LANGUAGES AND GROUPS

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PLAN

Word problems:

- introduce various classes of formal languages: regular, counter, context-free, indexed, ET0L, EDT0L, context-sensitive
- time and space complexity for Turing machine algorithms

Other problems:

- solutions to equations over groups

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Eg: F_2, free group on a, b, the set of all reduced words in $a^{\pm 1}, b^{\pm 1}$ with operation of *concatenate then reduce*

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- the complexity of the *Turing machine* (algorithm) that solves the decision problem
- the complexity of the set $\{w \in X^* \mid w =_G 1\}$ in terms of *formal language theory*

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- *logspace*: Turing machine with input, work and output tapes; on input w length n uses at most $O(\log n)$ squares of the work tape.

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Lipton and Zalcstein [19] gave the following logspace algorithm to solve the word problem for linear groups:

- on input $w \in X^*$, multiply I by the matrix for each letter and store the product with each entry $\mod p$ for some small number.
- if at the end the matrix is not I, then $w \neq_G 1$.
- if at the end the matrix is I for all small numbers, then $w =_G 1$.

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Now suppose G is infinite. If M is a FSA with n states accepting the word problem for G, let $w \in (X^{\pm 1})^*$ be a *geodesic* of length $> n$. Then w has prefix $w_1 w_2$ where both $w_1, w_1 w_2$ end at the same state of M.

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Then $w_1 w_1^{-1}$ and $w_1 w_2 w_1^{-1}$ both end at the same state, and since $w_1 w_1^{-1} =_G 1$ this is an accept state, which means $w_1 w_2 w_1^{-1} =_G 1$. Thus $w_2 =_G 1$ so w was not geodesic

Context-free languages can also be described in terms of a *grammar*:

- two finite alphabets T, N with $S \in N$
- finite set of rules $A \longrightarrow w \in (T \cup N)^*$ with $A \in N$
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Eg: $T = \{a^{\pm 1}, b^{\pm 1}\}, N = \{S\}$, rules

$$S \longrightarrow aSa^{-1}S \mid bSb^{-1}S \mid a^{-1}SaS \mid b^{-1}SbS \mid \epsilon$$

If a language L is context-free, there is a constant k (depending on the lengths of things in the grammar) so that for words w in L longer than k,

any derivation (drawn as a *parse tree*) must include the same nonterminal twice. This means $w = uvxyz$ and $uv^ixy^iz \in L$ for all $i \in \mathbb{N}$.

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Eg: $L = \{a^nb^na^n \mid n \in \mathbb{N}\}, L = \{ww \mid w \in \{a,b\}^*\}$ are not context-free.

(G, X) has a *Dehn's algorithm* if there exists a finite list

$$\{(u_i, v_i) \mid u_i, v_i \in X^*, u_i =_G v_i, |u_i| > |v_i|\}$$

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- problem: this might produce another u_j inside the stack

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Open problem: logspace? Not all hyperbolic groups are linear (see [1])

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See [9] for a nice discussion of this result.

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- nonterminals N, including a start symbol $S \in N$
- terminals A
- *flags* or *indices* F
- productions of the following types:
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Eg:

- $S \longrightarrow T_\$$
- $T \longrightarrow T_f \mid T_g \mid UU$
- $U_f \longrightarrow aU$
- $U_g \longrightarrow bU$
- $U_\$ \longrightarrow 1$

$$S \longrightarrow T_\$ \longrightarrow T_{\$f} \longrightarrow T_{\$fg} \longrightarrow T_{\$fgf} \longrightarrow U_{\$fgf}U_{\$fgf} \longrightarrow aU_{\$fg}aU_{\$fg}$$

Equivalent to indexed [2, 3].

Nested stack has a root node and a pointer which can read the current node

Moves:

- push (branch sideways if pointer not at a leaf)
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Subconjecture: word problem for $\mathbb{Z}^2$ is not indexed.

