Covering Random Points in a Unit Ball

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Choose random points $X_1, X_2, X_3, \ldots$ independently from a uniform distribution in a unit ball in $\mathbb{R}^m$. Call $X_n$ a dominator iff $\text{distance}(X_n, X_i) \leq 1$ for all $i < n$, i.e. the first $n$ points are all contained in the unit ball that is centered at the $n$'th point $X_n$. We prove that, with probability one, only finitely many of the points are dominators.

For the special case $m = 2$, we consider the unit disk graph $G_n$ determined by $n$ random points $X_1, X_2, \ldots, X_n$ in the unit disk. With asymptotic probability one, $G_n$ has a connected dominating set consisting of just two points.

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1 Introduction

Every finite set \( V \) of points in \( \mathbb{R}^m \), determines a unit ball graph \( G(V) = (V, E) \) as follows. The vertex set is \( V \), and an undirected edge \( e \in E \) connects vertices \( u, v \in V \) iff the distance \( d_m(u, v) \) is less than one. (Throughout this paper \( d_m(u, v) \) denotes the Euclidean distance between \( u, v \in \mathbb{R}^m \).) If \( u \) and \( v \) are connected by an edge, we say \( u \) and \( v \) are adjacent vertices in the graph \( G(V) \). The case \( m = 2 \) is particularly prominent in applications, and in this case \( G(V) \) is called a unit disk graph. Unit disk graphs have been used by many authors as simplified mathematical models for the interconnections between hosts in a wireless network, and random unit disk graphs have been used as stochastic models for these networks. For other examples of applications, see Marchette [7]. A recent survey of random unit ball graphs is Penrose [8].

A dominating set in any graph \( G = (V, E) \) is a subset \( C \subseteq V \) such that every vertex \( v \in V \) either is in the set \( C \), or is adjacent to a vertex in \( C \) [5]. We say \( C \) is a connected dominating set if \( C \) is a dominating set and the subgraph induced by \( C \) is connected. The following question arose naturally in the context of routing algorithms for certain wireless networks [9],[4],[3]. Suppose \( V \) is a set of \( n \) random points in the unit disk in \( \mathbb{R}^2 \), and let \( G_n = G(V) \) denote the corresponding unit disk graph. How large, typically, is the smallest connected dominating set for \( G_n \)? In this paper we show that with asymptotic probability one, the size of the smallest connected dominating set in \( G_n \) is two.

The paper is organized as follows. In Section 2, we prove a general result concerning unit ball graphs formed from random points \( X_1, X_2, X_3, \ldots \) chosen independently and uniformly in the unit ball in \( \mathbb{R}^m \). It follows from this general result that ‘one point does not suffice’: with asymptotic probability one, \( G_n \) does not have a one-point dominating set. In Section 3 we prove a geometric lemma which is required for the proof, in Section 4, of the existence, with high probability, of two-point connected dominating sets in \( G_n \).

2 One Vertex Dominating Sets

Suppose that \( X_1, X_2, X_3, \ldots \) is a sequence of random points chosen independently from a uniform distribution in a unit ball in \( \mathbb{R}^m \). Call \( X_n \) a dominator iff \( d_m(X_n, X_i) \leq 1 \) for all \( i < n \), i.e. all \( n \) points are contained in the unit ball that is centered at \( X_n \). Then we prove:

Theorem 1 With probability one, only finitely many of the points \( X_n \) are dominators.

Proof. Let \( A_n \) be the event that \( X_n \) is a dominator. By the Borel-Cantelli lemma, it suffices to prove that \( \sum_{n=1}^{\infty} \Pr(A_n) < \infty \). For positive real numbers \( r \) and positive integers \( m \geq 2 \), let \( V_m(r) \) be the volume of the \( a \) ball of radius \( r \) in \( \mathbb{R}^m \), i.e.

\[
V_m(r) = r^m V_m(1) = \frac{\pi^{\frac{m}{2}} r^m}{\Gamma(\frac{m}{2} + 1)}
\]  (1)
Let \( L(r) \) denote the volume of the intersection of two unit balls in \( \mathbb{R}^m \) whose centers are at distance \( r \) from each other. If the the distance from the point \( X_n \) to the origin is \( r \), then the conditional probability that the \( i \)’th point \( X_i \) is within distance one of \( X_n \) is \( \frac{L(r)}{V_m(1)} \). The distance between the origin and the random point \( X_n \) is a random variable with density \( f(r) = \frac{V_m'(r)}{V_m(1)} = mr^{m-1} \).

