Makanin-Razborov diagrams

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Les Diablerets

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Basic definitions: Equations

Let $F = F(A)$ be the free group on A, $F(X)$ be a free group with basis $X = \{x_1, \ldots, x_k\}$. Set $F[X] = F * F(X)$.

- An equation over F is an expression of the form

$$w = 1, \quad \text{where} \quad w \in F * F(X).$$

- A system of equations S over F is a collection of equations.
- Alternatively, an equation is an atomic formula in the language of groups (with constants).

Basic definitions: Solutions

- A solution of $S \in F[x_1, \ldots, x_k]$:

$$(g_1, \ldots, g_k) \in F^k \text{ so that } w(g_1, \ldots, g_k) = 1 \text{ in } F$$

  for all w in S.
- Equivalently, a solution of S is a homomorphism

$$\varphi : F[X] \to F \text{ so that } S \subseteq \ker(\varphi)$$

  that is a homomorphism

$$\varphi : F[X]/\langle\langle S \rangle\rangle = \langle A, X \mid S \rangle \to F$$

Let S be a system of equations:

1. Does S have a solution?
2. Can one describe the set of all solutions of S?

Linear equations

$S = \{xyz = 1\}$

$$F_{R(S)} = \langle x, y, z \mid xyz = 1 \rangle \simeq \langle x, y \rangle = F_2$$

$$Hom(F_2, F) \simeq F \times F$$

or

$$V(S) = \{(u, w, w^{-1}v^{-1}) \mid u, w \in F\}.$$

CONCLUSION: Linear equations always have solutions and we can describe their solution sets. In fact, if $F_{R(S)}$ is a free group, we understand $Hom(F_{R(S)}, F)$.

Quadratic equations: abelian case

$S = \{[x, y] = 1\}$ and so $F_{R(S)} = \langle x, y \mid [x, y] = 1\rangle = \mathbb{Z}^2$. The solution set $V(S) = \{(w^k, w^l) \mid w \in F, k, l \in \mathbb{Z}\}$.

Homomorphisms ϕ from $\mathbb{Z}^2 \to F$ map x to w^k and y to w^l
We can find an automorphism β of $\mathbb{Z}^2$ so that

$$\begin{array}{cccccc} \beta\phi : & \mathbb{Z}^2 & \to & \mathbb{Z}^2 & \to & F \\ & x & \to & \beta(x) & \to & w^{gcd(k,l)} \\ & y & \to & \beta(y) & \to & 1 \end{array}$$

And obtain the following diagram

$$\begin{array}{ccccc} \mathbb{Z}^2 & \to^{\beta} & \mathbb{Z}^2 & \twoheadrightarrow^{\pi} & \mathbb{Z} \\ \downarrow \phi & & \downarrow \beta\phi & & \downarrow \varphi \\ (w^k, w^l) & & (w^{gcd(k,l)}, 1) & & w^{gcd(k,l)} \end{array}$$

Abelian case

Every homomorphism $\phi : \mathbb{Z}^n \to F$ is the composition of an automorphism of $\mathbb{Z}^n$, the epimorphism $\pi : \mathbb{Z}^n \to \mathbb{Z}$ and a homomorphism $\varphi : \mathbb{Z} \to F$.

$$\circlearrowright \mathbb{Z}^n \twoheadrightarrow \mathbb{Z} \to F.$$

Quadratic equations: orientable case

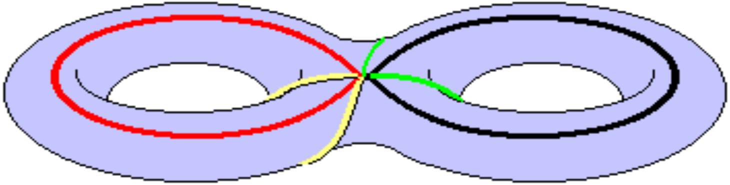
Consider the following quadratic equation in variables x, y, z, t:

$$[x, y][z, t] = 1$$

The group

$$G = \langle x, y, z, t \mid [x, y][z, t] = 1 \rangle$$

is the fundamental group of an orientable surface of genus 2.



