (\emptyset \vdash \beta \rightarrow \alpha \rightarrow \alpha)$$

The term P can also be assigned the typing

$$t_2 = (\emptyset \vdash \gamma \rightarrow \gamma \rightarrow \gamma)$$

which can be obtained from t_1 by a *substitution*:

$$S_2 = \{(\alpha \mapsto \gamma), (\beta \mapsto \gamma)\}$$

$$S_2(t_1) = t_2$$

More Typings for the Example (2)

(Memory from previous overheads)

$$P = (\lambda yx.x) \quad t_1 = (\emptyset \vdash \beta \rightarrow \alpha \rightarrow \alpha)$$

Another typing for $P = (\lambda yx.x)$ is

$$t_3 = (\{z : \delta \rightarrow \delta\} \vdash (\delta \rightarrow \delta) \rightarrow \delta \rightarrow \delta)$$

which can be obtained from t_1 by a substitution and a *weakening*:

$$S_3 = \{(\alpha \mapsto \delta), (\beta \mapsto \delta \rightarrow \delta)\}$$

$$W_3 = \{z : \delta \rightarrow \delta\}$$

$$W_3(S_3(t_1)) = t_3$$

Introducing Principal Typings

- In fact, it turns out that every typing t that can be assigned to P in STLC can be obtained from t_1 by applying some substitution S and some weakening W .

So t_1 is called a *principal typing* for P .

If $t_2 = W(S(t_1))$, then t_2 is often called an *instance* of t_1 .

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- STLC has principal typings, i.e., every typable term has one (see Hindley [1997] for references).
- In general, a principal typing in a system S for a term M is a typing assignable to M which *somehow* represents all other typings in S that can be assigned to M .
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Part of this talk is about clarifying the “*somehow*”.
- Principal typings turn out to be very useful when finding types for untyped programs, as the following example shows.

An Example Problem of Finding Types (1)

Consider these λ -terms:

$$M = \lambda z.NP$$

$$N = \lambda w.w(wz)$$

$$P = \lambda yx.x$$

A bottom-up analysis algorithm Inf for STLC would do this on M :

$$\text{Inf}(M) = \text{Cmb}_\lambda(z) = \begin{array}{c} \text{Cmb}_\lambda(z) \\ | \\ \text{Cmb}_@ \\ / \quad \backslash \\ \text{Inf}(N) \quad \text{Inf}(P) \end{array}$$

The diagram shows the recursive decomposition of the analysis of M . It starts with $\text{Inf}(M)$, which is equal to $\text{Cmb}_\lambda(z)$. This is further decomposed into a tree structure where $\text{Cmb}_\lambda(z)$ is the root, connected by a vertical line to $\text{Cmb}_@$. From $\text{Cmb}_@$, two diagonal lines branch out to $\text{Inf}(N)$ on the left and $\text{Inf}(P)$ on the right. A vertical line also connects $\text{Cmb}_\lambda(z)$ to $\text{Inf}(NP)$ in the middle part of the diagram.

The subalgorithms Cmb_λ and $\text{Cmb}_@$ are used to combine the results from recursively processing the subterms.

An Example Problem of Finding Types (2)

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$$\text{Inf}(M) = \left(\begin{array}{c} \text{Cmb}_\lambda(z) \\ | \\ \text{Cmb}_@ \\ / \quad \backslash \\ \text{Inf}(N) \quad \text{Inf}(P) \end{array} \right)$$

Suppose Inf is designed by a naïve person like me and $\text{Inf}(P) = t_2 = (\emptyset \vdash \gamma \rightarrow \gamma \rightarrow \gamma)$. What happens?

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Typing NP needs a typing for P like $t_4 = (\emptyset \vdash (\delta \rightarrow \delta) \rightarrow \delta \rightarrow \delta)$.

Can $\text{Cmb}_@$ figure out from t_2 that t_4 is also assignable to P ?

An Example Problem of Finding Types (3)

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$$t_2 = (\emptyset \vdash \gamma \rightarrow \gamma \rightarrow \gamma)$$

$$t_4 = (\emptyset \vdash (\delta \rightarrow \delta) \rightarrow \delta \rightarrow \delta)$$

If all we know about a term Q in STLC is that it can be assigned t_2 , can we conclude that we can also assign it t_4 ?

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If all we know about a term Q in STLC is that it can be assigned t_2 , can we conclude that we can also assign it t_4 ?

No, because here is a term which can be assigned t_2 but not t_4 :

$$Q = \lambda yx. (\lambda z.x)(\lambda w.wx(wyx))$$

The types of x and y are forced to be the same by the applications (wx) and (wy) . The term Q differs from P only by the addition of some **constraining subterms** which have no computational effect.

