Makanin-Razborov diagrams

Montserrat Casals-Ruiz, Ilya Kazachkov

Ikerbasque, UPV/EHU

Les Diablerets

March 7-10, 2016

Equations

- Equations are central in mathematics
- From Diophantus to Hilbert.
- Equations in logic.
- Tarski problems: the origin of the study of equations over free groups.

Equations over free groups

- R. Lyndon (1960) - one-variable equations;
- Yu. Hmelevskiĭ (1971, 1972) and Yu. Ozhigov (1983) - two variable equations;
- A. Razborov (1984) - no generalizations to 3 variables;
- A. Malcev (1962) - the commutator equation $[x, y] = [a, b]$;
- Commerford-Edmunds and Grigorchuk-Kurchanov - general quadratic equations;
- G. Makanin (1977, 1982) - decidability of compatibility;
- A. Razborov (1985, 1987) - description of solutions.
- and beyond...

Basic definitions: Equations

Let $F = F(A)$ be the free group on A, $F(X)$ be a free group with basis $X = \{x_1, \ldots, x_k\}$. Set $F[X] = F * F(X)$.

- An equation over F is an expression of the form

$$w = 1, \quad \text{where} \quad w \in F * F(X).$$

- A system of equations S over F is a collection of equations.
- Alternatively, an equation is an atomic formula in the language of groups (with constants).

Basic definitions: Equations

Let $F = F(A)$ be the free group on A, $F(X)$ be a free group with basis $X = \{x_1, \ldots, x_k\}$. Set $F[X] = F * F(X)$.

- An equation over F is an expression of the form

$$w = 1, \quad \text{where} \quad w \in F * F(X).$$

- A system of equations S over F is a collection of equations.
- Alternatively, an equation is an atomic formula in the language of groups (with constants).

Basic definitions: Equations

Let $F = F(A)$ be the free group on A, $F(X)$ be a free group with basis $X = \{x_1, \ldots, x_k\}$. Set $F[X] = F * F(X)$.

- An equation over F is an expression of the form

$$w = 1, \qquad \text{where} \qquad w \in F * F(X).$$

- A system of equations S over F is a collection of equations.
- Alternatively, an equation is an atomic formula in the language of groups (with constants).

Basic definitions: Equations

Let $F = F(A)$ be the free group on A, $F(X)$ be a free group with basis $X = \{x_1, \ldots, x_k\}$. Set $F[X] = F * F(X)$.

- An equation over F is an expression of the form

$$w = 1, \quad \text{where} \quad w \in F * F(X).$$

- A system of equations S over F is a collection of equations.
- Alternatively, an equation is an atomic formula in the language of groups (with constants).

Example:

$$x^{-1}y^{-1}xy = a^{-1}b^{-1}ab$$

$$([x, y][a, b]^{-1} = 1)$$

Basic definitions: Solutions

- A solution of $S \in F[x_1, \dots, x_k]$:

$$(g_1, \dots, g_k) \in F^k \ \text{ so that } \ w(g_1, \dots, g_k) = 1 \ \text{ in } \ F$$

  for all w in S.

- Equivalently, a solution of S is a homomorphism

$$\varphi : F[X] \to F \ \text{ so that } \ S \subseteq \ker(\varphi)$$

  that is a homomorphism

$$\varphi : F[X]/\langle\langle S \rangle\rangle = \langle A, X \mid S \rangle \to F$$

Basic definitions: Solutions

- A solution of $S \in F[x_1, \ldots, x_k]$:

$$(g_1, \ldots, g_k) \in F^k \ \text{ so that } \ w(g_1, \ldots, g_k) = 1 \ \text{ in } \ F$$

  for all w in S.
- Equivalently, a solution of S is a homomorphism

$$\varphi : F[X] \to F \ \text{ so that } \ S \subseteq \ker(\varphi)$$

  that is a homomorphism

$$\varphi : F[X]/\langle\langle S \rangle\rangle = \langle A, X \mid S \rangle \to F$$

Basic definitions: Algebraic Sets

The set $V(S) = \{p \in F^k \mid s(p) = 1 \ \ \forall s \in S\}$ is called the *algebraic set* defined by S.