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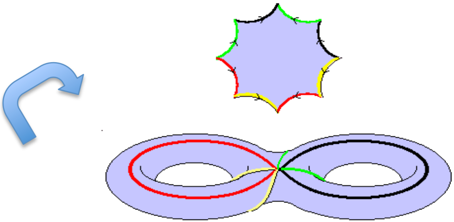
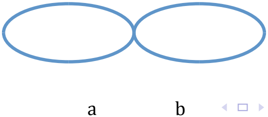


< x, y, z, t | [x,y][z,t]=1 >

$x \longrightarrow a$
$y \longrightarrow 1$
$z \longrightarrow b$
$t \longrightarrow 1$



Quadratic equations: non-orientable case

Consider the quadratic equation in variables x, y, z, t:

$$x^2y^2z^2t^2 = 1$$

The group

$$G = \langle x, y, z, t \mid x^2y^2z^2t^2 = 1 \rangle$$

is the fundamental group of a non-orientable surface of genus 4.

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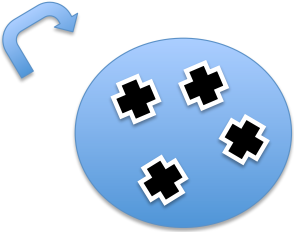
Theorem (Grigorchuk-Kurchanov, 1989)

Any homomorphism $\varphi : G \to F_2 = \langle a, b \rangle$ *factors through one of 3 non-equivalent epimorphisms.*

Quadratic equations: non-orientable case



$\langle x, y, z, t \mid x^2 y^2 z^2 t^2 = 1 \rangle$

$x \to a^{-1}b^{-1}$, $y \to ba$, $z \to a^{-1}$, $t \to a$

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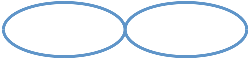

a b

Makanin-Razborov diagrams

$$
\begin{tikzcd}
 & & F_{R(S_0)} \arrow[loop above] \arrow[dl] \arrow[dr] \arrow[drrr] & & & \\
 & F_{R(S_1)} \arrow[loop above] \arrow[dl] \arrow[dr] & & F_{R(S_2)} \arrow[loop above] \arrow[dr] & & F_{R(S_3)} \arrow[loop above] \arrow[d] \\
F_{R(S_4)} \arrow[loop above] \arrow[d] & & F_{R(S_5)} \arrow[loop above] \arrow[d] & & F * F(Y_3) & F * F(Y_4) \\
F * F(Y_1) & & F * F(Y_2) & & &
\end{tikzcd}
$$

Equations in the free group

GAUSS ELIMINATION vs MAKANIN

- encode the problem
- use transformations to make the problem/objects "easier"
- stop when the problem is easy/obvious

Linear system of equations $S \to M$	System of equations over a free group $S \to \{M_1, \dots M_n\}$
Matrix	Generalised equation
Algorithm: "put 0" below the diagonal	Algorithm: "make variables free or constants"
Triangular matrices	System of equations $x_i = a_i$, where $a_i \in F$.

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Equations in the free group

GAUSS ELIMINATION	vs	MAKANIN
sequence of matrices		tree
finite (always terminates)		infinite
"solutions in matrices" correspond to solutions of the system.		for any S exists $K(S)$ such that solution is in $T_{K(S)}$.

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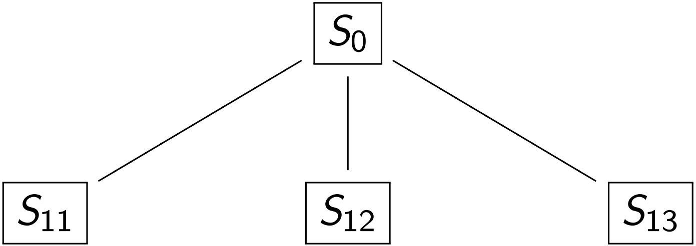
Algorithm: outline

Goal: Transform a system into finitely many simpler systems.