Hence

\[
\Pr(X_n \text{ is a dominator}) = \int_0^1 f(r) \left( \frac{L(r)}{V_m(1)} \right)^{n-1} dr.
\]

We split the integral into two. Let \( \xi = \frac{4(\log n)V_m(1)}{(n-1)V_{m-1}(1)} \). Then

\[
\Pr(X_n \text{ is a dominator}) = I_1 + I_2,
\]

where

\[
I_1 = m \int_0^\xi r^{m-1} \left( \frac{L(r)}{V_m(1)} \right)^{n-1} dr
\]

and

\[
I_2 = m \int_\xi^1 r^{m-1} \left( \frac{L(r)}{V_m(1)} \right)^{n-1} dr.
\]

For the first piece, we use the trivial estimate \( \frac{L(r)}{V_m(1)} \leq 1 \): for \( m \geq 2 \),

\[
I_1 \leq m \int_0^\xi r^{m-1} dr = \xi^m = O\left(\frac{\log^2 n}{n^2}\right).
\]

To estimate \( I_2 \), we use the following “well-known” formula for \( L(r) \):

\[
L(r) = 2 \int_{r/2}^1 V_{m-1}(\sqrt{1 - x^2}) dx = 2V_{m-1}(1) \int_{r/2}^1 (1 - x^2)^{m-1} x \ dx
\]

It is intuitively obvious that \( L(r) \) is decreasing, and this is easily confirmed by differentiating the right side of (6) to obtain

\[
L'(r) = -V_{m-1}(1) \cdot \left(1 - \frac{r^2}{4}\right)^{\frac{m-1}{2}} \leq 0
\]

for \( 0 \leq r \leq 1 \). By differentiating again, we also obtain

\[
L''(r) = V_{m-1}(1) \cdot \left(1 - \frac{r^2}{4}\right)^{\frac{m-1}{2}} \geq 0
\]
for $0 \leq r \leq 1$. Since $L(r) \leq L(\xi)$ for all $r \geq \xi$, and since $f$ is a density function, we have

$$I_2 \leq \left( \frac{L(\xi)}{V_m(1)} \right)^{n-1} \int \limits_{\xi}^{1} f(r) \, dr \leq \left( \frac{L(\xi)}{V_m(1)} \right)^{n-1}$$

(9)

To estimate the right side of (9), we note that it follows from (7) and (8) that there is some $0 < c_\xi < \xi$ such that

$$L(\xi) = L(0) + L'(c_\xi)\xi = V_m(1) - V_{m-1}(1) \cdot \left( 1 - \frac{c_\xi^2}{4} \right)^{\frac{m-1}{2}} \cdot \xi$$

(10)

Since $0 < c_\xi < \xi = o(1)$, we have $\left( 1 - \frac{c_\xi^2}{4} \right)^{\frac{m-1}{2}} > \frac{1}{2}$ for all sufficiently large $n$. So it follows from (10) that

$$L(\xi) \leq V_m(1) - \frac{V_{m-1}(1)}{2} \xi$$

(11)

for all sufficiently large $n$. Putting (11) back into the right side of (9), we get

$$I_2 \leq \left( 1 - \frac{\xi V_{m-1}(1)}{2V_m(1)} \right)^{n-1} = O\left( \frac{1}{n^2} \right).$$

(12)

Combining our estimates (5) and (12) for $I_1$ and $I_2$ respectively, we conclude that, for some positive constant $c$, and all sufficiently large $n$, $\Pr(\mathcal{A}_n) < \frac{c \log^2 n}{n^2}$. Hence $\sum \Pr(\mathcal{A}_n)$ converges.

Now let $\mathcal{G}_n^m \equiv \mathcal{G}(X_1, X_2, ..., X_n)$ denote the unit ball graph in $\mathbb{R}^m$ with random vertex set $V = \{X_1, X_2, ..., X_n\}$. (So, in particular, $\mathcal{G}_n = \mathcal{G}_2$.) Let $\mathcal{B}_n^m$ denote the event that $\mathcal{G}_n^m$ has a one-point dominating set.

**Corollary 2** For all $m \geq 2$ and for all sufficiently large $n$,

$$\Pr(\mathcal{B}_n^m) \leq \frac{c_m \log^2 n}{n}$$

where $c_m > 0$ is a positive constant which may depend on the dimension $m$ but does not depend on $n$.

