Formal conjugacy growth and hyperbolicity

Les Diablerets 2016

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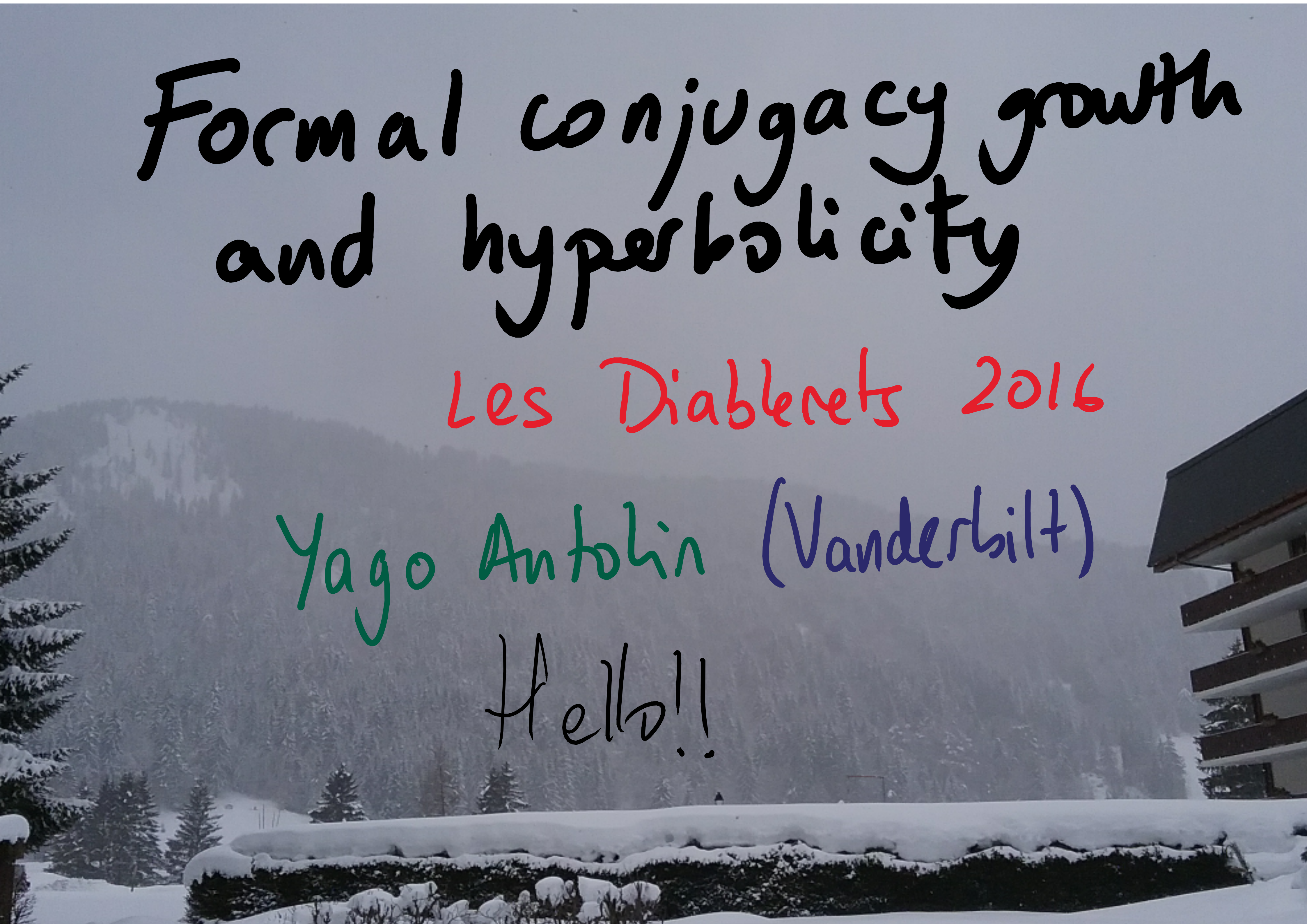
Hello!!



