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# Geodesic Language Complexity and Group Structure

Maranda Franke

University Of Nebraska–Lincoln

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# Introduction

In this talk we will explore some connections between geodesic language complexity and group structure.

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- What do properties of the geodesic language imply about the group?
- What do properties of the group imply about geodesic languages?

$Note$ : In this talk, all groups are finitely generated and all generating sets are finite and inverse closed.

# Languages

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The set of *regular languages* is the closure of finite sets under concatenation, union, intersection, complementation, and Kleene closure.

$L$ is regular if and only if $L$ can be recognized by a finite state automaton.

# Cayley Graphs and Geodesic Languages



## Definition

The *Cayley graph* of a group $G$ with generating set $A$, $\Gamma(G, A)$ is the directed graph with a vertex $\forall g \in G$ and an edge labeled $a$ from $g$ to $ga$ $\forall a \in A$, $\forall g \in G$.

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## Definition

The *geodesic language* of $G$ over $A$, $Geo(G, A)$, is the set of all geodesic words in $\Gamma(G, A)$.

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# Examples

$$G = F_2 =< x, y \mid \ >, \ A = \{x, y\}^{\pm}$$

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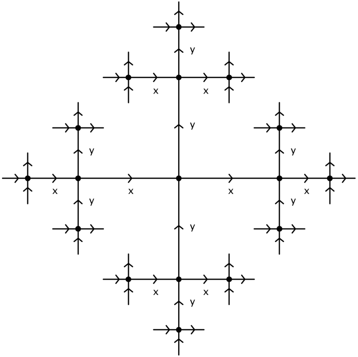

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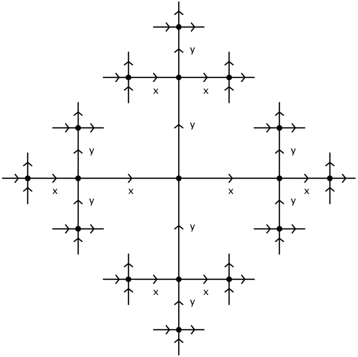


$Geo(G, A)$ is the set of all freely reduced words over $A$.

# Examples

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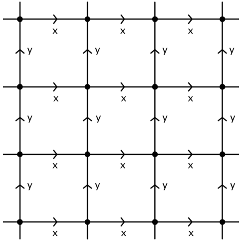

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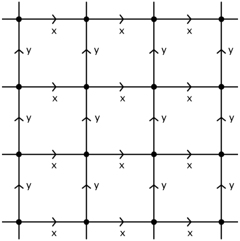


$Geo(G, A)$ is the set of all words in $A^*$ which do not contain both a generator and its inverse.

# Groups with Regular Geodesic Language

Groups with $Geo(G, A)$ regular for all generating sets:

- hyperbolic groups (*Cannon '80's*)
- abelian groups (*Neumann & Shapiro '95*)

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Groups with $Geo(G, A)$ regular for some generating set:

- Coxeter groups (*Howlett '93*)
- virtually abelian groups (*Neumann & Shapiro '95*)
- geometrically finite hyperbolic groups (*N & S '95*)
- Artin groups of finite type (*Charney & Meier '03*)
- Garside groups (*C & M '03*)
- Artin groups of large type (*Holt & Rees '11*)

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The second class is closed under graph products (*Loeffler, Meier, & Worthington '02*)

# Locally Excluding

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**Definition**

A group is *virtually free* if it has a finite index free subgroup.

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**Theorem (Gilman, Hermiller, Holt, & Rees, 2007)**

*$G$ is virtually free if and only if $Geo(G, A)$ is LE for some generating set $A$.*

# Piecewise Excluding



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**Theorem (Hermiller, Holt, & Rees, 2007)**

*If $G$ is abelian, then $Geo(G, A)$ is PE for all generating sets $A$.*

# Question 1

We know $G$ abelian $\Rightarrow Geo(G, A)$ is PE $\forall A$.

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**Remark**

*If $Geo(G, A)$ is PE, then $aa^{-1}$ must be an excluded piecewise subword $\forall a \in A$ as $aa^{-1} \notin Geo(G, A)$ $\forall a \in A$.*

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*Is $Geo(G, A)$ PE if and only if $G$ is abelian?*

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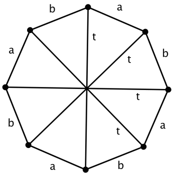
$D_8 =< a, b, t \mid a^2, b^2, (ab)^4, ababt >$ has PE geodesic language.