**Proof.** For each $n > 0$ and for $1 \leq i \leq n$, let $\mathcal{B}_n(X_i)$ denote the event that $X_i$ is a one-point dominating set of $\mathcal{G}_n^m$. Then we have, for all sufficiently large $n$,

$$\Pr(\mathcal{B}_n^m) = \Pr(\cup_{i=1}^{n} \mathcal{B}_n(X_i)) \leq \sum_{i=1}^{n} \Pr(\mathcal{B}_n(X_i)) = n \Pr(\mathcal{B}_n(X_i)) \leq \frac{c_m \log^2 n}{n}$$

since $X_1, X_2, ..., X_n$ are independent and identically distributed. We note that the last inequality follows from the bound obtained for $\Pr(\mathcal{A}_n)$ in the proof of Theorem 1. □
3 A Geometric Lemma

In this section and the next, we adopt the following notation. For any \( r > 0 \), and any \( v \in \mathbb{R}^2 \), let \( D_r(v) \) be the open unit disk centered at \( v \), and let \( \partial D_r(v) \) be the circle of radius \( r \) that bounds it. As observed in [6], a unit disk centered at a point \( o \) cannot be completely covered with two unit disks having centers at points other than \( o \): \( D_1(o) \not\subseteq D_1(u) \cup D_1(v) \) for \( u \neq o \neq v \). The purpose of this section is to prove Lemma 3, which provides an upper bound for the uncovered region’s area (when \( u \) and \( v \) are suitably situated).

Some notation is needed to state Lemma 3. Throughout this section, \( b \geq 3 \) will be an unspecified parameter. (When we actually apply Lemma 3 in Section 4, \( b \) will correspond to the number of vertices in a random disk graph.) In terms of \( b \), we define \( L_b = \lceil b^{1/3} (\log b)^2 \rceil, \delta = \delta_b = \frac{1}{\sqrt{b \log b}}, \) and \( \theta_b = \pi/L_b \). Let \( o = (0,0) \) be the origin in \( \mathbb{R}^2 \). We are essentially\(^1\) going to partition \( D_{\delta_b}(o) \) into \( 2L_b \) sectors as follows. For integers \( i \) such that \( 0 \leq i < L_b \), let \( Q_i \) be the sector consisting of those points \( (x, y) = (r \cos \theta, r \sin \theta) \) whose polar coordinates satisfy \( 0 < r \leq \delta \) and \( (i - \frac{1}{2}) \theta_b \leq \theta \leq (i + \frac{1}{2}) \theta_b \). Similarly let \( R_i \) consist of the points with \( 0 < r \leq \delta \) and \( (i - \frac{1}{2}) \theta_b \leq \theta - \pi \leq (i + \frac{1}{2}) \theta_b \). Note that the sectors \( Q_i \) and \( R_i \) are located symmetrically with respect to \( o \). Let \( \tilde{q}_i \) and \( \tilde{u}_i \) be the extreme points whose polar coordinates are respectively \( (\delta, (i - \frac{1}{2}) \theta_b) \) and \( (\delta, (i + \frac{1}{2}) \theta_b + \pi) \). Finally, for any points \( u, w \in D_1(o) \), let \( X(u, w) \) denote the area of \( (D_1(u) \cup D_1(w))^c \cap D_1(o) \), i.e. the area of the region in \( D_1(o) \) that is not covered by \( D_1(u) \cup D_1(w) \). Our goal in this section is to prove

**Lemma 3** There is a uniform constant \( C > 0 \) (independent of the parameter \( b \)) such that, for all \( i \), and for all \( q_i, u_i \in R_i \), we have \( X(q_i, u_i) \leq \frac{1}{\theta_b} \leq \frac{1}{\theta_b} \).

We prove four facts which together imply Lemma 3. In the first fact, we observe that for any \( v, w \in D_1(o) \) the omitted area \( X(v, w) \) increases if we move one (or both) of the two points \( v \) and \( w \) away from the origin along a radial line.

**Fact 1** Let \( v_1, v_2 \) and \( w_1, w_2 \) be four points in \( D_1(o) \) such that \( v_1 \) lies on the line segment \( \overline{v_1 w_1} \) and \( v_2 \) lies on the line segment \( \overline{v_2 w_2} \). Then \( X(v_2, w_2) \geq X(v_1, w_1) \).

**Proof.** It suffices to show that \( D_1(v_2) \cap D_1(o) \subseteq D_1(v_1) \cap D_1(o) \) and that \( D_1(w_2) \cap D_1(o) \subseteq D_1(w_1) \cap D_1(o) \). Suppose \( p \in D_1(v_2) \cap D_1(o) \). Since \( v_1 \) lies on the line segment from \( o \) to \( v_2 \), we have \( d(v_1, p) \leq \max(d(o, p), d(v_2, p)) \leq 1 \). Hence \( p \in D_1(v_1) \cap D_1(o) \). By a similar same argument, \( D_1(w_2) \cap D_1(o) \subseteq D_1(w_1) \cap D_1(o) \).

