Applied Analysis and PDE

6. Scalar conservation laws

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Conservation laws

- $u(x, t)$ – density of material at location x and time t
- $$\int_a^b u(x, t)dx \text{ – The total amount of material in interval } [a, b].$$
- $f(u) = f(u(x, t))$ – flux at location x and time t
  - Flux is the rate at which the material is passing the x at time t.
  - The flux is positive if the flow is in the positive x-direction.
- Conservation law (in integral form):

$$\frac{d}{dt}\int_a^b u(x, t)dx = f(u(a, t)) - f(u(b, t)).$$

- If u is differentiable

$$\int_a^b \frac{d}{dt} u(x,t) + (f(u(x,t)))_x \, dx = 0$$

since this holds for any interval $[a, b]$ we obtain the conservation law in differential form:

$$u_t + f(u)_x = 0.$$

Remarks:

- The conservation law in integral form holds also for non-smooth solutions u
- A *constitutive relation* or *equation of state* is required to determine the system.

Constitutive relations

- Fick's law:

$$f(u) = -u_x$$

giving the diffusion (heat) equation

$$u_t - u_{xx} = 0.$$

- Adding convection:

$$f(u) = -\epsilon u_x + u^2/2$$

gives Burgers' equation

$$u_t + uu_x = \epsilon u_{xx}$$

or with $\epsilon = 0$ the inviscid Burgers' equation

$$u_t + uu_x = 0.$$

Traffic flow

Let us derive a simple constitutive relation for traffic flow.

- $u(x,t)$ – the density (cars per mile) of cars on a road moving from left to right.
- If n is the maximum density we could model the speed at which people drive, by

$$m(n-u)$$

  where m is a constant.
- The rate at which cars pass a given point is the product of their speeds and the density so

$$f(u) = um(n-u).$$

- Conservation of cars leads to

$$\frac{d}{dt}\int_a^b u(x,t)dx = f(u(a,t)) - f(u(b,t)).$$

Characteristics and first order equations

Consider the linear advection equation

$$u_t + cu_x = 0, \qquad u(x,0) = u_0(x).$$

Note that

$$\frac{d}{dt}u(x(t),t) = u_t(x(t),t) + x'(t)u_x(x(t),t) = (x'(t) - c)u_x(x(t),t).$$

Hence u is constant ($\frac{d}{dt}u = 0$) along *characteristics* $x(t)$ given by

$$x'(t) = c \implies x(t) = ct + x(0).$$

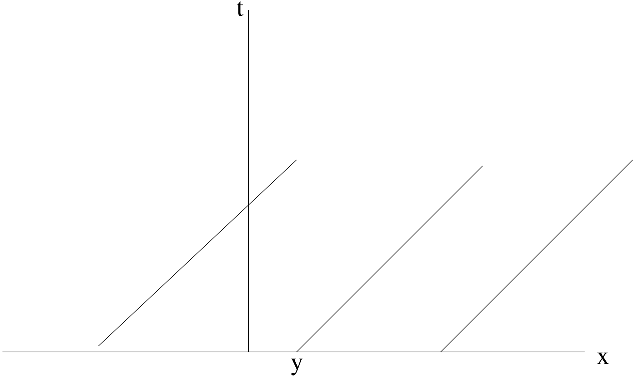
Denote $y = x(0)$ and solve for y in

$$x = ct + y \implies y = x - ct$$

to obtain solution at (x,t):

$$u(x,t) = u(y,0) = u_0(y) = u_0(x - ct).$$

Characteristics for the linear case



Non-homogeneous linear case

Consider

$$u_t(x,t) + cu_x(x,t) = g(u,x,t), \qquad u(x,0) = u_0(x).$$

Along characteristics $x'(t) = c$, u is the solution of an ODE:

$$\frac{d}{dt}u(x(t),t) = u_t(x(t),t) + x'(t)u_x(x(t),t) = g(u,x(t),t).$$

The nonlinear case

Let us consider

$$u_t + f(u)_x = 0, \qquad x \in \mathbb{R}, t > 0,$$
$$u(x,0) = u_0(x) \qquad x \in \mathbb{R}.$$

Note that the first equation can be written as

$$u_t + f'(u)u_x = 0.$$

Hence

$$\frac{d}{dt}u(x(t),t) = u_t(x,t) + x'u_x(x,t) = (x' - f'(u))u_x(x,t) = 0$$

along characteristics

$$\frac{d}{dt}x(t) = f'(u(x,t)).$$

Note that the characteristics are still straight lines.