Gilman and Shapiro: word problem accepted by deterministic limited erasing nested stack automaton if and only if virtually free

Gilman [15]

If $L \subseteq X^*$ is indexed and $m \in \mathbb{N}$, there is a constant $k > 0$ so that $|w| \geq k$ can be written as $w = w_1 \ldots w_r$

- $m < r \leq k$
- $|w_i| > 0$
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Doesn't help with word problem of $\mathbb{Z}^2$ though

An *ET0L-system* is a tuple $(C, T, \Delta, \#)$ where

- C is a finite alphabet
- $T \subseteq C$
- $\# \in C$
- $\Delta = \{f_1, \ldots, f_n\}$ with each $f_i : C \longrightarrow \mathcal{P}(C^*) \setminus \emptyset$.

A table f_i acts on a word $w \in C^*$ as follows: if $w = x_1 \ldots x_k$ with $x_i \in C$, for each x_j we can choose any $u_j \in f_i(x_j)$ and replace x_j by u_j.

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Eg: $C = \{a, b, A, B, \#\}, T = \{a, b\}$, tables
$f_1 = \{(a, \{a\}), (b, \{b, AA\}), (A, \{aA, \epsilon\}), (B, \{AA, B\}), (\#, \{B, \#\})\}$

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$$
\begin{aligned}
f_1 &= \{\{(a, \{a\}), (b, \{b\}), (\#, \{a\#a\#, \#\})\}, \\
f_2 &= \{\{(a, \{a\}), (b, \{b\}), (\#, \{b\#b\#, \#\})\}, \\
f_3 &= \{\{(a, \{a\}), (b, \{b\}), (\#, \{\epsilon, \#\})\},
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indexed:

- $S \notin C$, make production $S \longrightarrow \#_{\$}$
- make productions $S \longrightarrow S_{f_i}$ for each table
- then each derivation starts with $S \Rightarrow \#_{\$ f_{i_1} \dots f_{i_k}}$
- for each $c \in C, f_i \in \Delta$ make productions $c_{f_i} \longrightarrow w$ where $w \in f_i(c)$
- for each $t \in T$ make production $t_{\$} \longrightarrow t'$.

An ET0L system is *deterministic* (called EDT0L) if $|f_i(c)| = 1$ for each table and each $c \in C$, that is, there is no choice about how to rewrite different letters when you apply a table.

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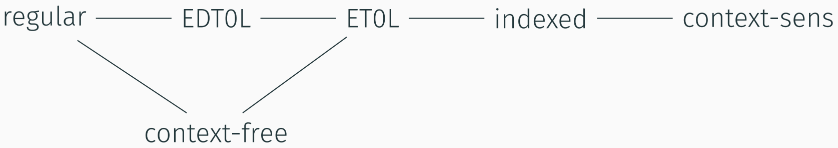
Shapiro [22] showed that every finitely generated subgroup of an automatic group is context-sensitive

— ie its a big class (includes group with undecidable conjugacy problem, non finitely presented etc.)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A formal language class $\mathcal{C}$ is *closed under inverse homomorphism* if $L \subseteq X^*$ in $\mathcal{C}$ and $\varphi : Y^* \longrightarrow X^*$ a letter homomorphism ($\varphi(y) = u \in X^*$ for each $y \in Y$) implies $\varphi^{-1}(L)$ is in $\mathcal{C}$.

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Fact: if a language class C is closed under inverse homomorphism and intersection with regular languages then (G, X) has word problem in C if and only if (G, Y) does

- regular word problem if and only if finite (Anisimov [4, 5])

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Open problems:

- indexed conjecture: virtually free
- ET0L conjecture: virtually free
- EDT0L conjecture: finite

Recall: $L \subseteq T^*$ is EDT0L if there is $T \subseteq C \ni \#$ and a finite set of tables f_i that rewrite each $c \in C$ by $u_c \in C^*$ in parallel.

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An alternative definition [6] restricts how the tables are applied.

A language $L \subseteq A^*$ is EDT0L if $\exists$

- $C \supseteq A$ an extended alphabet (finite)
- $\# \in C$
- R regular set of endomorphisms $h : C^* \longrightarrow C^*$

so that $L = \{h(\#) \mid h \in R\}$.