\( \square \)

\(^1\)It is not strictly correct to call this a partition of \( D_{\delta}(o) \) since the origin was omitted, the bounding circle was included, and some pairs of sectors have a non-empty intersection (with zero area).
Fact 2 Let $a,b$ be the two points where the circles $\partial D_1(p), \partial D_1(q)$ intersect. Then, $a,b \perp \overline{p,q}$, and the two line segments $a,b$ and $\overline{p,q}$ intersect at their midpoints.

Proof. This follows immediately from the fact that $d(p,a) = d(p,b) = d(q,a) = d(q,b) = 1$. □

Fact 3 Let $o_1,o_2$ be two points on the circle $x^2 + y^2 = \delta_0^2$. Then, $X(o_1,o_2)$ is a decreasing function of $\angle o_1oo_2$.

Proof. For convenience, we will use polar coordinates. Without loss of generality, let $o_1$ be the point with polar coordinates $(r_{o_1}, \phi_{o_1}) = (\delta_0, \pi)$. Let $o_2$ be an arbitrary point on the circle with the polar coordinates $(\delta_0, \phi_2)$. By symmetry, we only need to consider the case when $o_2$ is in the first or second quadrant; we may, without loss of generality, assume that $0 \leq \phi_2 \leq \pi$. We will show that $X(o_1,o_2)$ is an increasing function of $\phi_2$, then the result follows from the fact that $\angle o_1oo_2 = \pi - \phi_2$.

Let $a_1,b_1$ be the two points where the circles $\partial D_1(o_1)$ and $\partial D_1(o)$ intersect, with $a_1$ in the second quadrant and $b_1$ in the third quadrant.

Let $o^*$ be a point on the circle $x^2 + y^2 = \delta_0^2$ so that $\partial D_1(o^*)$ meets with both $\partial D_1(o)$ and $\partial D_1(o_1)$ at $a_1$. Let $b^*, d^*$ be the other intersection points of $\partial D_1(o^*)$ with $\partial D_1(o)$ and $\partial D_1(o_1)$, respectively. For convenience, let’s denote $\phi_{o^*}$ by $\phi^*$. Figure 1 illustrates the position of $\partial D_1(o_1), \partial D(o)$, and $\partial D_1(o^*)$ and their intersections.

![Figure 1: The position of the circle $\partial D_1(o^*)$](image)

As in the proof of Fact 2, we have $\overline{a_1,d^*} \perp \overline{o_1,o^*}$, $\overline{a_1,b^*} \perp \overline{o,o^*}$. Notice also
that \( o \) is on the line segment \( \overline{a_1, d^*} \). So,

\[
\angle b^*a_1o = \angle oo^*o_1 = \angle o^*o_1o = \frac{\phi^*}{2}. \tag{13}
\]

It follows that

\[
0 < \phi^*/2 < \pi/2, \text{ and, } \sin \frac{\phi^*}{2} = \delta_b.
\tag{14}
\]

Now, for the point \( o_2 \) with polar coordinates \((\delta_b, \phi_2)\), let \( a_2, b_2 \) denote the two points where \( \partial D_1(o_2) \) and \( \partial D_1(o) \) intersect, and let \( c_2, d_2 \) denote the two points where \( \partial D_1(o_2) \) and \( \partial D_1(o_1) \) intersect. There are two cases to consider: \( \phi_2 \leq \phi^* \), and \( \phi_2 \geq \phi^* \).

**Case 1.** \( \phi_2 \leq \phi^* \).

Notice that \( a_1, b_1 \) partitions the circle \( \partial D_1(o) \) into two arcs: the right section and the left section. When, \( \phi_2 \leq \phi^* \), as illustrated in Figure 2, \( a_2, b_2 \) are both on the right section of the circle \( \partial D_1(o) \) between \( a_1, b_1 \). Similarly, \( c_2, d_2 \) are both on the right section of the circle \( \partial D_1(o_1) \) between \( a_1, b_1 \). Clearly,

\[
X(o_1, o_2) = B_1 - (B_2 - B_3) = B_1 - B_2 + B_3,
\]

where

- \( B_1 = \text{area}(D_1(o_1)^c \cap D_1(o)) \)
• $B_2 = area(D_1(o) \cap D_1(o_2))$
• $B_3 = area(D_1(o_1) \cap D_1(o_2))$, the shaded area in Figure 2

Notice that $B_3$ is the only area that depends on $\phi_2$. We shall now give an expression for $B_3$. Let’s denote $\angle c_2 o_1 o_2 = \gamma$. Since $\angle o_2 o_1 o = \frac{\phi_2}{2}$, we have

$$0 < \gamma < \frac{\pi}{2}, \text{ and } \cos \gamma = \delta \cos \frac{\delta}{2}$$  

(15)

By symmetry, one can see that the shaded region is partitioned equally by the line $c_2, d_2$. So,

$$B_3 = 2\left(\frac{2\gamma}{\pi} - \frac{1}{2}(2 \sin \gamma)(\cos \gamma)\right) = 2\gamma - \sin 2\gamma.$$  