Growth in groups

$G = \langle X \rangle \quad |X| < \infty$

$B_X(n) = \{ g \in G \mid |g|_X \le n \}$ ball of radius n

Gromov '81 $|B_X(n)| \le$ polynomial in n

$\iff$

G virt. nilpotent

Grigorchuk '84 $\exists\, G \quad \text{poly} < |B_X(n)| < \exp$

Cannon '84 G hyperbolic $\implies \sum_{n \ge 0} |B_X(n)| z^n$ rational function

Conjugacy Growth in groups

$g \sim h \iff \exists\, c \in G \quad cgc^{-1} = h$

$\mathrm{Conj}(n) := B_X(n) \cap G/_{\sim}$

Questions

- $|\mathrm{Conj}(n)| \leq \text{poly} \overset{?}{\Longrightarrow} G$ virt nilp
- $\exists ?\, G \qquad \text{poly} < |\mathrm{Conj}(n)| < \exp$
- G hyperbolic $\underset{?}{\Longrightarrow} \sum_{n \geq 0} |\mathrm{Conj}(n)|\, z^n$ rational

Conjugacy Growth in groups

$$g \sim h \iff \exists\, c \in G \quad cgc^{-1} = h$$

$$\mathrm{Conj}(n) := B_X(n) \cap G/\!\sim$$

Questions

Yes, if G solv (Breuillard + Cornulier 2010)

- $|\mathrm{Conj}(n)| \leq \text{poly} \overset{?}{\Longrightarrow} G$ virt nilp

Yes (Hull + Osin 2013)

- $\exists ?\, G \quad \text{poly} < |\mathrm{Conj}(n)| < \exp$
- G hyperbolic $\underset{?}{\Longrightarrow} \sum_{n \geq 0} |\mathrm{Conj}(n)|\, z^n$ rational

Rivin's Conjecture '90

G hyperbolic then

$$\sum_{n \geq 0} |\mathrm{conj}(n)| z^n \in \mathbb{Z}[[z]] \text{ rational}$$

$$\Longleftrightarrow$$

G is virtually cyclic

Rivin's Conjecture '90

G hyperbolic then

$$\sum_{n\geqslant 0} |\mathrm{conj}(n)|\, z^n \in \mathbb{Z}[[z]] \text{ rational}$$

$$\Longleftrightarrow$$

G is virtually cyclic

Ciobanu + Hermiller + Holt + Rees '14 proved $\Leftarrow$

Thm 1 A + Ciobanu '15 $\Rightarrow$

Thm 1 A+ Ciobanu

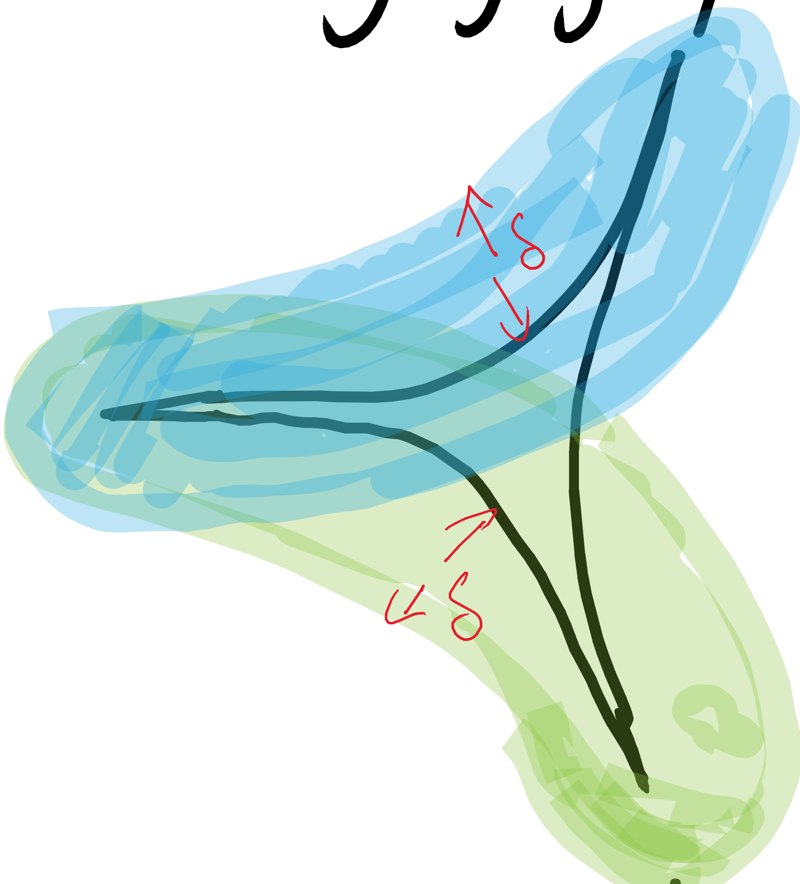
G hyperbolic non virt cyclic, then the following are transcendental over $\mathbb{Q}(z)$

- $\sum_{n \geq 0} |\mathrm{conj}(n)| z^n$
- $\sum_{n \geq 0} |\mathrm{pconj}(n)| z^n$ $\qquad \mathrm{pconj}(n) = \mathrm{conj}(n) \cap \{\text{non powers}\}$
- $\sum_{n \geq 0} |\mathrm{comm}(n)| z^n$ $\qquad \mathrm{comm}(n) = B_X(n) \cap G/\approx$