Equivalently, the algebraic set is the set

$$Hom(F_{R(S)}, F)$$

where

$$F_{R(S)} = \langle A, X \mid S \rangle.$$

Remark: We consider F-homomorphism: homomorphisms $\varphi : F_{R(S)} \to F$ that are the identity on constants, i.e. $\varphi\mid_F = id\mid_F$.

Basic definitions: Algebraic Sets

The set $V(S) = \{p \in F^k | \; s(p) = 1 \;\; \forall \, s \in S\}$ is called the *algebraic set* defined by S.
Equivalently, the algebraic set is the set

$$Hom(F_{R(S)}, F)$$

where

$$F_{R(S)} = \langle A, X \mid S \rangle.$$

Remark: We consider F-homomorphism: homomorphisms $\varphi : F_{R(S)} \to F$ that are the identity on constants, i.e. $\varphi \mid_F = id \mid_F$.

Basic definitions: Algebraic Sets

The set $V(S) = \{p \in F^k | \; s(p) = 1 \;\; \forall \, s \in S\}$ is called the *algebraic set* defined by S.
Equivalently, the algebraic set is the set

$$Hom(F_{R(S)}, F)$$

where

$$F_{R(S)} = \langle A, X \mid S \rangle.$$

Remark: We consider F-homomorphism: homomorphisms $\varphi : F_{R(S)} \to F$ that are the identity on constants, i.e. $\varphi \mid_F = id \mid_F$.

Basic definitions: Algebraic Sets

The set $V(S) = \{p \in F^k | \ s(p) = 1 \ \ \forall\, s \in S\}$ is called the *algebraic set* defined by S.
Equivalently, the algebraic set is the set

$$Hom(F_{R(S)}, F)$$

where

$$F_{R(S)} = \langle A, X \mid S \rangle.$$

Remark: We consider F-homomorphism: homomorphisms $\varphi : F_{R(S)} \to F$ that are the identity on constants, i.e. $\varphi \mid_F = id \mid_F$.

Examples of Algebraic Sets

1. Every element $g \in G$ is an algebraic set $\{g\}$:
   $S = \{x = g\}$, $V_G(S) = \{g\}$, $F_{R(S)} = \langle A, x \mid x = g \rangle \simeq F$.
2. Every element $g \in G^n$ is an algebraic set $\{(g_1, \ldots, g_n)\}$:
   $S = \{x_1 = g_1, \ldots, x_n = g_n\}$, $V_G(S) = \{(g_1, \ldots, g_n)\}$.
3. The centraliser $C_G(M)$ of every subset $M \subseteq G$ is algebraic,
   $S = \{[x, m] = 1 | \ m \in M\}$.
4. The whole affine space G^n is the algebraic set defined by the system $S = \{1 = 1\}$.
5. Then the empty set $\emptyset$ is not algebraic of any coefficient-free system.

Examples of Algebraic Sets

1. Every element $g \in G$ is an algebraic set $\{g\}$:
   $S = \{x = g\}$, $V_G(S) = \{g\}$, $F_{R(S)} = \langle A, x \mid x = g\rangle \simeq F$.
2. Every element $g \in G^n$ is an algebraic set $\{(g_1, \dots, g_n)\}$:
   $S = \{x_1 = g_1, \dots, x_n = g_n\}$, $V_G(S) = \{(g_1, \dots, g_n)\}$.
3. The centraliser $C_G(M)$ of every subset $M \subseteq G$ is algebraic,
   $S = \{[x, m] = 1 | \ m \in M\}$.
4. The whole affine space G^n is the algebraic set defined by the system $S = \{1 = 1\}$.
5. Then the empty set $\emptyset$ is not algebraic of any coefficient-free system.

Examples of Algebraic Sets

1. Every element $g \in G$ is an algebraic set $\{g\}$:
   $S = \{x = g\}$, $V_G(S) = \{g\}$, $F_{R(S)} = \langle A, x \mid x = g \rangle \simeq F$.
2. Every element $g \in G^n$ is an algebraic set $\{(g_1, \ldots, g_n)\}$:
   $S = \{x_1 = g_1, \ldots, x_n = g_n\}$, $V_G(S) = \{(g_1, \ldots, g_n)\}$.
3. The centraliser $C_G(M)$ of every subset $M \subseteq G$ is algebraic,
   $S = \{[x, m] = 1 \mid m \in M\}$.
4. The whole affine space G^n is the algebraic set defined by the system $S = \{1 = 1\}$.
5. Then the empty set $\emptyset$ is not algebraic of any coefficient-free system.