Here, the first term is the area of the sector $D_1(o_1)$ that extends from $c_2$ to $d_2$, and the second term is the area of the triangle$(c_2, o_1, d_2)$. From the above two equations, we have

$$\frac{dX(o_1, o_2)}{d\phi_2} = \frac{dB_3}{d\phi_2} = \frac{dB_3}{dy} \cdot dy = (1 - \cos 2\gamma) \cdot \delta \sin \frac{\delta}{2} > 0.$$  

Here the last inequality follows from the fact that $0 < \frac{\delta}{2}, \gamma < \frac{\pi}{2}$. Thus $X(o_1, o_2)$ is an increasing function in $\phi_2$.

Case 2. $\phi_2 > \phi^*$.

One can see from Figure 3 that

$$X(o_1, o_2) = B_1 - (B_2 - B_3) = B_1 - B_2 + B_3$$  

Where $B_1, B_2$ are defined the same as those in the case 1, but

$$B_3 = area(D_1(o_1) \cap D_1(o_2) \cap D_1(o))$$  

the shaded area in Figure 3

Again, $B_3$ is the only area that depends on $\phi_2$. We will now give an expression for $B_3$. We show first that $\angle c_2 a_1 o_2 = \angle a_2 o c_2$ by showing that $\phi_2 - \phi_1 = \phi_2 - \phi_1$. Then, it follows that the region with area $B_3$ is split in half by the line segment $c_2, d_2$. From Figure 1, one can see that

$$\phi_1 = \phi^* + \left(\frac{\pi}{2} - \angle b^* a_1 o\right) = \phi^* + \left(\frac{\pi}{2} - \frac{\phi^*}{2}\right) = \frac{\pi}{2} + \frac{\phi^*}{2}$$  

(16)

To find $\phi_2$, observe that, as in the proof of Fact 2, we have $a_2, b_2 \perp o_1, o_2$. So, $\sin \angle b_2 a_2 o = \frac{b_2}{o_2}$. Comparing with (14), we see that $\sin \angle b_2 a_2 o = \frac{\phi^*}{2}$. This implies that $\angle b_2 a_2 o = \frac{\phi^*}{2}$. Thus,

$$\phi_2 = \phi_2 + \left(\frac{\pi}{2} - \angle b_2 a_2 o\right) = \phi_2 + \left(\frac{\pi}{2} - \frac{\phi^*}{2}\right)$$  

(17)

Lastly, using the fact that $c_2, o \perp o_1, o_2$, we have

$$\phi_c = \pi - \left(\frac{\pi}{2} - \angle o_2 a_1 o\right) = \pi - \left(\frac{\pi}{2} - \frac{\phi_2}{2}\right) = \frac{\pi}{2} + \frac{\phi_2}{2}$$  

(18)
Figure 3: The case when $\phi_2 > \phi^*$

It follows that $\phi_{c_2} - \phi_{a_1} = \phi_{c_2} - \phi_{c_2} = \frac{\phi_2}{2} - \frac{\phi^*}{2}$. Using that the circle $\partial D_1(o_1)$ in the polar system is

$$r = \sqrt{1 - \delta_b^2 \sin^2 \phi - \delta_b \cos \phi}$$

and that

$$\phi_{d_2} = -(\pi - \phi_{c_2}) = -\left(\frac{\pi}{2} - \frac{\phi_2}{2}\right)$$

we get

$$B_3 = 2 \left( \int_{\phi_2 - \frac{\phi^*}{2}}^{\frac{\pi}{2} + \frac{\phi^*}{2}} \int_0^{\sqrt{1 - \delta_b^2 \sin^2 \phi - \delta_b \cos \phi}} r dr d\phi + \frac{\phi_2 - \phi^*}{2\pi} \cdot \pi \right)$$

