Applied Analysis and PDE
6. Scalar conservation laws: Additional remarks

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Outline

Traffic problem

Weak solutions

Equal-Area Principle

Traffic flow

Recall the traffic flow problem:

$$u_t + f(u)_x = 0 \qquad \text{where } f(u) = u(1-u).$$

Uniform flow being stopped at $x = 0$ and $t > 0$ by a red light leads to IVP/BVP:

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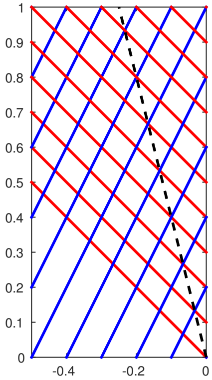
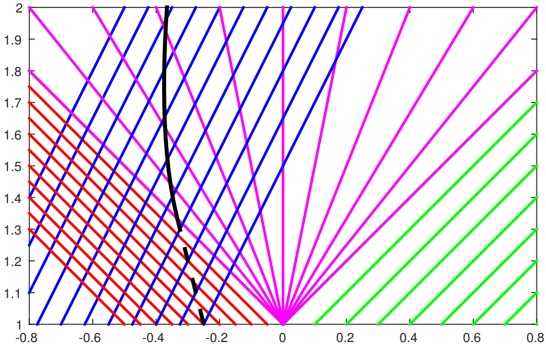


Traffic continued

Now suppose light turns green at $t_0 > 0$ giving initial value problem for $t > t_0$ and $x \in \mathbb{R}$ with

$$u(x, t_0) = \begin{cases} u_\ell & \text{if } x < a_0 \\ 1 & \text{if } a_0 < x < 0 \\ 0 & \text{if } x > 0, \end{cases}$$

where a_0 is the position of the shock on previous slide at time t_0.



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Classical solutions

- A more formal approach to the concept of discontinuous solutions leads to *weak solutions*.
- Consider again the conservation law

$$\begin{aligned} u_t + f(u)_x = 0, \qquad & x \in \mathbb{R}, t > 0, \\ u(x,0) = u_0(x), \qquad & x \in \mathbb{R}. \end{aligned} \tag{1}$$

- A smooth solution u satisfying the above equation is called a *classical solution*.
- We show next how to obtain *weak solutions* of the above equations.

Test functions

Let $v \in C^{\infty}_{\text{comp}}(\mathbb{R}, [0, \infty))$ where $C^{\infty}_{\text{comp}}(\mathbb{R}, [0, \infty))$ is the set of infinitely smooth functions with compact support, i.e.

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Testing (1) with v and integrating by parts gives

$$\begin{aligned} 0 &= \int_0^\infty \int_{\mathbb{R}} u_t v + f(u)_x v dx dt \\ &= -\int_{\mathbb{R}} u_0(x) v(x,0) dx - \int_0^\infty \int_{\mathbb{R}} (u v_t + f(u) v_x) dx dt. \end{aligned}$$

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$$\int_0^\infty \int_{\mathbb{R}} (uv_t + f(u)v_x)\,dx\,dt = -\int_{\mathbb{R}} u_0(x)v(x,0)\,dx, \qquad \forall v \in C^\infty_{\mathrm{comp}}. \tag{2}$$

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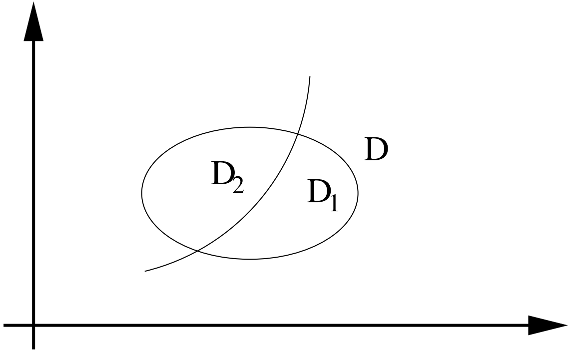
- Any solution of (2) that's sufficiently smooth is a classical solution of (1).
- However, a weak solution can be much less smooth and in that sense generalises the notion of a solution.
- Does the weak solution satisfy the jump condition?