Example

$$\begin{aligned} u_t + uu_x &= 0, \qquad && x \in \mathbb{R}, t > 0, \\ u(x,0) &= x, \qquad && x \in \mathbb{R}. \end{aligned}$$

Nonlinear case ctd.

We come back to

$$u_t + f(u)_x = 0, \qquad x \in \mathbb{R}, t > 0,$$
$$u(x,0) = u_0(x) \qquad x \in \mathbb{R}.$$

with extra assumption $f''(u) > 0$, $\forall u$, (so truly nonlinear). Recall

$$\frac{d}{dt}x(t) = f'(u(x,t)) = const \implies x(t) - f'(u)t = x(0).$$

are the characteristics along which $u(x(t),t)$ is constant. Hence

$$u(x,t) = u(y,0) = u_0(y)$$

where

$$y = x - f'(u_0(y))t \implies y_x = 1 - y_x u_0'(y) f''(u_0(y))t$$
$$\implies y_x = \frac{1}{1 + u_0'(y) f''(u_0(y))t}$$

Further

$$u_x = y_x u_0'(y) = \frac{u_0'(y)}{1 + u_0'(y) f''(u_0(y)) t}.$$

Remarks:

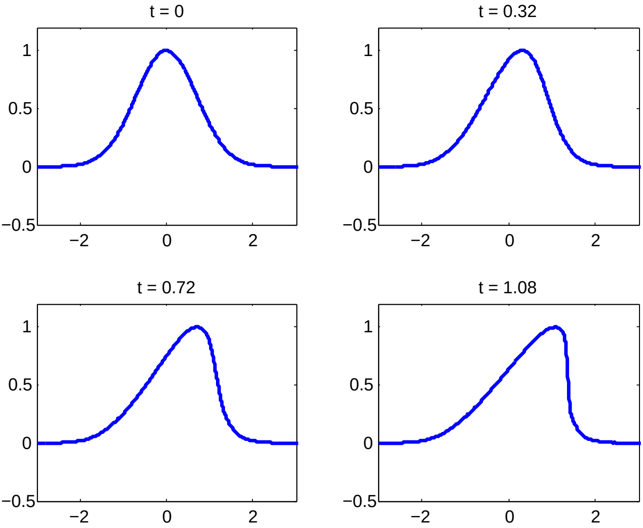
- If $u_0' > 0$ (recall $f'' > 0$) then solution exists for all $t > 0$.
- If $u_0'(y) < 0$ for some y, the characteristics intersect. This will happen at the earliest time $t > 0$ such that

$$1 + u_0'(y) f''(u_0(y)) t = 0.$$

Example:

$$u_t + uu_x = 0 \qquad x \in \mathbb{R},\ t > 0$$
$$u(x,0) = u_0(x) = e^{-x^2}.$$

Burgers' equation: shock development



Example:

$$u_t + uu_x = 0 \qquad x \in \mathbb{R},\ t > 0$$

$$u(x,0) = u_0(x) = \begin{cases} 1 & x \le 0 \\ 1 - k^{-1}x & 0 < x < k \\ 0 & x \ge k \end{cases}$$

Solution:

Note: characteristics all intersect at (k, k) so a smooth solution cannot exist for $t > k$.

It's clear that $u(x,t) = 1$ for $x \leq t \leq k$ and $u(x,t) = 0$ for $x \geq k$ and $t \leq k$.

In the triangle $t < x < k$ we have

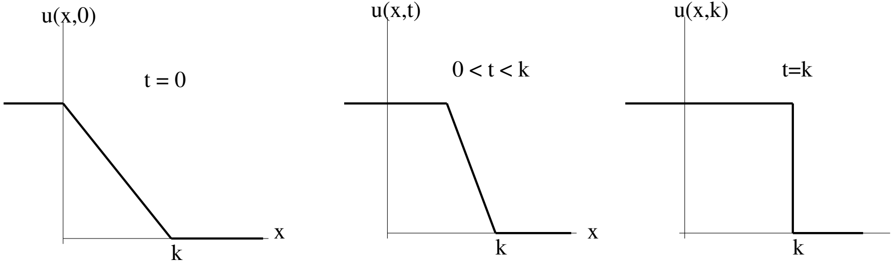
$$x = t(1 - k^{-1}y) + y \implies y = \frac{x - t}{1 - k^{-1}t}.$$