$g \approx h \iff \exists c \in G,\ n, m \in \mathbb{Z} - \{0\}$
$g^n = c h^m c^{-1}$

Hyperbolic groups

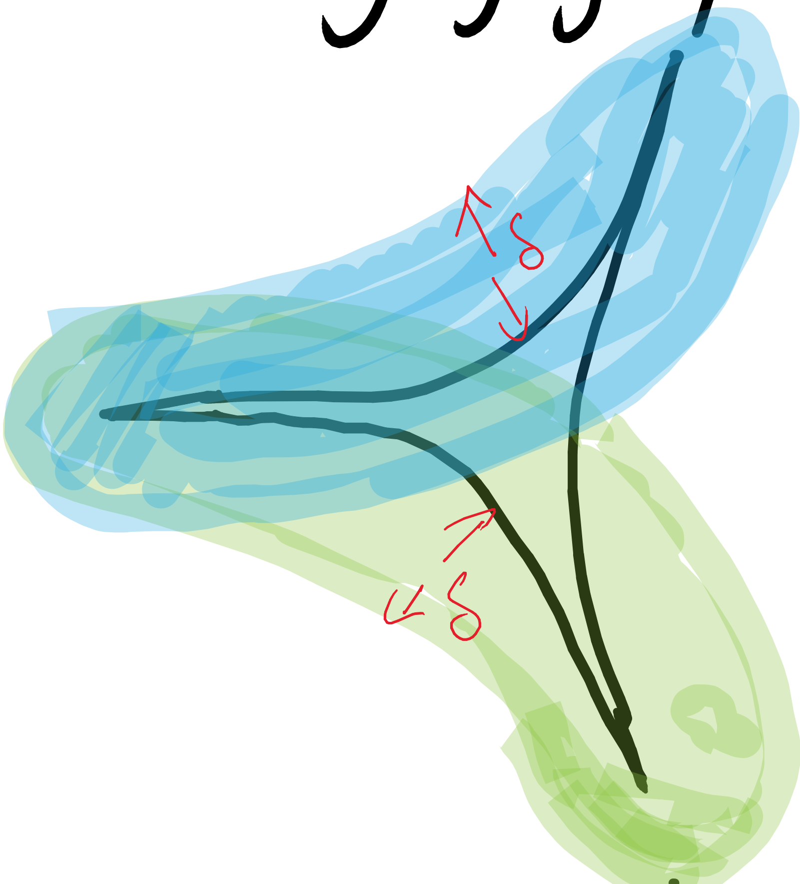
G is hyperbolic if $\exists\, \delta \geq 0$ such that in the Cayley graph geodesic triangles are δ thin



geodesic triangle

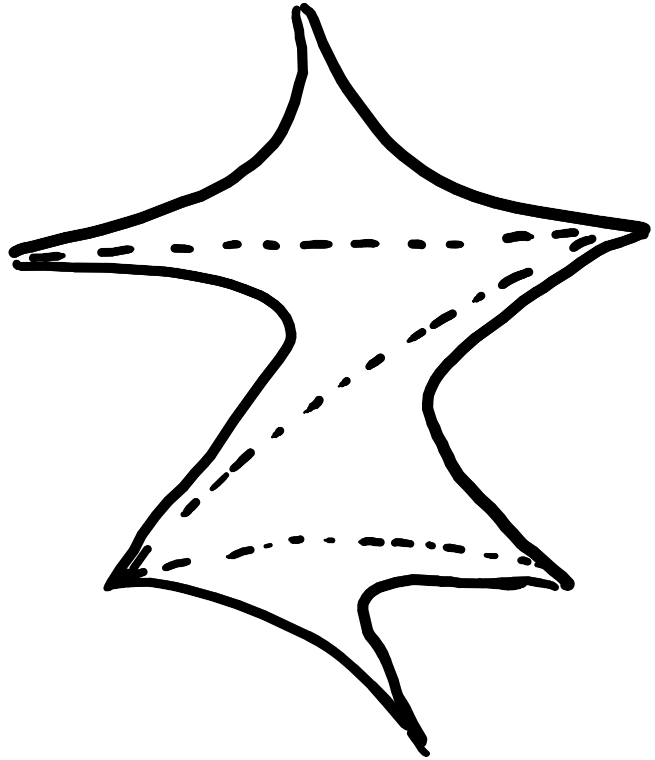
Hyperbolic groups

G is **hyperbolic** if $\exists\, \delta \geq 0$ such that in the Cayley graph geodesic triangles are δ-thin