Problem: on input

$$aXXb = YYbX$$

find substitutions for X, Y by words in a, b, a^{-1}, b^{-1} so that both sides are equal in the free group on $A_+ = \{a, b\}$.

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How hard is the problem of finding all solutions to some equation:

- the complexity of the *Turing machine* (algorithm)
- the complexity of the solution set in terms of *formal language theory*

Context-sensitive: input $(u_1, \ldots, u_k)$ of length $n = \sum |u_i|$, can decide

- u_i reduced words in $A_\pm$
- $X_i \longrightarrow u_i$ is a solution

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But this does not tell us much — the *emptiness problem* for context-sensitive languages in undecidable [18]

Theorem (Ciobanu-Diekert-E [8])

For any equation over a free group, the set

$$\{(u_1, \ldots, u_k) \mid u_i \text{ reduced}, X_i \longrightarrow u_i \text{ is a solution}\}$$

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We construct in NSPACE($n \log n$) a finite direct labeled graph where

- nodes are modified versions of the equation
- edges are labeled by letter homomorphisms
- every solution in reduced words is encoded by some path from an initial to final node in the graph.

Here is a naïve first attempt.

- Input: $XaYbaXa = bYb^3ZP$ equation in a free monoid.
- Guess the first letter of some variable, and replace. Eg: $Y \longrightarrow \overline{a}Y$.
- Guess $Y \longrightarrow 1$.
- Repeat. If there is a solution, this method will find it!

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- $Y \longrightarrow \overline{a}Y$ increases the length of the equation (there is no cancellation) — can get arbitrarily long.
- to ensure solutions are reduced words, need to keep track of letters popped out of variables

To keep the equation length bounded, we can try to *compress* constants using new constants. Eg: $ab \longrightarrow c$, $aa \longrightarrow d$, $aa \longrightarrow a$.

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Issues:

- might need many new constants.
- $aa \longrightarrow d$ means that $ad = da$, so we are no longer in a free monoid.
- $aa \longrightarrow a$ only works if all blocks of a have *even* length

Edges correspond to making one of the following moves on an equation

pop

- $X \longrightarrow aX,\ \overline{X} \longrightarrow \overline{X}\,\overline{a}$
- $X \longrightarrow 1,\ \overline{X} \longrightarrow 1$

split

- $X \longrightarrow X'X,\ \overline{X} \longrightarrow \overline{X}\,\overline{X'}$

compress

- $aa \longrightarrow a,\ \overline{a}\,\overline{a} \longrightarrow \overline{a}$
- $aa \longrightarrow c,\ \overline{a}\,\overline{a} \longrightarrow \overline{c}$
- $ab \longrightarrow c,\ \overline{b}\,\overline{a} \longrightarrow \overline{c}$

CONSTRAINTS

We need to introduce a symbol $\#$ that serves various purposes, but require it not appear in any solution $X_i \longrightarrow u$

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Let $N = A_\pm \times A_\pm \cup \{0,1\}$ with multiplication

$$\begin{array}{l} 0 \cdot x = 0 = x \cdot 0 \\ 1 \cdot x = x = x \cdot 1 \end{array} \quad \text{and} \quad (a,b)\cdot(c,d) = \begin{cases} (a,d) & b \neq \overline{c} \\ 0 & b = \overline{c} \end{cases}$$

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Then define the morphism $\mu : (A_\pm \cup \{\#\})^* \longrightarrow N$ by $\mu(a) = (a, a), \mu(1) = 1$ and $\mu(\#) = 0$.

If $u \in (A_{\pm} \cup \{\#\})^*$ then $\mu(u) = 0$ if u contains $\#$ or is not reduced, is 1 if and only if $u = 1$, and otherwise $\mu(u) = (a, b)$ where a, b are the first and last letters of u.

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Given an equation in $A_\pm, \#$ and variables X_i we *guess* the first and last letters of each X_i (or that $X_i \longrightarrow 1$) by guessing a value for $\mu(X_i)$. As we modify our equation (pop and compress moves), we make sure that the μ values are consistent, and updated as we pop and compress letters

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In this way, if there is a way to consistently find a solution following our procedure, then it is guaranteed to be in reduced words not using $\#$.