$$\boxed{S_0}$$

Algorithm: outline



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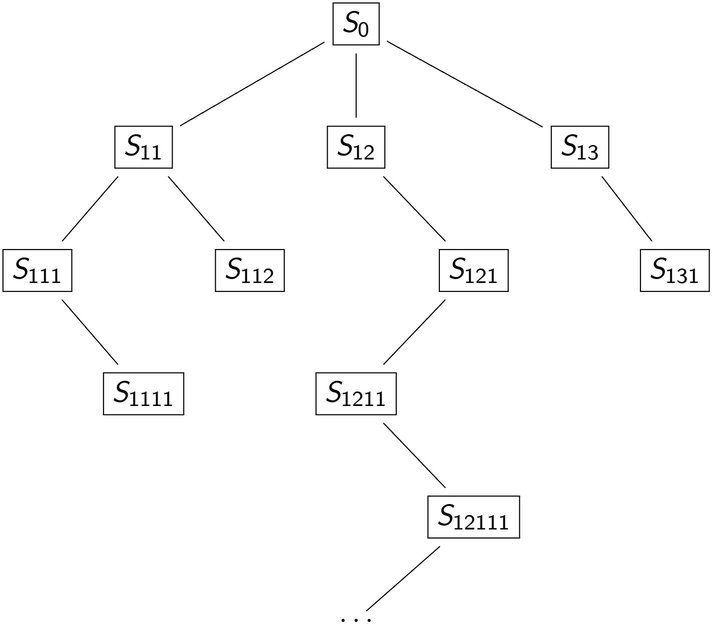
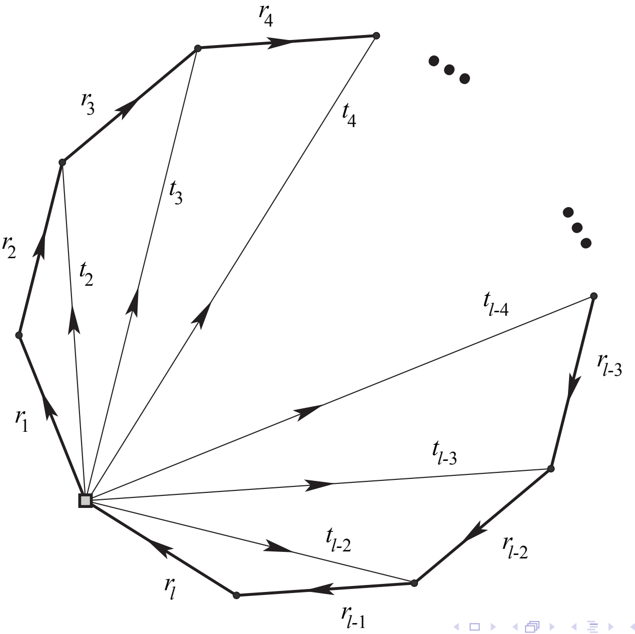


From system of equations to a single equation

Systems of equations are not more complicated than one equation:

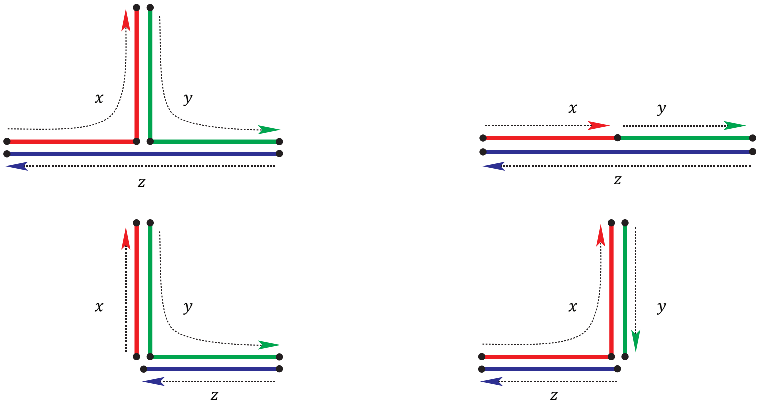
$$U = 1, V = 1 \Leftrightarrow U^2 a U^2 a^{-1} = (VbVb^{-1})^2$$