Principal Typings Solve the Example Problem

(Memory from previous overheads)

$$N = \lambda w.w(wz)$$

$$P = \lambda yx.x$$

$$t_1 = (\emptyset \vdash \beta \rightarrow \alpha \rightarrow \alpha)$$

$$t_2 = (\emptyset \vdash \gamma \rightarrow \gamma \rightarrow \gamma)$$

$$t_4 = (\emptyset \vdash (\delta \rightarrow \delta) \rightarrow \delta \rightarrow \delta)$$

$$\text{Inf}(NP) = \left(\begin{array}{c} \text{Cmb}_@ \\ \swarrow \quad \searrow \\ \text{Inf}(N) \quad \text{Inf}(P) \end{array} \right)$$

Suppose instead of $\text{Inf}(P) = t_2$ we make $\text{Inf}(P) = t_1$, the principal typing of P . Can $\text{Cmb}_@$ deduce that it can legally use t_4 , the typing needed for NP ?

By principality, $t_4 = W(S(t_1))$ for some S and W . Is this enough?

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Yes, because for any substitution or weakening X , if $M' : t$ is derivable in STLC, then $M' : X(t)$ is also derivable.

PT Example with Intersection Types (1)

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Another:

$$t_2 = (\emptyset \vdash ((\alpha_1 \cap \alpha_2) \rightarrow (\delta \times \delta)) \rightarrow ((\gamma_1 \rightarrow \alpha_1) \cap (\gamma_2 \rightarrow \alpha_2)) \rightarrow (\gamma_1 \cap \gamma_2) \rightarrow (\delta \times \delta))$$

There is an *expansion* E and a substitution S such that $t_2 = S(E(t_1))$ (see van Bakel [1995] for references).

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Other operations used in obtaining other typings from the principal typing in systems with intersection types include *coverings* and *liftings*.

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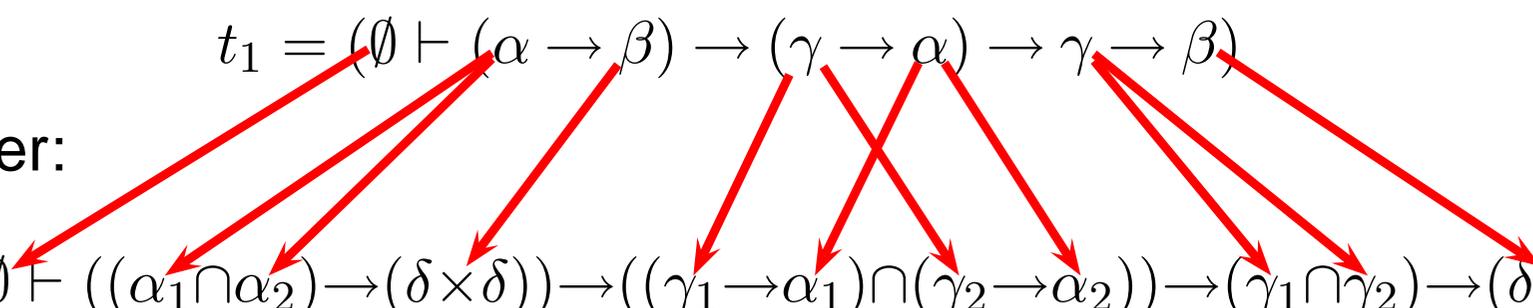
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PT Example with Intersection Types (2)

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$$M = \lambda xyz.x(yz)$$

$$t_1 = (\emptyset \vdash (\alpha \rightarrow \beta) \rightarrow (\gamma \rightarrow \alpha) \rightarrow \gamma \rightarrow \beta)$$

$$\begin{aligned} t_2 &= (\emptyset \vdash ((\alpha_1 \cap \alpha_2) \rightarrow (\delta \times \delta)) \\ &\quad \rightarrow ((\gamma_1 \rightarrow \alpha_1) \cap (\gamma_2 \rightarrow \alpha_2)) \\ &\quad \rightarrow (\gamma_1 \cap \gamma_2) \rightarrow (\delta \times \delta)) \end{aligned}$$

The example may be clearer in a system with expansion variables [Kfoury and Wells, 1999]:

$$M' = \lambda xyz.x(G(y(Fz)))$$

$$t'_1 = (\emptyset \vdash ((G\alpha) \rightarrow \beta) \rightarrow (G((F\gamma) \rightarrow \alpha)) \rightarrow (G(F\gamma)) \rightarrow \beta)$$

$$S' = \{F_1 \mapsto \square, F_2 \mapsto \square, G \mapsto \square \cap \square, \beta \mapsto \delta \times \delta\}$$

Then $t_2 = S'(t'_1)$. Expansion and substitution are integrated.

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- A paper with “principal” in the title currently submitted by a good researcher to a major conference with false claims:
 - Claim: All typings for a term M in STLC are obtained from its principal typing using *only substitution*.
 - Claim: In HM when $M : (\Gamma \vdash \tau)$, the type τ is principal for M iff τ is a *subtype* of any type τ' such that $M : (\Gamma \vdash \tau')$.

Technical Definitions

Let the metavariable S range over type systems.

Let $S \triangleright M : t$ mean $M : t$ is derivable by the rules of S , in which case the system S *assigns* the typing t to the term M .

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Now a new ordering on typings is defined. Let $t_1 \leq_S t_2$ mean $\text{Terms}_S(t_1) \subseteq \text{Terms}_S(t_2)$, in which case the typing t_1 is *at least as strong as* t_2 .