$$= \int_{\phi_2 - \frac{\phi^*}{2}}^{\frac{\pi}{2} + \frac{\phi^*}{2}} \left( 1 - \delta_b^2 \sin^2 \phi + \delta_b^2 \cos^2 \phi - 2 \delta_b \cos \phi \sqrt{1 - \delta_b^2 \sin^2 \phi} \right) d\phi + \frac{\phi_2 - \phi^*}{2}$$
Thus,\[ \frac{dX(\phi_1, \phi_2)}{d\phi_2} = \frac{dB_b}{d\phi_2} = -\frac{1}{2} \left[ 1 - \delta_b^2 \sin^2(\frac{\phi_1}{2} + \frac{\phi_2}{2}) + \delta_b^2 \cos^2(\frac{\phi_1}{2} + \frac{\phi_2}{2}) \right] - 2\delta_b \cos(\frac{\phi_1}{2} + \frac{\phi_2}{2}) \sqrt{1 - \delta_b^2 \sin^2(\frac{\phi_1}{2} + \frac{\phi_2}{2})} + \frac{1}{2} \] \[ = \frac{1}{2} \left[ \delta_b^2 \cos^2(\frac{\phi_1}{2}) - \delta_b^2 \sin^2(\frac{\phi_1}{2}) + 2\delta_b \sin(\frac{\phi_1}{2}) \sqrt{1 - \delta_b^2 \cos^2(\frac{\phi_1}{2})} \right] \] \[ = \frac{1}{2} \left[ -\delta_b \sin(\frac{\phi_1}{2}) - \sqrt{1 - \delta_b^2 \cos^2(\frac{\phi_1}{2})} \right] \geq 0 \]

The last inequality follows because \( 0 \leq \delta_b \sin(\frac{\phi_1}{2}) \leq 1 \), \( 0 \leq \sqrt{1 - \delta_b^2 \cos^2(\frac{\phi_1}{2})} \leq 1 \), and thus \( (\delta_b \sin(\frac{\phi_1}{2}) - \sqrt{1 - \delta_b^2 \cos^2(\frac{\phi_1}{2})})^2 < 1 \). \( \square \)

**Fact 4** Uniformly for all \( i \), we have \( X(q_i, \tilde{u}_i) = O(\frac{1}{b \log^3 b}) \).

**Proof.** Without loss of generality, let \( i = 0 \) and \( v = (0, 0) \). To simplify notation, define \( x_b = \delta_b \cos(-\frac{1}{2} \theta_b), y_b = \delta_b \sin(-\frac{1}{2} \theta_b) \). Let \( (\xi, \eta) \) be the point in the first quadrant where the circles \( x^2 + y^2 = 1 \) and \( (x-x_b)^2 + (y-y_b)^2 = 1 \) meet. Then
\[
X(q_0, \tilde{u}_0) \leq 4 \int_0^\xi \sqrt{1 - x^2} - (y_b + \sqrt{1 - (x-x_b)^2}) \, dx \]
\[
= -4y_b \xi + 4 \int_0^\xi \frac{-2xx_b + x_b^2}{\sqrt{1 - x^2} + \sqrt{1 - (x-x_b)^2}} \, dx \]

Hence we have
\[
X(q_0, \tilde{u}_0) = O(x_b \xi^2) + O(x_b^2 \xi) + O(\xi \eta^2) + O(\xi^2 \eta). \tag{20} \]

Note that \( x_b^2 + y_b^2 = \delta_b^2 = \frac{1}{b \log^3 b} \), that \( \xi^2 + \eta^2 = 1 \), that \( (\xi-x_b)^2 + (\eta-y_b)^2 = 1 \), that \( x_b = \delta_b(1 + O(\theta_b^2)) \), and that \( y_b = \frac{\delta_b \theta_b}{2}(1 + O(\theta_b^2)) \). Combining these equations, we get \( \xi = O(\delta_b) \). Putting this estimate back into (20), we get
\[
X(q_0, \tilde{u}_0) = O(\frac{1}{b \log^3 b}). \tag{21} \]

\( \square \)

### 4 Two Point Dominating Sets

Let \( n \) be an integer such that \( n \geq 3 \), and let \( L_n = \lfloor n^{1/3}(\log n)^2 \rfloor \) and \( \delta_n = \frac{1}{n^{1/3} \log n} \). Select \( n \) points \( X_1, X_2, ..., X_n \) independently and uniform randomly from the unit disk \( D_1(o) \) and form the unit disk graph \( G_n (\equiv G^n_n) \) by putting an
edge between two of the n points iff the distance between them is less than 1. Our goal in this section is to prove that, with high probability, \( G_n \) contains a dominating set consisting of two vertices of \( G_n \) that are adjacent to each other.

For \( 0 \leq i < L_n \), let \( Q_i, R_i \) denote the sectors of \( D_{\delta_n}(o) \) as defined in the previous section and let \( N(Q_i), N(R_i) \) respectively be the number of vertices of \( G_n \) that lie in \( Q_i \) and \( R_i \). Let \( \tau_n = \sum_{i=0}^{L_n-1} I_i \) where, in this section only, the indicator variable \( I_i = 1 \) if and only if \( N(R_i) = N(Q_i) = 1 \) (and otherwise \( I_i = 0 \)).