From one equation to a triangular system



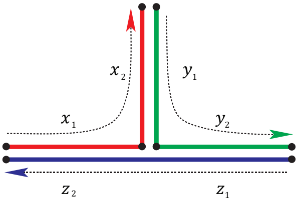
Reduction to generalised equations

WLOG all equations of a system are triangular: $xyz = 1$.
Cancellation schemes for the equation $xyz = 1$ in the free group:



Reduction to generalised equations

To each cancellation scheme one can associate a system of equations



$$\xrightarrow[\left\{\begin{array}{l} x = x_1 x_2 \\ y = y_1 y_2 \\ z = z_1 z_2 \end{array}\right.]{\hspace{6em}} \qquad \left\{\begin{array}{l} x_1 = z_2^{-1}, \\ x_2 = y_1^{-1}, \\ y_2 = z_1^{-1}. \end{array}\right.$$

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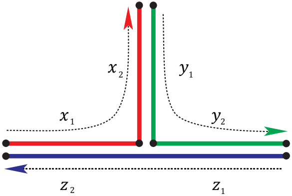

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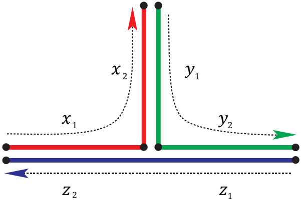

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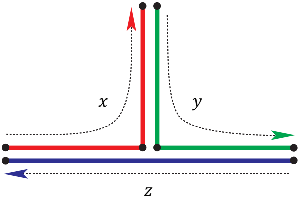
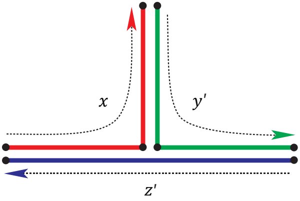


$$\begin{cases} x_1 = z_2^{-1}, \\ x_2 = y_1^{-1}, \\ y_2 = z_1^{-1}. \end{cases}$$

$$\Longrightarrow \qquad x_1 x_2 = x_1' x_2'$$



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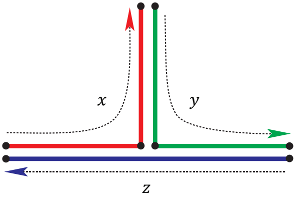

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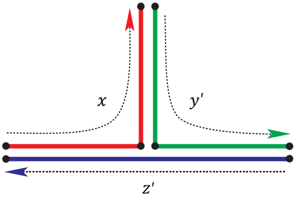

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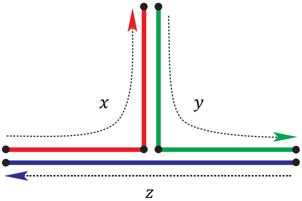

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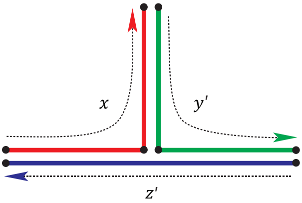

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Summarising:

- Begin from system, pass to 1 equation and to a triangular system.
- Consider all possible cancellation schemes for all equations of the system and obtain MANY systems of equations.
- The equations in these systems become graphical.
- Every solution of the original system defines a solution of one of the new systems.
- Any solution of any of the new systems lifts to a solution of the original system.

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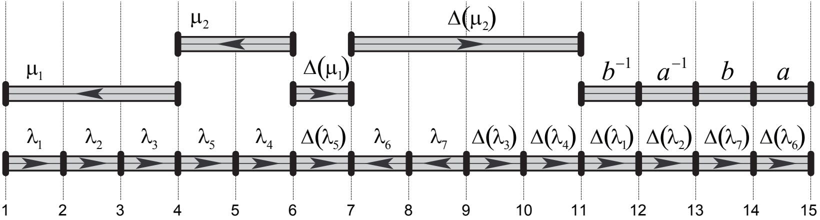
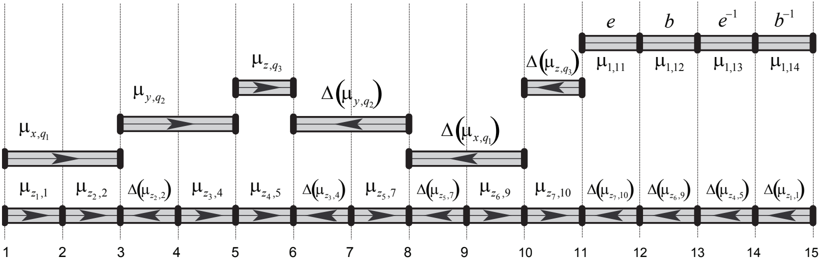
Generalised equations: example