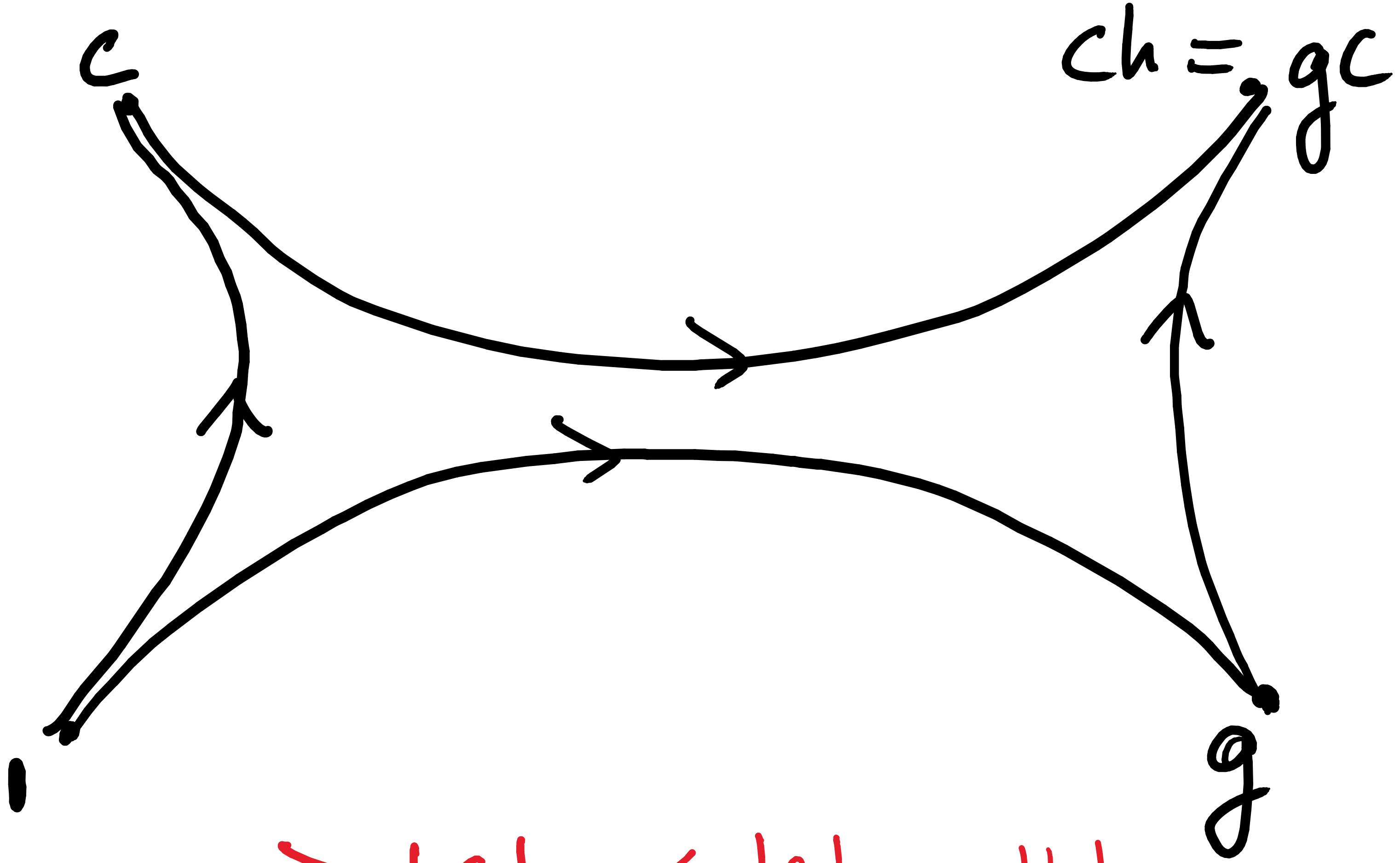
geodesic triangle

Hexagons are 4δ thin



Conjugacy in hyperbolic groups

$gc = ch \rightsquigarrow$ geodesic 4-gon in Cayley graph

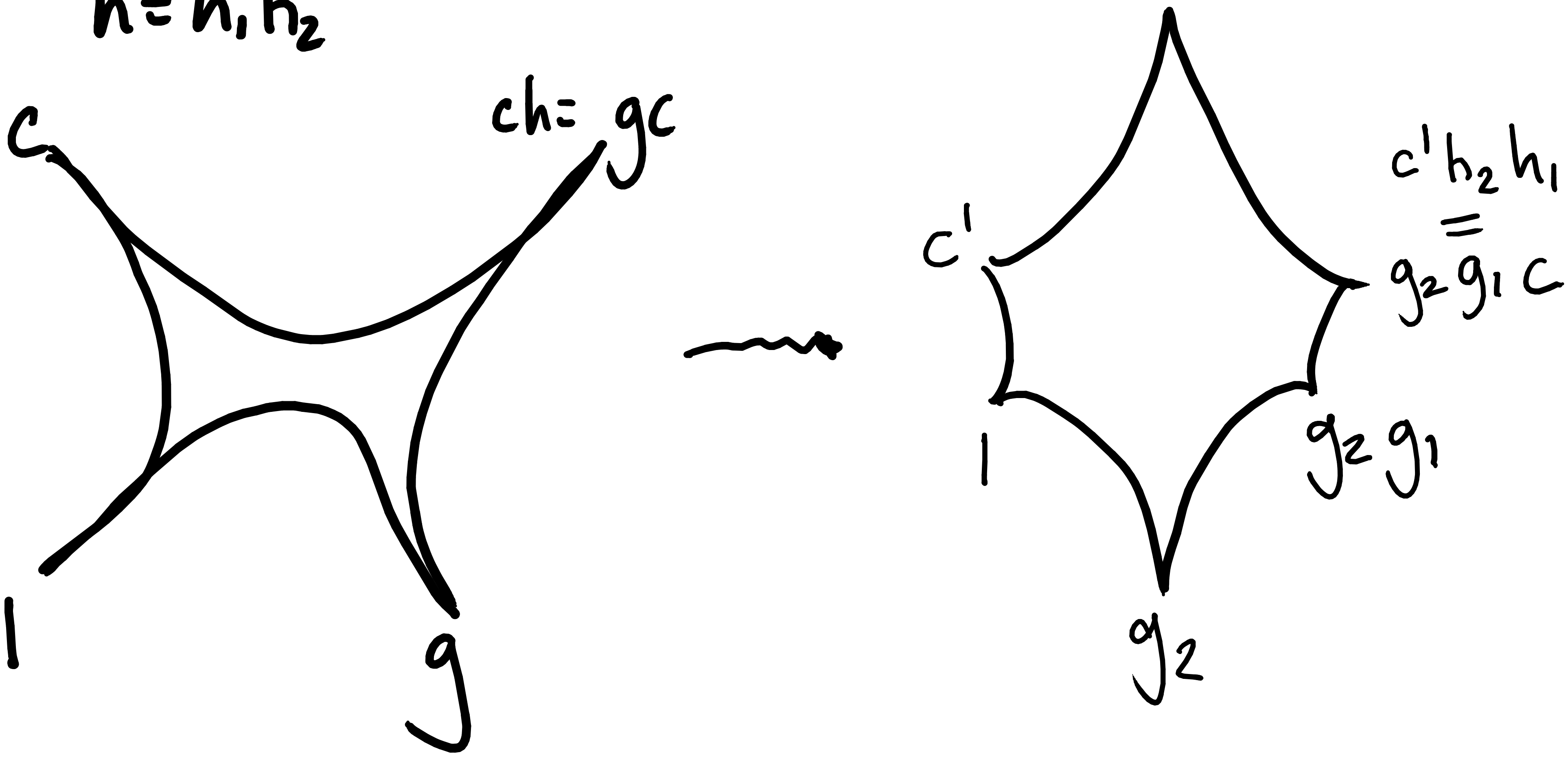


$$\Rightarrow |c|_X \leq |g|_X + |h|_X + 4\delta$$

Conjugacy up to cyclic permutation

$g = g_1 g_2$

$h = h_1 h_2$

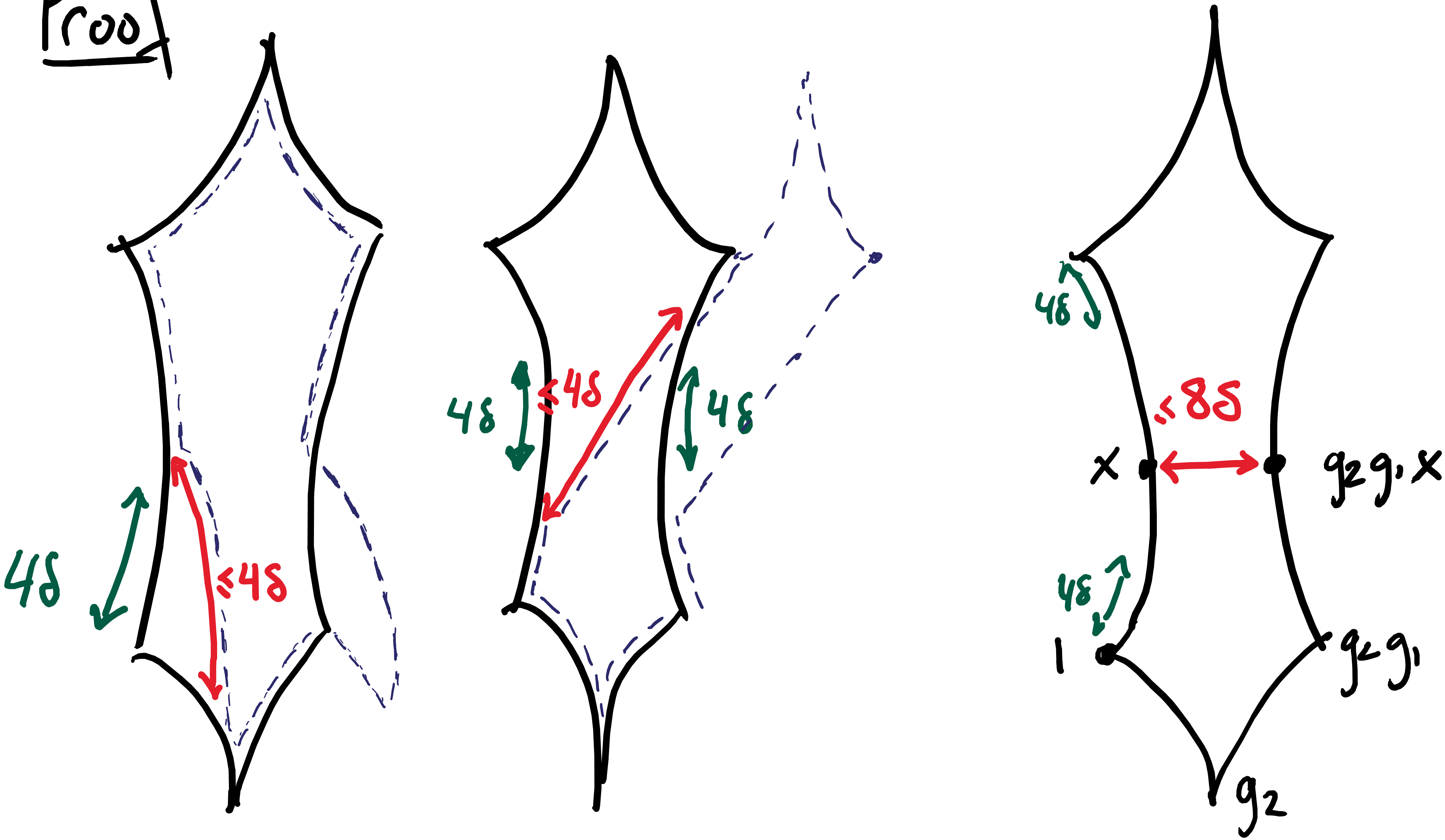


Lemma

Up to cyclic permutation of g and h

$$|C| \leq 8\delta + |B_x(8\delta)|$$

Proof



G hyperbolic

- g cyclic geodesic, $|g|_X = n$
- Lemma $\Rightarrow$ $\#\{h \mid h \sim g, h \text{ cyclic geo}\}$