Examples of Algebraic Sets

1. Every element $g \in G$ is an algebraic set $\{g\}$:
   $S = \{x = g\}$, $V_G(S) = \{g\}$, $F_{R(S)} = \langle A, x \mid x = g\rangle \simeq F$.
2. Every element $g \in G^n$ is an algebraic set $\{(g_1, \ldots, g_n)\}$:
   $S = \{x_1 = g_1, \ldots, x_n = g_n\}$, $V_G(S) = \{(g_1, \ldots, g_n)\}$.
3. The centraliser $C_G(M)$ of every subset $M \subseteq G$ is algebraic,
   $S = \{[x, m] = 1 \mid m \in M\}$.
4. The whole affine space G^n is the algebraic set defined by the system $S = \{1 = 1\}$.
5. Then the empty set $\emptyset$ is not algebraic of any coefficient-free system.

Examples of Algebraic Sets

1. Every element $g \in G$ is an algebraic set $\{g\}$:
   $S = \{x = g\}$, $V_G(S) = \{g\}$, $F_{R(S)} = \langle A, x \mid x = g \rangle \simeq F$.
2. Every element $g \in G^n$ is an algebraic set $\{(g_1, \ldots, g_n)\}$:
   $S = \{x_1 = g_1, \ldots, x_n = g_n\}$, $V_G(S) = \{(g_1, \ldots, g_n)\}$.
3. The centraliser $C_G(M)$ of every subset $M \subseteq G$ is algebraic,
   $S = \{[x, m] = 1 \mid m \in M\}$.
4. The whole affine space G^n is the algebraic set defined by the system $S = \{1 = 1\}$.
5. Then the empty set $\varnothing$ is not algebraic of any coefficient-free system.

Basic definitions: Algebraic Geometry

In fact, there is a well-developed algebraic geometry theory over groups.

- A coordinate group of S is:

$$F_{R(S)} = F[X] \Big/ \bigcap_{\varphi \text{ solution}} \ker(\varphi)$$

- Solutions of S are homomorphisms from $F_{R(S)}$ to F.
- Every algebraic set $V(S)$ can be identified with the set $Hom(F_{R(S)}, F)$.
- there is a one-to-one correspondence between varieties $V(S)$ and coordinate groups $F_{R(S)}$.

N.B: The group $\langle A, X \mid S \rangle$ and the coordinate group $F_{R(S)}$ do not always coincide.

Basic definitions: Algebraic Geometry

In fact, there is a well-developed algebraic geometry theory over groups.

- A coordinate group of S is:

$$F_{R(S)} = F[X] \Big/ \bigcap_{\varphi \text{ solution}} \ker(\varphi)$$

- Solutions of S are homomorphisms from $F_{R(S)}$ to F.

- Every algebraic set $V(S)$ can be identified with the set $Hom(F_{R(S)}, F)$.
- there is a one-to-one correspondence between varieties $V(S)$ and coordinate groups $F_{R(S)}$.

N.B: The group $\langle A, X \mid S \rangle$ and the coordinate group $F_{R(S)}$ do not always coincide.

Basic definitions: Algebraic Geometry

In fact, there is a well-developed algebraic geometry theory over groups.

- A coordinate group of S is:

$$F_{R(S)} = F[X] \Big/ \bigcap_{\varphi \text{ solution}} \ker(\varphi)$$

- Solutions of S are homomorphisms from $F_{R(S)}$ to F.
- Every algebraic set $V(S)$ can be identified with the set $Hom(F_{R(S)}, F)$.
- there is a one-to-one correspondence between varieties $V(S)$ and coordinate groups $F_{R(S)}$.

N.B: The group $\langle A, X \mid S \rangle$ and the coordinate group $F_{R(S)}$ do not always coincide.

Basic definitions: Algebraic Geometry

In fact, there is a well-developed algebraic geometry theory over groups.

- A coordinate group of S is:

$$F_{R(S)} = F[X] \Big/ \bigcap_{\varphi \text{ solution}} \ker(\varphi)$$

- Solutions of S are homomorphisms from $F_{R(S)}$ to F.
- Every algebraic set $V(S)$ can be identified with the set $Hom(F_{R(S)}, F)$.
- there is a one-to-one correspondence between varieties $V(S)$ and coordinate groups $F_{R(S)}$.