A key difference between our result and Makanin-Razborov diagrams is that we are able to ensure that solutions are in reduced words, ie or maps are simply letter homomorphisms (so EDT0L) not group homomorphisms, where controlling the form of solutions seems difficult,

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The result is ET0L (and not EDT0L for free groups of rank two or more).

Thank you

REFERENCES I

- http://www.math.utah.edu/~bestvina/eprints/questions.pdf, http://mathoverflow.net/questions/110208/understanding-groups-that-are-not-linear.

- A. V. Aho.
  Indexed grammars—an extension of context-free grammars.
  J. ACM, 15:647–671, 1968.

- A. V. Aho.
  Nested stack automata.
  J. ACM, 16:383–406, 1969.

A. V. Anisimov.
Group languages.
Kibernetika, 4/1971:18–24, 1971.
English translation in Cybernetics and Systems Analysis 4 (1973), 594-601.

A. V. Anisimov and F. D. Seifert.
Zur algebraischen Charakteristik der durch kontext-freie Sprachen definierten Gruppen.
Elektron. Informationsverarbeit. Kybernetik, 11(10–12):695–702, 1975.

P. R. Asveld.
Controlled iteration grammars and full hyper-AFL's.
Information and control, 34, 1977.

J. W. Cannon.
The combinatorial structure of cocompact discrete hyperbolic groups.
Geom. Dedicata, 16(2):123–148, 1984.

L. Ciobanu, V. Diekert, and M. Elder.
Solution sets for equations over free groups are EDT0L languages.
In *Automata, languages, and programming. Part II*, volume 9135 of *Lecture Notes in Comput. Sci.*, pages 134–145. Springer, Heidelberg, 2015.

V. Diekert and A. Weiß.
Context-free groups and their structure trees.
Internat. J. Algebra Comput., 23(3):611–642, 2013.

A. Ehrenfeucht and G. Rozenberg.
The number of occurrences of letters versus their distribution in some EOL languages.
Information and Control, 26:256–271, 1974.

A. Ehrenfeucht and G. Rozenberg.
On inverse homomorphic images of deterministic ETOL languages.
In *Automata, languages, development*, pages 179–189. North-Holland, Amsterdam, 1976.

A. Ehrenfeucht and G. Rozenberg.
On some context free languages that are not deterministic ETOL languages.
RAIRO Informat. Théor., 11(4):273–291, i, 1977.

M. Elder, M. Kambites, and G. Ostheimer.
On groups and counter automata.
Internat. J. Algebra Comput., 18(8):1345–1364, 2008.

R. Gilman and M. Shapiro.
On groups whose word problem is solved by a nested stack automaton.
1998; http://arxiv.org/abs/math/9812028v1.

R. H. Gilman.
A shrinking lemma for indexed languages.
Theoret. Comput. Sci., 163(1-2):277–281, 1996.

R. I. Grigorchuk and A. Machì.
An example of an indexed language of intermediate growth.
Theoret. Comput. Sci., 215(1-2):325–327, 1999.

T. Herbst.
On a subclass of context-free groups.
Informatique Theorique et Applications, 25:255–272, 1991.

J. E. Hopcroft and J. D. Ullman.
Introduction to automata theory, languages, and computation.
Addison-Wesley Publishing Co., Reading, Mass., 1979.
Addison-Wesley Series in Computer Science.

R. J. Lipton and Y. Zalcstein.
Word problems solvable in logspace.
J. Assoc. Comput. Mach., 24(3):522–526, 1977.

D. E. Muller and P. E. Schupp.
Groups, the theory of ends, and context-free languages.
J. Comput. System Sci., 26(3):295–310, 1983.

B. Rozoy.
The Dyck language D_i^* is not generated by any matrix grammar of finite index.
Information and Computation, 74:64–89, 1987.

M. Shapiro.
A note on context-sensitive languages and word problems.
Internat. J. Algebra Comput., 4(4):493–497, 1994.