Hence

$$u(x,t) = u_0(y) = 1 - k^{-1}y = \frac{x - k}{t - k}, \qquad \text{for } t < x < k.$$

So finally (for $t \leq k$)

$$u(x,t) = \begin{cases} 1 & x \leq t \\ \frac{x-k}{t-k} & t < x < k \\ 0 & x \geq k \end{cases}$$


u(x,0)
t = 0
x
k
u(x,t)
0 < t < k
x
k
u(x,k)
t=k
k

Example:

$$u_t + uu_x = 0\,, \qquad x \in \mathbb{R}\,,\ t > 0$$

$$u(x,0) = u_0(x) = \begin{cases} 0 & x \leq 0 \\ k^{-1}x & 0 < x < k \\ 1 & x \geq k \end{cases}$$

where $k > 0$.
Since $u_0(x)$ is an increasing function we have a solution for all $t > 0$:

$$u(x,t) = \begin{cases} 0 & x \leq 0 \\ \frac{x}{t+k} & 0 < x < k+t \\ 1 & x \geq k+t \end{cases}$$

Discontinuous solutions

Example:

$$u_t + uu_x = 0 \qquad x \in \mathbb{R},\ t > 0$$
$$u(x,0) = u_0(x) = \begin{cases} 1 & x < 0 \\ 0 & x > 0 \end{cases}$$

Characteristics:

To determine the curve $x = y(t)$ across which u is discontinuous we will again make use of the conservation law.

Consider again a general flux $f(u)$ and set

$$I(t) = \int_a^b u(x,t)\,dx = \int_a^{y(t)} u(x,t)\,dx + \int_{y(t)}^b u(x,t)\,dx$$

Then

$$\frac{dI(t)}{dt} = \int_a^{y(t)} u_t(x,t)\,dx + su_\ell + \int_{y(t)}^b u_t(x,t)\,dx - su_r$$

where the notation $s = y'(t)$ is used.

Since $u(x,t)$ is smooth for $x < y(t)$ and $x > y(t)$ we have

$$\int_a^{y(t)} u_t(x,t)\,dx = f_a - f_\ell \qquad \text{and} \qquad \int_{y(t)}^b u_t(x,t)\,dx = -f_b + f_r$$

where we use the notation

$$f(u_\ell) = f_\ell\,, \quad f(u_r) = f_r\,, \quad f(u(a)) = f_a\,, \quad f(u(b)) = f_b$$

Hence we obtain

$$\frac{dI(t)}{dt} = f_a - f_\ell + su_\ell - f_b + f_r - su_r$$

The conservation law is

$$\frac{dI(t)}{dt} = f_a - f_b$$

and combining all this we obtain

$$s[u] = [f]$$

where

$$[u] = u_r - u_\ell \qquad \text{and} \qquad [f] = f_r - f_\ell$$

This is called the **jump condition** *(or Rankine-Hugoniot condition in fluid mechanics). The curve $y = x(t)$ is called the shock and the discontinuity is called a shock wave.*

For Burgers' equation $f(u) = u^2/2$ the jump condition is

$$s(u_r - u_\ell) = \frac{u_r^2 - u_\ell^2}{2} \implies s = y' = \frac{u_r + u_\ell}{2}$$

Back to the initial example with

$$u(x,0) = u_0(x) = \begin{cases} 1 & x < 0 \\ 0 & x > 0 \end{cases}$$

Hence $y(t) = st = \frac{1}{2}t$ and the discontinuous solution is

$$u(x,t) = \begin{cases} 1 & x < st \\ 0 & x > st \end{cases} .$$

Example: Consider Burgers' equation with initial data

$$u(x,0) = u_0(x) = \begin{cases} 1 & x \leq 0 \\ 1 - k^{-1}x & 0 < x < k \\ 0 & x \geq k \end{cases}$$

We have already shown that for $t < k$

$$u(x,t) = \begin{cases} 1 & x \leq t \\ \frac{x-k}{t-k} & t < x < k \\ 0 & x \geq k \end{cases}$$

and that u developed a singularity at $t = k$.

For $t = k$ we have that

$$u(x,k) = \begin{cases} 1 & x < k \\ 0 & x > k \end{cases}$$

so we take $u_\ell = 1$ and $u_r = 0$ in the jump condition to get $y' = 1/2$. Hence

$$y(t) = \frac{t+k}{2}$$

So the solution for $t \geq k$ is

$$u(x,t) = \begin{cases} 1 & x < (t+k)/2 \\ 0 & x > (t+k)/2 \end{cases},$$

i.e., the step function continues to travel to the right.