N.B: The group $\langle A, X \mid S \rangle$ and the coordinate group $F_{R(S)}$ do not always coincide.

Basic definitions: Algebraic Geometry

In fact, there is a well-developed algebraic geometry theory over groups.

- A coordinate group of S is:

$$F_{R(S)} = F[X] \Big/ \bigcap_{\varphi \text{ solution}} \ker(\varphi)$$

- Solutions of S are homomorphisms from $F_{R(S)}$ to F.
- Every algebraic set $V(S)$ can be identified with the set $Hom(F_{R(S)}, F)$.
- there is a one-to-one correspondence between varieties $V(S)$ and coordinate groups $F_{R(S)}$.

N.B: The group $\langle A, X \mid S \rangle$ and the coordinate group $F_{R(S)}$ do not always coincide.

Let S be a system of equations:

1. Does S have a solution?
2. Can one describe the set of all solutions of S?

Linear equations

Linear equations = variables occur at most once in the system of equations.
(Convention: x, x^{-1} are occurrences of the same variable)

Example 1: $S = \{x = g, y = g' \mid g, g' \in F\}$, $V(S) = \{(g, g')\}$.

Linear equations

Linear equations = variables occur at most once in the system of equations.
(Convention: x, x^{-1} are occurrences of the same variable)

Example 2: $S = \{xyz = 1\}$

$$F_{R(S)} = \langle x, y, z \mid xyz = 1\rangle \simeq \langle x, y\rangle = F_2$$

$$Hom(F_2, F) \simeq F \times F$$

or

$$V(S) = \{(u, w, w^{-1}v^{-1}) \mid u, w \in F\}.$$

Linear equations

Linear equations = variables occur at most once in the system of equations.
(Convention: x, x^{-1} are occurrences of the same variable)

CONCLUSION:
Linear equations always have solutions and we can describe their solution sets. In fact, if $F_{R(S)}$ is a free group, we understand $Hom(F_{R(S)}, F)$.

Quadratic equations

Quadratic equation = every variable occurs at most twice in the system of equations.

Example 1. Not all quadratic systems of equations are compatible! $S = \{x = a, x = b\}$, $V(S) = \{\emptyset\}$.

Quadratic equations

Quadratic equation = every variable occurs at most twice in the system of equations.

Example 2. $S = \{[x, g] = 1\}$, $g \in F$.

$$V(S) = C(g) = \{\sqrt{g}^n \mid n \in \mathbb{Z}\}.$$

The set $V(S)$ can be "parametrised" by a word g^t, where t is an integer variable.

N.B: Lyndon showed that the solution set of one variable equations admit a "parametrization". These ideas extended to 2-variable equations but it does not work for 3-variable equations.

Quadratic equations: abelian case

Example 3. $S = \{[x, y] = 1\}$ and so
$F_{R(S)} = \langle x, y \mid [x, y] = 1 \rangle = \mathbb{Z}^2$. The solution set
$V(S) = \{(w^k, w^l) \mid w \in F, k, l \in \mathbb{Z}\}$.

Quadratic equations: abelian case

Example 3. $S = \{[x, y] = 1\}$ and so
$F_{R(S)} = \langle x, y \mid [x, y] = 1\rangle = \mathbb{Z}^2$. The solution set
$V(S) = \{(w^k, w^l) \mid w \in F, k, l \in \mathbb{Z}\}$.

Homomorphisms ϕ from $\mathbb{Z}^2 \to F$ map x to w^k and y to w^l

Quadratic equations: abelian case

Homomorphisms ϕ from $\mathbb{Z}^2 \to F$ map x to w^k and y to w^l

Let us do an α automorphism of $\mathbb{Z}^2$:

$$\begin{array}{rccc} \alpha: & \mathbb{Z}^2 & \to & \mathbb{Z}^2 \\ & x & \to & x \\ & y & \to & yx^{-1} \end{array}$$