Let $S = \{z_1 z_2 z_3 = z_5; z_5 = z_7^{-1} z_6^{-1} z_3 z_4; z_1 = a^{-1}; z_2 = b^{-1}; z_6 = a; z_7 = b\}$



Generalised equations: example

b	ac	$c^{-1}a^{-1}$	d	e	d^{-1}	ac	$c^{-1}a^{-1}$	b^{-1}	e^{-1}	e	b	e^{-1}	b^{-1}
h'_1	h'_2	h'_3	h'_4	h'_5	h'_6	h'_7	h'_8	h'_9	h'_{10}	h'_{11}	h'_{12}	h'_{13}	h'_{14}

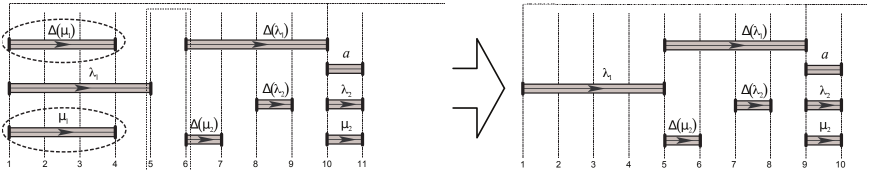


Generalised equations: moves

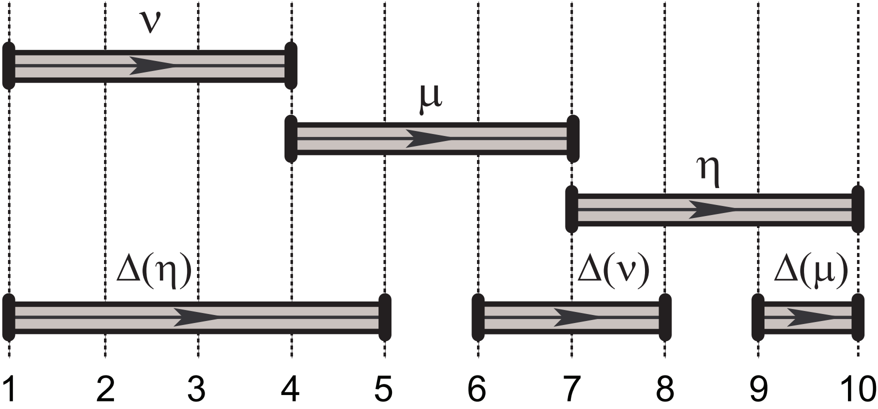
Two types of moves:

1. Simplify the generalised equation whenever possible;
2. Move bases to the right.

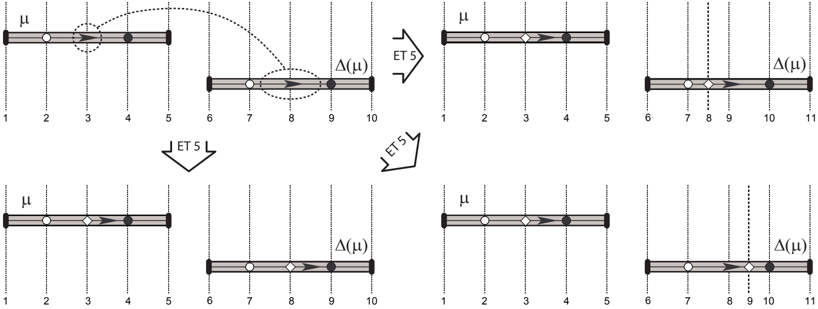
Generalised equations: moves



Generalised equations: moves



Dividing the problem



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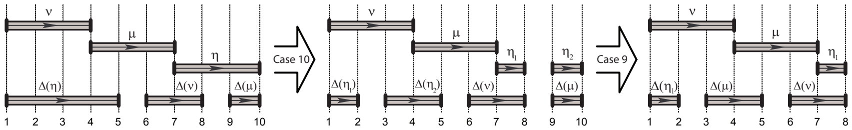