**Lemma 4** \( \Pr \left( \tau_n < \frac{n^{1/3}}{\log n} \right) = O\left( \frac{\log n}{n^{1/3}} \right) \)

**Proof.** Let 
\[ p = \frac{\text{Area}(Q_i)}{\text{Area}(D_1(0))} = \frac{\pi \delta_n^2}{2L_n} = \frac{1}{2n \log n} \left( 1 + O \left( \frac{1}{n^{1/3} \log^2 n} \right) \right). \] 
Then 
\[ E(I_i) = n(n-1)p^2(1-2p)^{n-2}, \] 
and 
\[ E(\tau_n) = L_n n(n-1)p^2(1-2p)^{n-2} = \frac{n^{1/3}}{4(\log n)^6} \left( 1 + O \left( \frac{1}{n^{1/3} \log^2 n} \right) \right). \] 

Similarly, for \( i \neq j \) 
\[ E(I_i I_j) = n(n-1)(n-2)(n-3)p^4(1-4p)^{n-4}. \] 
Since \( \tau_n = \sum_{i=0}^{L_n-1} I_i \), and the \( I_i \)'s are identically distributed, we have 
\[ \text{Var}(\tau_n) = L_n(L_n-1)E(I_1 I_2) + L_n E(I_1) - (E(\tau))^2. \] 
Combining this identity with the expression for \( E(I_i) \) in (23), the expression for \( E(I_i I_j) \) in (25), and the definitions for \( L_n, \delta_n \) and \( p \), we get 
\[ \text{Var}(\tau_n) = E(\tau_n) \left( 1 + O \left( \frac{1}{(\log n)^8} \right) \right). \] 
The lemma now follows by Chebyshev’s inequality.

**Theorem 5** There is a constant \( c > 0 \) such that, with probability greater than \( 1 - \frac{c}{(\log n)^7} \), the random graph \( G_n \) has a connected dominating set that consists of two vertices in \( D_{\delta_n}(o) \).

**Proof.** Let \( T_n \subseteq \{0, 1, 2, 3, \ldots, L_n - 1\} \) be the random subset of indices such that \( i \in T_n \) iff \( N(Q_i) = N(R_i) = 1 \). If \( T_n \neq \emptyset \), define \( Y = \min T_n \) to be the smallest of the indices in \( T_n \); otherwise, if \( T_n = \emptyset \), set \( Y = -1 \). Define the indicator
random variable $X_n$ as follows: If $\tau_n = |T_n| = 0$ then $X_n = 0$; otherwise, if $T_n = \{i_1, i_2, \ldots, i_{\tau_n}\}$ and $i_1 < i_2 < \ldots < i_{\tau_n}$, then $X_n = 1$ iff $Q_{i_1} \cup R_{i_1}$ contains a two-point connected dominating set for $G_n$.

Let $V = \{v_1, v_2, \ldots, v_n\}$ be the set of vertices of $G_n$, selected independently and uniformly randomly from $D_1(o)$. Define $Z = V \cap D_{\delta_n}(o)$ to be set of vertices that lie near the origin $o$, and let $Z = |Z|$ be the number of these points. Then

$$\Pr(X_n = 0) \leq \Pr \left( X_n = 0, \tau_n \neq 0, Z \leq \frac{2n^{1/3}}{(\log n)^2} \right) + \Pr(\tau_n = 0)$$

$$+ \Pr \left( Z > \frac{2n^{1/3}}{(\log n)^2} \right). \quad (27)$$

Note that $Z$ has a binomial distribution: $Z \overset{d}{=} Bin(n, \delta_n^2)$. If $\beta = \frac{2n^{1/3}}{(\log n)^2}$, then by Chernoff’s inequality,

$$\Pr(Z \geq \beta) \leq \exp\left(-\frac{n^{1/3}}{4(\log n)^2}\right). \quad (28)$$

By Lemma 4, $\Pr(\tau_n = 0) = O\left(\frac{\log n}{n^{1/3}}\right)$. Therefore

$$\Pr(X_n = 0) \leq \Pr(X_n = 0, \tau_n \neq 0, Z \leq \beta) + O\left(\frac{\log n}{n^{1/3}}\right). \quad (29)$$

Now we decompose the first term on the right side of (29) according to the value of $Y$.

$$\Pr(X_n = 0, \tau_n \neq 0, Z \leq \beta) = \sum_{k=0}^{L_n-1} \Pr(X_n = 0|Y = k, Z \leq \beta)\Pr(Y = k, Z \leq \beta). \quad (30)$$

(The redundant condition $\tau_n \neq 0$ need not be included on the right side of (30) because it a consequence of the condition $Y \geq 0$.) We have

$$\Pr(X_n = 0|Y = k, Z \leq \beta) = \sum_S \Pr(X_n = 0|Z = S, Y = k)\Pr(Z = S|Y = k, Z \leq \beta) \quad (31)$$

where the sum is over subsets $S \subseteq [n]$ such that $2 \leq |S| \leq \beta$.

$$\Pr(X_n = 0|Z = S, Y = k) = 1 - \Pr(X_n = 1|Z = S, Y = k), \quad (32)$$

so it is enough to find a lower bound for $\Pr(X_n = 1|Z = S, Y = k)$.