Example: If u is a smooth solution of

$$u_t + \left(\frac{u^2}{2}\right)_x = 0$$

and $w = u^2$, then

$$w_t + \left(\frac{2w^{3/2}}{3}\right)_x = 0$$

These equations have different discontinuous solutions, for example for

$$u(x,0) = w(x,0) = \begin{cases} 1 & x < 0 \\ 0 & x > 0 \end{cases}$$

the jump conditions give $s = 1/2$ and $s = 2/3$ respectively.

It is crucial to know the underlying conservation law!

Non-uniqueness

Example: Consider again Burgers' equation

$$u_t + uu_x = 0 \qquad x \in \mathbb{R},\ t > 0$$
$$u(x,0) = u_0(x) = \begin{cases} 0 & x < 0 \\ 1 & x > 0 \end{cases}$$

The method of characteristics determines the solution for all $t > 0$ except in the region $0 < x < t$.

An infinite number of solutions satisfying the jump conditions exist:

$$u_\alpha(x,t) = \begin{cases} 0 & x < \alpha t/2 \\ \alpha & \alpha t/2 < x < (1+\alpha)t/2 \\ 1 & x > (1+\alpha)t/2 \end{cases} \qquad \text{for any } \alpha \in (0,1).$$

However, we can also fill the "empty" triangle by a smooth function.

Suppose we smooth out the initial data in order to fill the empty triangle with characteristics:

This suggests to look for solutions of the form $u(x,t) = g(x/t)$ for $0 < x < t$. Then

$$0 = u_t + uu_x = -\frac{x}{t^2}g' + \frac{1}{t}gg' = (-\frac{x}{t^2} + \frac{1}{t}g)g'.$$

So either

- $g = \mathit{const}$ giving a discontinuous solution, but not the one given by the jump condition.
- or $g(z) = z$.

So a smooth solution (the so-called *rarefaction wave*) is given by

$$u_2(x,t) = \begin{cases} 0 & x < 0 \\ x/t & 0 < x < t \\ 1 & x > t \end{cases}$$

For this problem the rarefaction solution can also be obtained directly:

Take initial data

$$u(x,0) = u_0(x) = \begin{cases} 0 & x \leq 0 \\ k^{-1}x & 0 < x < k \\ 1 & x \geq k \end{cases}$$

As $k \to 0$ the initial data converges to the step function and the solution

$$u(x,t) = \begin{cases} 0 & x < 0 \\ \frac{x}{k+t} & 0 < x < t+k \\ 1 & x > t+k \end{cases}$$

converges to u_2.

Entropy condition

Which solution should we choose!?

- Discontinuities were introduced because of colliding characteristics.
- *Therefore we only accept discontinuities that separate two characteristics that otherwise would inpinge on each other.*
- For Burgers' equation this means that a discontinuity needs to satisfy the *entropy condition*

$$u_\ell > s > u_r.$$

- For the general conservation law

$$u_t + f(u)_x = 0$$

  the entropy condition is

$$f'(u_\ell) > s > f'(u_r).$$

  If $f'' > 0$, this again implies

$$u_\ell > u_r.$$

Summary: Riemann problem

$$u_t + f(u)_x = 0, \quad u(x,0) = u_0(x) = \begin{cases} u_\ell & x < 0 \\ u_r & x > 0 \end{cases}$$

and $f'' > 0$.

1. If $u_\ell > u_r$ the admissible solution is

$$u(x,t) = \begin{cases} u_\ell & x < st \\ u_r & x > st \end{cases}$$

   where the shock speed s satisfies $s = \frac{f(u_r) - f(u_\ell)}{u_r - u_\ell}$.

2. If $u_\ell < u_r$, we look for a rarefaction wave

$$u(x,t) = \bar{u}(x/t) = \bar{u}(z), \quad z = x/t.$$

   A calculation shows that $f'(\bar{u}) = z$ and

$$u(x,t) = \begin{cases} u_\ell & x < f'(u_\ell)t \\ \bar{u}(x/t) & f'(u_\ell)t < x < f'(u_r)t \\ u_r & x > f'(u_r)t \end{cases}$$

Viscosity solution

We say that u is a viscosity solution if it is the limit as $\epsilon \to 0^+$ of the solution $v = v^\epsilon$ of the parabolic problem

$$v_t + f(v)_x = \epsilon v_{xx}, \qquad x \in \mathbb{R}\,,\ t > 0$$
$$v(x,0) = u_0(x)$$

The fact that v^ϵ converges to the admissible solution is proved by Bianchini and Bressan in 2005.