Jump condition

Consider the situation (x-t space):



with the curve separating D_1 and D_2 given by

$$x = y(t)$$

and hence the exterior normal to D_1 on $y(t)$ by

$$\nu = \frac{(-1, y'(t))}{\sqrt{1 + y'(t)^2}}.$$

Jump condition ctd.

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Let v be compactly supported in D, then using the divergence theorem and since u is a solution inside D_1 and D_2:

$$\begin{aligned}
0 &= \int\int_D (uv_t + f(u)v_x)dxdt \\
&= \int\int_{D_1} (uv_t + f(u)v_x)dxdt + \int\int_{D_2} (uv_t + f(u)v_x)dxdt \\
&= -\int\int_{D_1} (u_t + f(u)_x)v)dxdt - \int\int_{D_2} (u_t + f(u)_x)vdxdt \\
&+ \int_\Gamma v(u_\ell y'(t) - f(u_\ell))dt - \int_\Gamma v(u_r y'(t) - f(u_r))dt \\
&= \int_\Gamma v([u]y'(t) - [f(u)])dt.
\end{aligned}$$

Hence

$$0 = [u]y'(t) - [f(u)] \implies s[u] = [f(u)].$$

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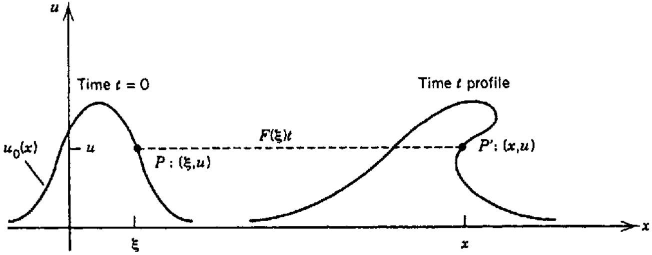
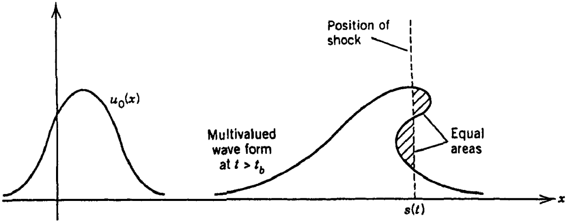
- First follow the characteristics allowing for multivalued solutions (all figures taken from Logan):



Equal-Area Principle ctd.

- The location of the shock $z = y(t)$ at time t is the position at which a vertical line cuts off equal area lobes of the multivalued wavelet:



Equal-Area Principle ctd.

- Why does this work?

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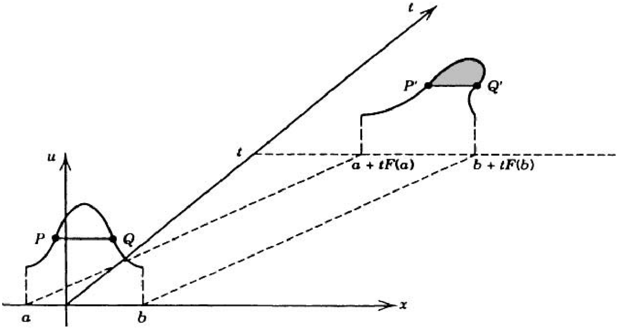
- Any horizontal line segment PQ at $t = 0$ has the same length as $P'Q'$ at time t because points on the wave at the same height u move at the same speed $c(u) = f'(u)$ so that the area under a curve segment remains constant:



Exercise

As a trivial example of the principle consider

$$u_t + uu_x = 0, \qquad u(x,0) = \begin{cases} 1 & x < 0 \\ 0 & x > 0 \end{cases}$$

and compute the solution at $t = 1$.