Quadratic equations: abelian case

Homomorphisms ϕ from $\mathbb{Z}^2 \to F$ map x to w^k and y to w^l

Let us do an α automorphism of $\mathbb{Z}^2$:

$$\begin{array}{rccc} \alpha: & \mathbb{Z}^2 & \to & \mathbb{Z}^2 \\ & x & \to & x \\ & y & \to & yx^{-1} \end{array}$$

Then

$$\begin{array}{rccccc} \alpha\phi: & \mathbb{Z}^2 & \to & \mathbb{Z}^2 & \to & F \\ & x & \to & x & \to & w^k \\ & y & \to & yx^{-1} & \to & w^{l-k} \end{array}$$

Quadratic equations: abelian case

Homomorphisms ϕ from $\mathbb{Z}^2 \to F$ map x to w^k and y to w^l

Let us do an α automorphism of $\mathbb{Z}^2$:

$$\begin{array}{rccc} \alpha: & \mathbb{Z}^2 & \to & \mathbb{Z}^2 \\ & x & \to & x \\ & y & \to & yx^{-1} \end{array}$$

Then

$$\begin{array}{rccccc} \alpha\phi: & \mathbb{Z}^2 & \to & \mathbb{Z}^2 & \to & F \\ & x & \to & x & \to & w^k \\ & y & \to & yx^{-1} & \to & w^{l-k} \end{array}$$

Arguing this way we find an automorphism β of $\mathbb{Z}^2$ so that

$$\begin{array}{rccccc} \beta\phi: & \mathbb{Z}^2 & \to & \mathbb{Z}^2 & \to & F \\ & x & \to & \beta(x) & \to & w^{gcd(k,l)} \\ & y & \to & \beta(y) & \to & 1 \end{array}$$

Quadratic equations: abelian case

Arguing this way we find an automorphism β of $\mathbb{Z}^2$ so that

$$\begin{array}{cccccc} \beta\phi : & \mathbb{Z}^2 & \to & \mathbb{Z}^2 & \to & F \\ & x & \to & \beta(x) & \to & w^{gcd(k,l)} \\ & y & \to & \beta(y) & \to & 1 \end{array}$$

Define

$$\pi : \mathbb{Z}^2 = \langle x, y \rangle \to \mathbb{Z} = \langle x \rangle$$

and

$$\varphi : \mathbb{Z} \to F$$

$$x \to w^{gcd(k,l)}$$

Then $\phi = \beta^{-1}(\beta\phi)$ and $\beta\phi = \pi\varphi$ and so

$$\begin{array}{ccccc} \mathbb{Z}^2 & \to^{\beta} & \mathbb{Z}^2 & \twoheadrightarrow^{\pi} & \mathbb{Z} \\ \downarrow \phi & & \downarrow \beta\phi & & \downarrow \varphi \\ (w^k, w^l) & & (w^{gcd(k,l)}, 1) & & w^{gcd(k,l)} \end{array}$$

Abelian case

CONCLUSION:
Every homomorphism $\phi : \mathbb{Z}^n \to F$ is the composition of an automorphism of $\mathbb{Z}^n$, the epimorphism $\pi : \mathbb{Z}^n \to \mathbb{Z}$ and a homomorphism $\varphi : \mathbb{Z} \to F$.

$$\circlearrowright \mathbb{Z}^n \twoheadrightarrow \mathbb{Z} \to F.$$

Quadratic equations

Fact. Any quadratic equation is "equivalent" to one of the following two types:

- $[x_1, y_1] \dots [x_n, y_n] = 1$ (orientable case)
- $x_1^2 x_2^2 \dots x_n^2 = 1$ (non-orientable case)

By "equivalent" we mean: if $S = 1$ is quadratic and $S \in F(X)$, then there exists an automorphism ψ of $F(X)$ such that $\psi(S)$ is of the form above.

Quadratic equations

Fact. Any quadratic equation is "equivalent" to one of the following two types:

- $[x_1, y_1] \dots [x_n, y_n] = 1$ (orientable case)
- $x_1^2 x_2^2 \dots x_n^2 = 1$ (non-orientable case)

By "equivalent" we mean: if $S = 1$ is quadratic and $S \in F(X)$, then there exists an automorphism ψ of $F(X)$ such that $\psi(S)$ is of the form above.