We will give a heuristic argument that the viscosity solution is admissible, i.e., that

$$f'(u_\ell) > s > f'(u_r);$$

and that the jump condition $s[u] = [f]$ is satisfied at a discontinuity.

Suppose that $u(x,t)$ has a singularity of the type

$$u(x,t) = \begin{cases} u_\ell & x < st \\ u_r & x > st \end{cases}$$

It is natural to assume that near the singularity

$$v(x,t) \approx \bar{v}\left(\frac{x - st}{\epsilon}\right)$$

Substituting this into the original equation gives

$$-s\bar{v}' + f(\bar{v})' = \bar{v}''$$

Integrating gives

$$\bar{v}' = -s\bar{v} + f(\bar{v}) + C \tag{1}$$

where C is a constant.

Note for $v^\epsilon \to u$ to hold we need

$$\lim_{z\to-\infty} \bar{v}(z) = u_\ell \qquad \text{and} \qquad \lim_{z\to\infty} \bar{v}(z) = u_r.$$

Hence u_ℓ and u_r are equilibria of the ODE (1) and

$$C = su_\ell - f(u_\ell) = su_r - f(u_r)$$

In particular, this implies the jump condition

$$s(u_r - u_\ell) = f(u_r) - f(u_\ell).$$

Coming back to

$$\bar{v}' = -s(\bar{v} - u_\ell) + f(\bar{v}) - f(u_\ell) \equiv H(\bar{v})$$

with $\bar{v}(-\infty) = u_\ell$ and $\bar{v}(\infty) = u_r$. Assuming first $u_\ell > u_r$ the phase portrait is

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|--------<--------|
R                 L
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It follows that $H(\bar{v}) < 0$ for $u_r < \bar{v} < u_\ell$ and

$$H'(u_\ell) \geq 0\,, \quad H'(u_r) \leq 0$$

Since $H'(\bar{v}) = f'(\bar{v}) - s$ we obtain the entropy condition.

If $u_\ell < u_r$ then the phase portrait is

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|-------->--------|
L                 R
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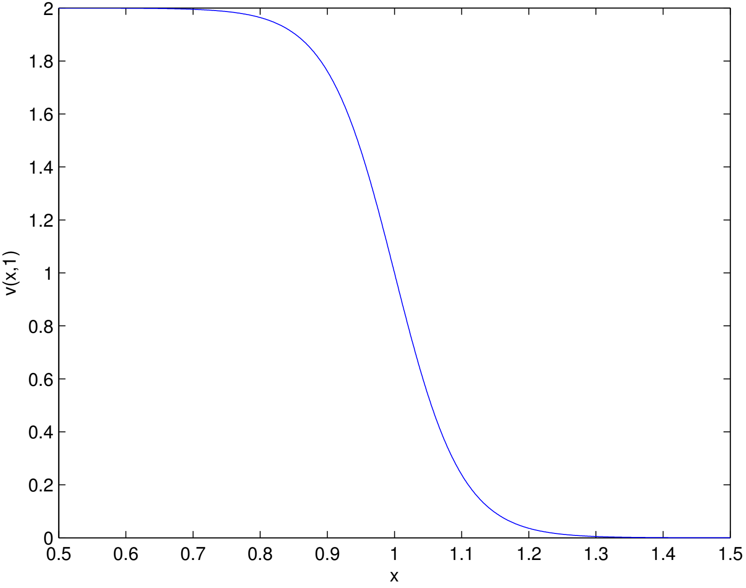
giving $H(\bar{v}) > 0$ for $u_\ell < \bar{v} < u_r$ and again the entropy condition.

For the special case $f(u) = u^2/2$, $\bar{v}$ can be found analytically: For $u_\ell = 2$ and $u_r = 0$

$$v(x,t) = \bar{v}(z) = \frac{2}{1 + \exp\left(\frac{x-t}{\epsilon}\right)}$$

The approximation to the solution v^ϵ with $\epsilon = 0.05$ at time $t = 1$



The Cole-Hopf Transformation

We study again the viscous Burgers' equation:

$$v_t + vv_x = \epsilon v_{xx}\,, \qquad x \in \mathbb{R},\ t > 0$$
$$v(x,0) = v_0(x)$$

Cole and Hopf in the early 1950s found the analytic solution. They did this by showing that the change of variables

$$v = \frac{-2\epsilon w_x}{w}$$

shows that w is the solution of a 1-D heat equation.