Observations on Systems with PTs

The following holds for each system S with principal typings that I know.

In S , each typable term M can be assigned a typing t which is *principal for M* in the sense that for every other typing t' assignable to M , there exist operations O_1, \dots, O_n such that

- $t' = O_n(\dots(O_1(t))\dots)$, and
- for any term N , if $S \triangleright N : t$, then $S \triangleright N : t_i$ where $t_i = O_i(\dots(O_1(t))\dots)$ for $1 \leq i \leq n$.

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Observation: If t is principal for M , then $t \leq_S t'$ for every $t' \in \text{Typings}_S(M)$.

A New System-Independent Definition

A typing t is *principal for M* in system S iff

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- $t \leq_S t'$ exactly when an analysis algorithm can correctly replace a typing t inferred for a term M by t' *without inspecting M again*.
- If t is principal for M , then t is minimal in $\text{Typings}_S(M)$, so an analysis algorithm possessing t has the mathematical potential to infer any typing for M in a larger context.

Remarks on Principal Typings for Analysis

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- Principal typings allow *compositional* type analysis, where analyzing a fragment uses only the *results* for its subfragments, which can be analyzed independently in any order.
- Compositional analysis *helps* with separate compilation and modularity and with making a complete/terminating analysis algorithm.

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- A typing t is Γ -principal for M iff
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Principality for HM: If M is Γ -typable, there is a Γ -principal t for M .

HM Does Not Have Principal Typings (1)

A counterexample is the term (xx) .

Suppose $\text{HM} \triangleright (xx) : t$ where $t = (\Gamma \vdash \tau)$. It will be proven that t is not principal for (xx) . The strategy is to find t' and M such that:

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Easy: $\text{HM} \triangleright (xx) : t'$.

So t' is another typing for (xx) , but it will be seen that $t \not\leq_{\text{HM}} t'$.

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- To measure the length of the leftmost path in a type viewed as a tree: Let $\text{LLen}(\alpha) = 0$, $\text{LLen}(\rho \rightarrow \rho') = 1 + \text{LLen}(\rho)$, and $\text{LLen}(\forall \alpha.\rho) = \text{LLen}(\rho)$.

HM Does Not Have Principal Typings (2)

Some useful auxiliary machinery:

- Let $K = (\lambda xy.x)$.
- To force identical derived result types for two subterms:
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- To have a term whose derived result type must have a fixed LLen value: Let y_1, y_2, \dots be fresh. Let $\text{FixLLen}(0) = c$ for base type constant c . Let $\text{FixLLen}(i + 1) = (\lambda x.K y_{i+1} \text{Unify}(x, \text{FixLLen}(i)))$ for fresh x .
Easy: If $\text{HM} \triangleright \text{FixLLen}(i) : (\hat{\Gamma} \vdash \hat{\tau})$, then $\text{LLen}(\hat{\tau}) = i$.

HM Does Not Have Principal Typings (3)

(Memory from previous overheads)

$$\begin{array}{lll} \text{HM} \triangleright (xx) : t & t = (\Gamma \vdash \tau) & \Gamma(x) = \forall \vec{\alpha}. \sigma \\ \text{HM} \triangleright (xx) : t' & t' = (\Gamma' \vdash \tau') & \Gamma' = \{x : \sigma'\} \\ & & \sigma' = \forall. (\sigma \rightarrow \beta) \end{array}$$

Now the term M of the counterexample is defined. Let $k = \text{LLen}(\Gamma(x))$. Let $M = \text{K}(xx)(\lambda y_1 \dots y_k. x \text{FixLLen}(k - 1))$.

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The finale: $\text{HM} \not\triangleright M : t'$, because $\text{LLen}(\Gamma'(x)) = k + 1$ and can not shrink by instantiation to the needed k to allow typing the subterm $(x \text{FixLLen}(k - 1))$.

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Therefore, $t \not\leq_{\text{HM}} t'$ and t is not a principal typing in HM for (xx) . Because t is arbitrary, (xx) has no HM-principal typing.

Implications of HM's Lack of PTs

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- Not use HM typings for intermediate results. E.g., the principal typing of (xx) in the Chap. 1 system of Damas [1985]:

$$(\{x:\alpha, x:\alpha \rightarrow \beta\} \vdash \beta)$$

This is essentially intersection types, i.e.:

$$(\{x:\alpha \cap (\alpha \rightarrow \beta)\} \vdash \beta)$$

Essentially the same was done by Shao and Appel [1993] and Bernstein and Stark [1995].

System F Does Not Have Principal Typings

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- Unlike for HM, the term M makes one dimension of a type too small for use with t' without relying on a ground-type constant but instead uses methods of Wells [1999].

Conclusions

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Because these systems are commonly used as the basis of other type systems, this has wide implications.

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- Unfortunately, neither HM nor F have principal typings.
Because these systems are commonly used as the basis of other type systems, this has wide implications.
- If principal typings are needed, one can often obtain them by adding intersection types.

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