Quadratic equations

Example:

$$S = \{x^{-1}yxy = 1\}, S \in F(x, y)$$

Consider the automorphism:

$$\begin{array}{cccc} \psi : & F(x,y) & \rightarrow & F(x,y) \\ & x & \rightarrow & x^{-1} \\ & y & \rightarrow & xy \end{array}$$

then $\psi(S) = x\ xy\ x^{-1}\ xy = x^2y^2$

Quadratic equations: orientable case

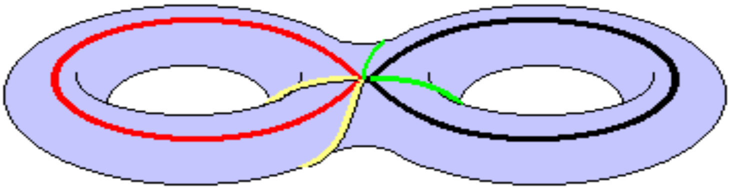
Consider the following quadratic equation in variables x, y, z, t:

$$[x, y][z, t] = 1$$

The group

$$G = \langle x, y, z, t \mid [x, y][z, t] = 1 \rangle$$

is the fundamental group of an orientable surface of genus 2.



Quadratic equations: orientable case

Fact (Grigorchuk-Kurchanov, Commerford-Edmunds)

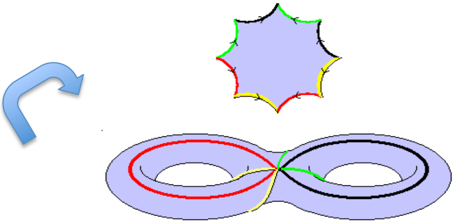
Any homomorphism $\varphi : G \to F_2 = \langle a, b\rangle$ factors through the epimorphism

$$\begin{array}{rccc} \pi : & G & \to & F_2 \\ & x & \to & a \\ & y & \to & 1 \\ & z & \to & b \\ & t & \to & 1 \end{array}$$

if pre-composed with an automorphism of the surface group G.

Note that the group of automorphisms of the surface group G is well-understood.

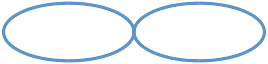
Quadratic equations: orientable case



$< x, y, z, t \mid [x,y][z,t]=1 >$

$\downarrow$

$x \longrightarrow a$
$y \longrightarrow 1$
$z \longrightarrow b$
$t \longrightarrow 1$



a b

Quadratic equations: non-orientable case

In general, one epimorphism and automorphisms are not enough to describe all the homomorphisms of quadratic equations.

Consider the following quadratic equation in variables x, y, z, t:

$$x^2y^2z^2t^2 = 1$$

The group

$$G = \langle x, y, z, t \mid x^2y^2z^2t^2 = 1 \rangle$$

is the fundamental group of a non-orientable surface of genus 4.

Quadratic equations: non-orientable case

$$G = \langle x, y, z, t \mid x^2y^2z^2t^2 = 1 \rangle$$

Theorem (Grigorchuk-Kurchanov, 1989)

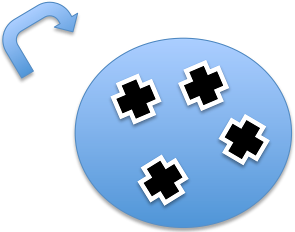
Any homomorphism $\varphi : G \to F_2 = \langle a, b \rangle$ *factors through one of the following epimorphisms:*

$$\begin{array}{llcc} \pi_1 : & G & \to & F_2 \\ & x & \to & a^{-1}c^{-1} \\ & y & \to & ca \\ & z & \to & a^{-1} \\ & t & \to & a \end{array} \quad \begin{array}{llcc} \pi_2 : & G & \to & F_2 \\ & x & \to & a^{-1} \\ & y & \to & a \\ & z & \to & c^{-1} \\ & t & \to & c \end{array} \quad \begin{array}{llcc} \pi_3 : & G & \to & F_2 \\ & x & \to & c^{-1}a^{-1} \\ & y & \to & ac \\ & z & \to & c^{-1} \\ & t & \to & c \end{array}$$

if pre-composed with an automorphism of the surface group G.

The epimorphisms are NOT equivalent: there is no automorphism of G such that $\varphi\pi_i = \pi_j$, $i \neq j$.