We do the change of variables in two steps: First set

$$v = z_x$$

Then

$$z_{tx} + \left(\frac{z_x^2}{2}\right)_x = \epsilon z_{xxx}$$

which can be integrated to

$$z_t + \frac{z_x^2}{2} = \epsilon z_{xx}$$

Now set $z = -2\epsilon \log w$. Then

$$z_t = -\frac{2\epsilon w_t}{w}, \qquad z_x = -\frac{2\epsilon w_x}{w}, \qquad z_{xx} = \frac{2\epsilon w_x^2}{w^2} - \frac{2\epsilon w_{xx}}{w}$$

and a calculation shows that

$$w_t = \epsilon w_{xx}.$$

To find the initial condition $w(x, 0) = w_0(x)$ we solve

$$-2\epsilon w_0'(x) = v_0(x) w_0(x)$$

to get

$$w_0(x) = \exp\left(-\frac{1}{2\epsilon}\int_0^x v_0(y)\, dy\right).$$

Analytic solution of the 1D heat equation is given by the formula

$$w(x,t) = \left(\frac{1}{4\pi\epsilon t}\right)^{1/2} \int_{-\infty}^{\infty} w_0(y) \exp\left(-\frac{(x-y)^2}{4\epsilon t}\right)\, dy.$$

Hence the solution to the original problem is

$$v(x,t) = \frac{\int_{-\infty}^{\infty} \left(\frac{x-y}{t}\right)\, e^{-G(y,x,t)/2\epsilon}\, dy}{\int_{-\infty}^{\infty} e^{-G(y,x,t)/2\epsilon}\, dy}$$

where

$$G(y,x,t) = \frac{(x-y)^2}{2t} + \int_0^y v_0(s)\, ds.$$

Example: Solution of inviscid Burgers' equation with initial data

$$u_0(x) = \begin{cases} 2 & x < 0 \\ 0 & x > 0 \end{cases}$$

is

$$u(x,t) = \begin{cases} 2 & x < t \\ 0 & x > t \end{cases}.$$

In the previous section we approximated the corresponding v^ϵ by

$$v^\epsilon(x,t) \approx \frac{2}{1 + \exp\left(\frac{x-t}{\epsilon}\right)}.$$

Now we can use the exact solution

$$v^\epsilon(x,t) = \frac{2}{1 + Q(x,t)\exp\left(\frac{x-t}{\epsilon}\right)}$$

where $Q = q_1/q_2$ and

$$q_1 = \int_{-x/\sqrt{4\epsilon t}}^{\infty} e^{-s^2}\, ds\,, \qquad q_2 = \int_{(x-2t)/\sqrt{4\epsilon t}}^{\infty} e^{-s^2}\, ds.$$

Traffic flow

See the Models stream for more detail on the model.

- $\rho(x,t)$ – traffic density
- $q(x,t)$ – flux, i.e., number of cars passing x at time t
- The conservation law is

$$\frac{d}{dt}\int_a^b \rho(x,t)\,dx = q(a,t) - q(b,t)$$

  or in differential form

$$\rho(x,t)_t + q(x,t)_x = 0.$$

- It's reasonable to assume the

$$q = V(\rho)\rho$$

  where $V(\rho)$ is the local traffic speed.
- $V(\rho)$ is a decreasing function, the simplest choice being

$$V(\rho)\rho = m\rho(n-\rho)$$

  where m and n are constants.

By scaling ρ and x we obtain the normalised conservation law

$$u_t + f(u)_x = 0$$

where

$$f(u) = u(1-u).$$

Hence $f''(u) = -2 < 0$, so

$$f'(u_\ell) > f'(u_r)$$

implies that at an admissible discontinuity

$$u_\ell < u_r.$$

Example: Consider the traffic flow problem with

$$u(x,0) = u_0(x) = \begin{cases} 3/4 & x < 0 \\ 1/4 & x > 0 \end{cases}$$

Then the admissible solution is

$$u(x,t) = \begin{cases} 3/4 & x \leq -t/2 \\ 1/2 - x/2t & -t/2 < x < t/2 \\ 1/4 & x \geq t/2 \end{cases}$$

Some comments on the traffic problem

- Flux is maximum at density $u = 1/2$.
- At density $u = 1$ no movement.

Suppose we have uniform density, but then the flow hits a red light: