Notes 6

CONTRACTION MAPPING THEOREM

Let \((X, d)\) be a metric space.

**Definition.** A function \(T : X \rightarrow X\) is called a **contraction mapping** if there exists \(k \in \mathbb{R}\) such that \(0 < k < 1\) and 
\[ d(T(x), T(y)) \leq kd(x, y) \]
for any \(x, y \in X\).

**Example.**
(a) Let \(X = \mathbb{R}\) and suppose \(f : \mathbb{R} \rightarrow \mathbb{R}\) is such that \(|f'(x)| < k\) for all \(x \in \mathbb{R}\) and some \(k \in (0, 1)\). Then \(f\) is a contraction mapping.

(b) Let \(X = C[0, 1]\) and suppose \(T : C[0, 1] \rightarrow C[0, 1]\) is given by
\[ (T(u))(x) = \frac{1}{2} \int_0^x u(s) \, ds, \]
where \(u \in C[0, 1]\). Then \(T\) is a contraction mapping.

**Contraction Mapping Theorem.** If \((X, d)\) is a complete metric space and \(T : X \rightarrow X\) is a contraction mapping, then \(T\) has one and only one fixed point (i.e., there exists exactly one \(x \in X\) such that \(T(x) = x\)).

**Sketch of the proof the Contraction Mapping Theorem.** The proof proceeds in several steps:

(a) starting from an arbitrary \(x_0 \in X\), construct the sequence \(\{x_n\} \subseteq X\) by taking \(x_n = T(x_{n-1})\);

(b) prove that \(d(x_n, x_{n+1}) \leq k^n d(x_0, x_1)\), and therefore \(d(x_m, x_n) = \frac{k^n - k^m}{1 - k} d(x_0, x_1)\)
for any \(n, m \in \mathbb{N}\), \(m > n\);

(c) conclude that \(\{x_n\}\) is Cauchy, and thus, since \(X\) is complete, converges;

(d) prove that \(x = \lim_{n \to \infty} x_n\) is a fixed point of \(T\);

(e) prove that \(x\) is the only fixed point of \(T\).

**Remark.** The Theorem does not hold if \(X\) is not complete or if \(T\) satisfies only \(d(T(x), T(y)) \leq d(x, y)\).

The Contraction Mapping Theorem is widely used for proving the existence and uniqueness of solutions of integral and differential equations.