To simplify notation, let $\gamma = X(\bar{q}_0, \bar{u}_0)$, and recall that $\gamma = O\left(\frac{1}{n \log n}\right)$.

In this section of the paper, define $|D_{\delta_n}(o)| = \frac{\pi}{n^{2/3}(\log n)^2}$ to be the area of the disk $D_{\delta_n}(o)$, and $|D_1(o)| = \pi = \text{area of the unit disk centered at } o$. An important observation is that, once we have specified $n - |S| = \text{the number of}$
points that fall outside $D_{\delta_n}(o)$, the locations in $D_{\delta_n}(o)^c$ of these $n - |S|$ points are independent of the locations of the $|S|$ points in $D_{\delta_n}(o)$. Hence

$$\Pr(X_n = 1|Z = S, Y = k) \geq \frac{\left(1 - \frac{|D_{\delta_n}(o)|}{|D_1(o)|} - \gamma \right)^{n-|S|}}{\left(1 - \frac{|D_{\delta_n}(o)|}{|D_1(o)|}\right)^{n-|S|}}$$

(33)

$$\geq \left(1 - \frac{C}{n(\log n)^3}\right)^{n-|S|} \geq 1 - \frac{C'}{(\log n)^3}$$

(34)

for some constants $C$ and $C'$ which are independent of $Z, Y$. Hence

$$\Pr(X_n = 0) \leq \frac{c}{(\log n)^3}$$

(35)

for some positive constant $c$ that does not depend on $n$.

□

We note that the result obtained in Theorem 5 depends on a delicate trade-off: We must choose $\delta_n$ small enough and $L_n$ large enough to guarantee that for any $q \in Q_i$ and any $u \in R_i$, where $(Q_i, R_i)$ is a pair of opposite sectors of $D_{\delta_n}(o)$, there is high probability that none of the points $X_1, X_2, ..., X_n$ lie in the ‘uncovered’ region $(D_1(q) \cup D_1(u))^c \cap D_1(o)$. On the other hand, $\delta_n$ must not be so small or $L_n$ so large that we cannot find (with high probability) some pair of opposite sectors $(Q_i, R_i)$ such that there is some $X_j \in Q_i$ and $X_k \in R_i$. The necessity for this ‘trade-off’ stems from the fact that a unit disk centered at a point $o$ cannot be completely covered with two unit disks having centers at points other than $o$, i.e. $D_1(o) \not\subseteq D_1(u) \cup D_1(v)$ for $u \neq o \neq v$.

5 Final Comments

The original question posed in the introduction concerned the typical size of a minimum connected dominating set in the random disk graph $G_n^2$. Theorem 5 establishes that, with asymptotic probability one, $G_n^2$ has a two-point connected dominating set. By Theorem 1, this two-point dominating set is also a minimum connected dominating set (with asymptotic probability one).

Theorem 5 was difficult because a unit disk, centered at a point $o$, cannot be completely covered with two unit disks having centers at points other than $o$. In contrast, one can easily find three points $u, v, w \in D_1(o) \setminus \{o\}$ such that $D_1(o) \subseteq D_1(u) \cup D_1(v) \cup D_1(w)$. Using this fact, the authors show in [4] that there is some $\alpha$, with $0 < \alpha < 1$, such that for every $k \geq 3$ the probability that there does not exist a $k$-point connected dominating set in $G_n^2$ is less than $3\alpha^n$. This exponential probability bound was used to analyze the performance of the Rule $k$ local algorithm for constructing a connected dominating set in a wireless network model when $k \geq 3$. By comparing the exponential $O(\alpha^n)$ probability bound for $k \geq 3$ with the $O\left(\frac{1}{\log n}\right)$ bound for $k = 2$, we gain some insight into
the empirical observation that the Rule $k$ algorithm does not perform as well for $k = 2$ as it does for $k \geq 3$.

Finally, we have not determined the typical size of the minimum connected dominating set for dimensions $m > 2$. The case $m = 2$ was already challenging, and we did not see how to extend our methods to the general case. We did prove in [3] that, when $m = 3$, the probability that there does not exist a 4-point CDS is exponentially small. Therefore, with high probability the smallest CDS in $G_m^n$ consists of either 2 or 3 vertices. It is reasonable to conjecture that an analogous statement holds for all $m \geq 2$: with asymptotic probability 1, $G_m^n$ has an $m$ point CDS.

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References