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**Proposition (F., 2015)**

*Let $G$ be abelian and $H$ be finite. If $K = H \rtimes G$, then there is a generating set $A$ such that $Geo(K, A)$ is PE.*

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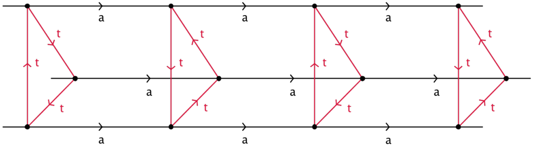
$\mathbb{Z}/3\mathbb{Z} \rtimes \mathbb{Z}$



# Question 2

$G$ abelian $\Rightarrow Geo(G, A)$ is PE $\forall A$

$Geo(G, A)$ PE $\not\Rightarrow G$ abelian

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**Question (from Murray Elder)**

*Is $Geo(G, A)$ PE for all generating sets $A$ if and only if $G$ is abelian?*

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*A group $G$ with $Geo(G, A)$ PE for all generating sets A need not be abelian.*

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$\mathbb{H} =< i, j, k \mid i^2 = j^2 = k^2 = ijk,\ i^4 = 1 >$ (the quaternions) has PE geodesic language for all generating sets.

# Virtually Abelian Groups



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A group is *virtually abelian* if it has a finite index abelian subgroup.

**Theorem (Neumann & Shapiro, 1995)**

*If G is abelian, $Geo(G, A)$ is regular for all generating sets $A$.*

*If G is virtually abelian, there is a generating set $A$ such that $Geo(G, A)$ is regular.*

# Cannon's Example

A virtually abelian group need not have $Geo(G, A)$ regular for all generating sets $A$:

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**Cannon's Example:**

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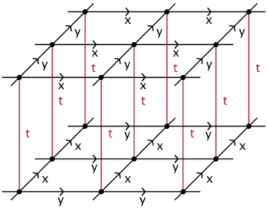

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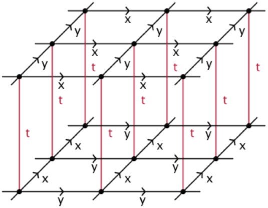


$\mathbb{Z}^2 \rtimes \mathbb{Z}/2\mathbb{Z} =< x, c, d, t \mid cx^{-2}, t^2, [x, x^{-1}d], txtd^{-1}x >$
does *not* have regular geodesic language

# Piecewise Testable

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**Definition**

$L$ is *piecewise testable*, PT, if $L$ is a Boolean combination of languages of the form $A^*a_1A^* \cdots A^*a_nA^*$, where $n \geq 0$ and $a_i \in A\ \forall i \in [n]$.

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Informally, $L$ is PT if membership can be decided by consideration of piecewise subwords.

Note $PE \subset PT$.

## Theorem (Hermiller, Holt, & Rees, 2007)

*If $G$ is virtually abelian, then there is a generating set $A$ for which $Geo(G, A)$ is PT.*

# Question 3

$G$ abelian $\Rightarrow Geo(G, A)$ PE
$Geo(G, A)$ PE $\forall A \not\Rightarrow G$ abelian
$G$ virtually abelian $\Rightarrow \exists A$ s.t. $Geo(G, A)$ is PT $\supset$ PE

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**Question**

*If G is virtually abelian, must $G$ have a generating set $A$ for which $Geo(G, A)$ is PE?*

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*If G is virtually abelian, must $G$ have a generating set $A$ for which $Geo(G, A)$ is PE?*

No once again.

**Theorem (F., 2015)**

*If G is virtually abelian, there need not exist a generating set $A$ for which $Geo(G, A)$ is PE.*

# Proof Outline

Recall that if $awa^{-1}$ is geodesic for any $a \in A$, $w \in A^*$, then $Geo(G, A)$ cannot be PE.

# Proof Outline



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Let $G = \mathbb{Z} \rtimes \mathbb{Z}/2\mathbb{Z} =< x, y \mid y^2, yxy = x^{-1} >$.

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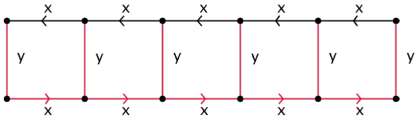
Consider the normal form set $N = \{x^n y^\epsilon \mid n \in \mathbb{Z},\ \epsilon \in \{0, 1\}\}$.



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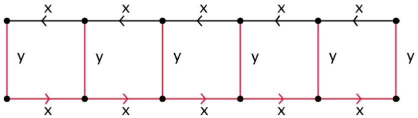


Identify each $a \in A$ with its unique representative in $N$.

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Case 1: $\exists \alpha \in A$ such that $\alpha =_G x^n$ for some $n \neq 0$.

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<u>Case 1:</u> $\exists \alpha \in A$ such that $\alpha =_G x^n$ for some $n \neq 0$.

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Let $b$ be the generator of the form $x^m y$ with $m$ maximal.

# Proof Outline

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Case 1: $\exists \alpha \in A$ such that $\alpha =_G x^n$ for some $n \neq 0$.

Let $a$ be the generator of the form $x^n$ with $n$ maximal. Note $n > 0$ as $A$ is inverse closed.

Let $b$ be the generator of the form $x^m y$ with $m$ maximal. Note at least one such generator must exist for $A$ to generate $G$.

# Proof Outline



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Suppose $aba^{-1}$ is not geodesic.

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Suppose $aba^{-1}$ is not geodesic.
Then there must either be a single generator or a product of two generators $=_G aba^{-1} =_G x^n x^m y x^{-n} =_G x^{2n+m} y$.