  $$\leq (\underbrace{B_X(8\delta)+8\delta}_{\text{constant}})\, n$$
- allmost all elements of $B_X(n)$ are cyclic geodesics of length n.
- $$\frac{|B_X(n)|}{n} \simeq |\mathrm{conj}(n)|$$

Thm Coornaert '93

G hyp and non-virt cyclic. $\exists \lambda > 1$, and $A, B > 0$ s.t. $\forall n \gg 0$

$$A \lambda^n \leq |B_X(n)| \leq B \lambda^n$$

Thm 2 A + Ciobanu

Same hypothesis: $\forall n \gg 0$

$$A \frac{\lambda^n}{n} \leq |\mathrm{comm}(n)| \leq |\mathrm{pconj}(n)| \leq |\mathrm{conj}(n)| \leq B \frac{\lambda^n}{n}$$

Thm 2 $\Rightarrow$ Thm 1

$$A\frac{\lambda^n}{n} \le |\mathrm{conj}(n)| \le B\frac{\lambda^n}{n} \quad (\text{thm } 2)$$

$$A\frac{(\lambda z)^n}{n} \le |\mathrm{conj}(n)| z^n \le B\frac{(\lambda z)^n}{n}$$

$$-A\log(1-\lambda z) \le \sum_{n\geq 0} |\mathrm{conj}(n)| z^n \le -B\log(1-\lambda z)$$

essential singularity $z=\frac{1}{\lambda}$

$\square$

Recall

Thm 1 G hyp and not virt cyclic $\Longrightarrow$

$\sum_{n \geq 0} |\mathrm{conj}(n)| z^n$ is transcendental over $\mathbb{Q}(z)$

Thm Chomsky + Schutzemberger

$\mathcal{L} \subseteq X^*$ unambiguous context-free $\Rightarrow \sum_{w \in \mathcal{L}} z^{\ell(w)}$ is algebraic

Corollary No language of minimal length conjugacy representatives of G is unambiguous context-free.