Quadratic equations: non-orientable case

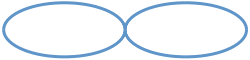


$$\langle x, y, z, t \mid x^2 y^2 z^2 t^2 = 1 \rangle$$

$x \longrightarrow a^{-1}b^{-1}$
$y \longrightarrow ba$
$z \longrightarrow a^{-1}$
$t \longrightarrow a$

$x \longrightarrow a^{-1}$
$y \longrightarrow a$
$z \longrightarrow b^{-1}$
$t \longrightarrow b$

$x \longrightarrow b^{-1} a^{-1}$
$y \longrightarrow ab$
$z \longrightarrow b^{-1}$
$t \longrightarrow b$



a b

Quadratic equations

Theorem (Commerford-Edmunds, Grigorchuk-Kurchanov '89)

Let S be a quadratic equation and $G = \langle X \mid S \rangle$. Then there exist finitely many epimorphisms $\pi_1, \ldots, \pi_k$ such that any homomorphism from G to a free group F factors through one of the epimorphism $\pi_i, i \in \{1, \ldots, k\}$ if pre-composed with an automorphism of G.

$$\begin{tikzcd} & \circlearrowright G \arrow[dl] \arrow[d] \arrow[dr] & \\ F_{r_1} & F_{r_2} & F_{r_k} \end{tikzcd}$$

General equations

If S is an equation, in general it is not true that automorphisms of $G = \langle X \mid S \rangle$ and finitely many epimorphisms suffice to describe the set of homomorphisms.

The Makanin-Razborov process "detects" pieces that are quadratic or abelian or free. Homomorphisms of these pieces factor up to automorphism through finitely many proper quotients. The process repeats the analysis of the quotients. After finitely many steps, the quotients are free groups.

General equations

$G = \langle a, b, u, v, x, y, z, t \mid S \rangle$ where

$S = [[[[u, v], a][[u, w], b], ab][[v, w], aba], abba][[x, y][z, t]w'', abaa]$

Makanin-Razborov (+ Kharlampovich-Miasnikov):

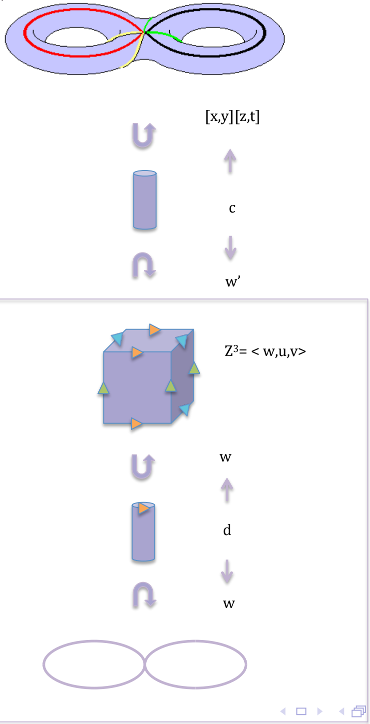
$$G \simeq \left\langle a, b, u, v, x, y, z, t \mid R(a, b, u, v), [x, y][z, t] = w' \right\rangle$$

The process produces finitely many quotients (NOT necessarily free groups) such that any homomorphism from G to F factors up to automorphism through one of these quotients.

$$\pi : G \to G_1 = \langle a, b, u, v, \mid R(a, b, u, v) \rangle$$

Using Makanin-Razborov:

$$G_1 \simeq \langle a, b, u, v \mid [u, v] = 1, [u, w] = 1, [v, w] = 1, w = w(a, b) \rangle$$


[x,y][z,t]
c
w’
Z^3= < w,u,v>
w
d
w

Makanin-Razborov diagrams

$$
\begin{tikzcd}
 & & F_{R(S_0)} \arrow[loop above] \arrow[dl] \arrow[dr] \arrow[drrr] & & & \\
 & F_{R(S_1)} \arrow[loop above] \arrow[dl] \arrow[dr] & & F_{R(S_2)} \arrow[loop above] \arrow[dr] & & F_{R(S_3)} \arrow[loop above] \arrow[d] \\
F_{R(S_4)} \arrow[loop above] \arrow[d] & & F_{R(S_5)} \arrow[loop above] \arrow[d] & & F * F(Y_3) & F * F(Y_4) \\
F * F(Y_1) & & F * F(Y_2) & & &
\end{tikzcd}
$$