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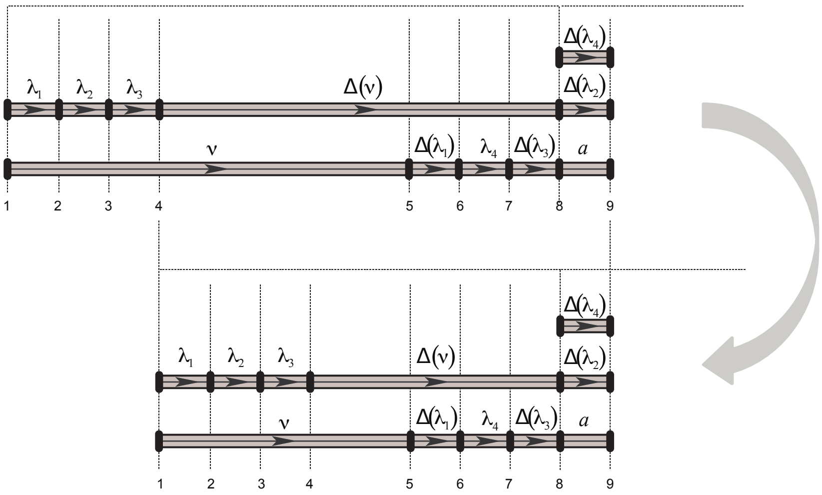
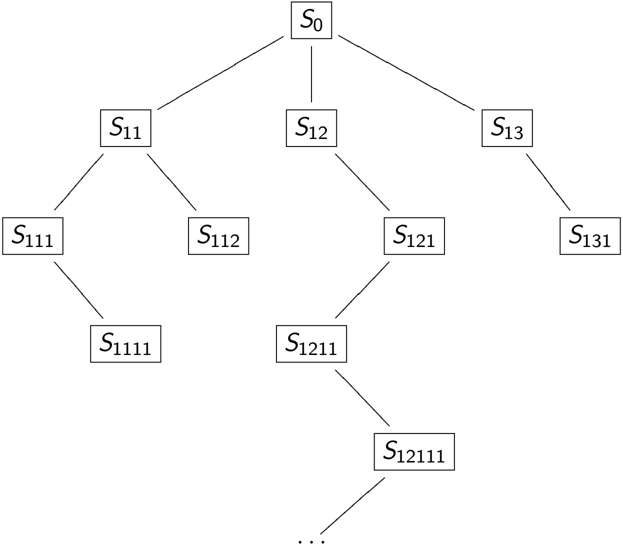


Description of algebraic sets



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\begin{tikzcd}
 & & & F_{R(S_0)} \arrow[dll] \arrow[d] \arrow[drr] & & & \\
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F_{R(S_{111})} \arrow[dr] & & F_{R(S_{112})} & & F_{R(S_{121})} \arrow[dl] & & F_{R(S_{131})} \\
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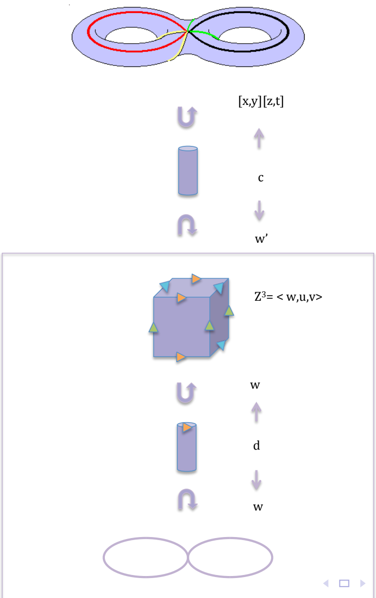
\end{tikzcd}
$$

Group-theoretic structure

Matrix to affine subspace MR diagram to tower



Applications

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2. Complexity.

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