Beyond hyperbolic groups

- $G = \langle X \rangle$, $|X| < \infty$ non necesarily hyp
- $H \leqslant G$ hyperbolic non virt cyclic

- $\mathcal{L}\mathrm{conj}(H)$: Language min X-length H-conj rep of elements of H
- $\mathcal{L}\mathrm{conj}(G)$: same for G

complexity $\mathcal{L}\mathrm{conj}(H) \leqslant$ complexity $\mathcal{L}\mathrm{conj}(G)$??

Thm 3 A+Ciobanu

$G = \langle X \rangle$, $|X| < \infty$

$H \leq G$ hyperbolic and not virt cyclic.

- H almost Fratini embeded
- H Morse
- H BCD embedded

Then $\mathcal{L}_{conj}(G)$ is not unambiguous context-free.

Thm 3 A+Ciobanu

$G = \langle X \rangle$, $|X| < \infty$

$H \leq G$ hyperbolic and not virt cyclic.

- H almost Fratini embeded ($\forall h \in H$ $h^G \cap H = h^H$)
- H Morse (geodesics in $\Gamma(G,X)$ with endpoints in H stay close to H)
- H BCD embedded

Then $\mathcal{L}_{conj}(G)$ is not unambiguous context-free.

BCD embedded

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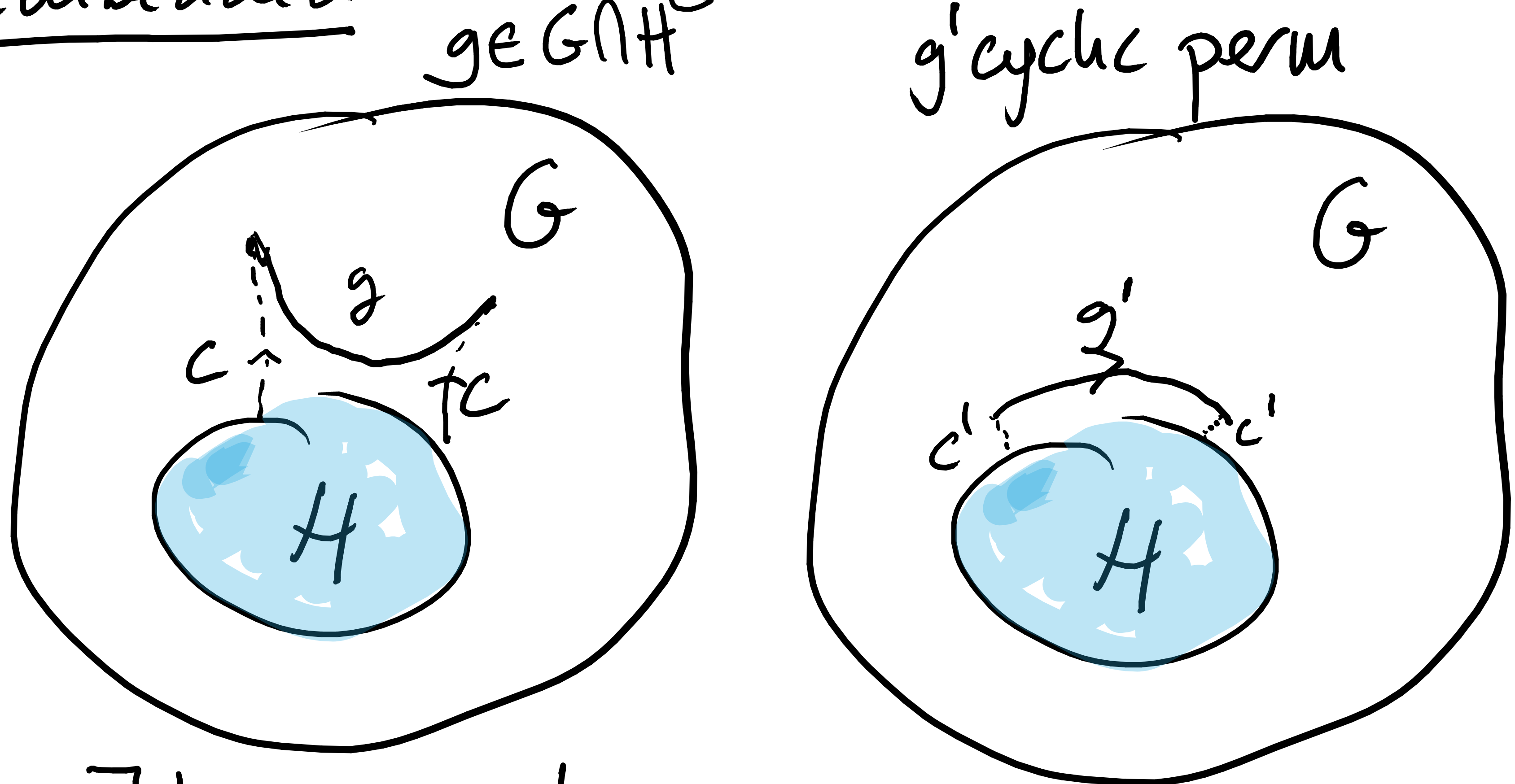


Def $\exists K > 0$ s.t.:

$\forall\, u \in CycGeo(G,X) \cap H^G$

$\exists c \in G,\ |c|_X \leq K,\ \exists u'$ cyc perm of u s.t. $u' \in H^c$

(Fake) Proof of Thm3

$H = \langle Y \rangle$, $|Y| < \infty$

$\mathrm{CycGeo}(H, Y)$ is regular (Ciobanu + Hermiller + Holt + Rees)

$$M = \left\{ (u,v) \in (X \times Y)^* \;\middle|\; \begin{array}{l} v \in \mathrm{CycGeo}(H,Y) \\ uc = cv \text{ for some } c \in G,\ |c|_X < K \end{array} \right\}$$

H Morse $\Rightarrow$ u, v fellow travel $\Rightarrow$ M regular

$\mathcal{L} = M \cap \{ (u,v) \in (X \times Y^*) \mid u \in \mathcal{L}_{\mathrm{conj}}(G) \}$ is regular

BCD-embd $\Rightarrow$ $\pi_2(\mathcal{L})^G \cap H = H$

almost Fratini $\Rightarrow$ $|\pi_2(\mathcal{L}) \,\Delta\, \mathcal{L}_{\mathrm{conj}}(H)| < \infty$ $\square$

Thm 4 A + Ciobanu

G f.g acylindrically hyperbolic,

Then no language of min length conjugacy rep is unambiguous context-free.

Examples of acylindrically hyperbolic groups

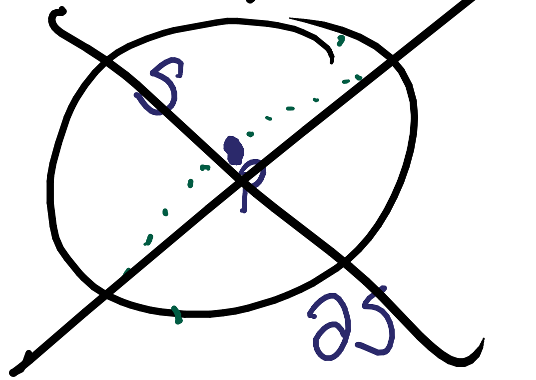
1. non virt cyclic subgroups of hyperbolic gps
2. Free products with finite amalgamation ($\neq \mathbb{Z}_2 * \mathbb{Z}_2$)
3. non v.c. relatively hyperbolic groups
4. $Out(F_n)$
5. $MCG(\Sigma_g)$
6. RAAGS non directly decomposable
7. $C'(\frac{1}{6})$ small cancellation gps
8. 3-generator 1-relator gps

...

<u>Def</u> G is acylindrically hyperbolic if

acts by isometries on an hyperbolic space S

non-elementary



$G \curvearrowright S$ has more than 2 limit points in ∂S

and acylindrically.

$\Leftrightarrow \forall \varepsilon>0\ \exists N, R>0$ s.t. $\forall x,y \in S$ with $d_S(x,y)>R$

$$\#\{g \in G \mid d_S(x,gx)<\varepsilon \text{ and } d_S(y,gy)<\varepsilon\} < N$$

Strengthened Rivin's Conjecture

G acylindrically hyperbolic $\Rightarrow$

$$\sum_{n \geq 0} |\mathrm{conj}(n)| z^n \text{ is transcendental.}$$

Question: Is there a lang of conj rep of $F_2 \times F_2$ U.C.F for some gen set? and $\sum_{n \geq 0} |\mathrm{conj}(n)| z^n$ algebraic?

Thank you!!