# Proof Outline

If there was a single generator $=_G x^{2n+m}y$, then we have a contradiction as $n > 0$ and $b = x^m y$ was chosen to have $m$ maximal.


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# Proof Outline

If there was a single generator $=_G x^{2n+m}y$, then we have a contradiction as $n > 0$ and $b = x^m y$ was chosen to have $m$ maximal.

If there was a product of generators $=_G x^{2n+m}y$, one must be of the form $x^k$ and the other of the form $x^l y$.

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Note we may assume $x^k x^l y =_G x^{k+l} y =_G x^{2n+m} y$ as $x^l y x^k =_G x^{l-k} y = x^{-k} x^l y$ and $x^k \in A \implies x^{-k} \in A$.

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By our assumptions, $k \leq n$ and $l \leq m \implies k + l \leq n + m < 2n + m$, another contradiction.

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By our assumptions, $k \leq n$ and $l \leq m \implies k + l \leq n + m < 2n + m$, another contradiction.

Thus, $aba^{-1}$ must be geodesic.

# Proof Outline

Case 2: $\nexists \alpha \in A$ such that $\alpha =_G x^n$ for some $n \neq 0$.

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Case 2: $\nexists \alpha \in A$ such that $\alpha =_G x^n$ for some $n \neq 0$.

Let $a$ be the generator of the form $x^n y$ with $n$ maximal.
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# Proof Outline

Case 2: $\nexists \alpha \in A$ such that $\alpha =_G x^n$ for some $n \neq 0$.

Let $a$ be the generator of the form $x^n y$ with $n$ maximal.
Let $b$ be the generator of the form $x^m y$ with $m$ minimal.

*show $aba^{-1}$ is geodesic*

# Proof Outline



Case 2: $\nexists \alpha \in A$ such that $\alpha =_G x^n$ for some $n \neq 0$.

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*show $aba^{-1}$ is geodesic*

In either case, $aba^{-1} \in Geo(G, A)$ implies $aa^{-1}$ can't be an excluded piecewise subword in $Geo(G, A)$.

# Proof Outline

<u>Case 2:</u> $\nexists \alpha \in A$ such that $\alpha =_G x^n$ for some $n \neq 0$.

Let $a$ be the generator of the form $x^n y$ with $n$ maximal.
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*show $aba^{-1}$ is geodesic*

In either case, $aba^{-1} \in Geo(G, A)$ implies $aa^{-1}$ can't be an excluded piecewise subword in $Geo(G, A)$.

Hence $Geo(G, A)$ is not PE.

# Extensions



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**Proposition (F., 2015)**

*Let $G$ be an extension $1 \to H \to G \xrightarrow{\pi} K \to 1$ and $A$ a generating set for $G$. If $Geo(K, \pi(A))$ has $awa^{-1}$ geodesic for some $a \in \pi(A)$, $w \in \pi(A)^*$, then $Geo(G, A)$ is not PE.*

# Extensions

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**Proposition (F., 2015)**

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$\mathbb{Z}^2 \rtimes \mathbb{Z}/2\mathbb{Z}$ (Cannon's example) is another group that has no generating set which produces a PE geodesic language.

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**Proposition (F., 2015)**

*Let $G$ be an extension $1 \to H \to G \xrightarrow{\pi} K \to 1$ and $A$ a generating set for $G$. If $Geo(K, \pi(A))$ has $awa^{-1}$ geodesic for some $a \in \pi(A),\ w \in \pi(A)^*$, then $Geo(G, A)$ is not PE.*

$\mathbb{Z}^2 \rtimes \mathbb{Z}/2\mathbb{Z}$ (Cannon's example) is another group that has no generating set which produces a PE geodesic language.

Extensions of $\mathbb{Z}^2 \rtimes \mathbb{Z}/2\mathbb{Z}$ will also have non-PE geodesic language for any generating set.

# Continuing Work



## Question

*If $Geo(G, A)$ is not PE, then is there some $a \in A$ and $w \in A^*$ so that $awa^{-1}$ is geodesic?*

# Continuing Work



**Question**

*If $Geo(G, A)$ is not PE, then is there some $a \in A$ and $w \in A^*$ so that $awa^{-1}$ is geodesic?*

**Conjecture**

*If $G = \mathbb{Z}^n \rtimes_\alpha F$ where $|F| < \infty$ and $\alpha$ is non-trivial, then $Geo(G, A)$ is not PE for any generating set $A$.*

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**Question**

*If $Geo(G, A)$ is not PE, then is there some $a \in A$ and $w \in A^*$ so that $awa^{-1}$ is geodesic?*

**Conjecture**

*If $G = \mathbb{Z}^n \rtimes_\alpha F$ where $|F| < \infty$ and $\alpha$ is non-trivial, then $Geo(G, A)$ is not PE for any generating set $A$.*

**Open Question**

*Is $Geo(G, A)$ PT for some generating set $A$ if and only if $G$ is virtually abelian?*

# Thanks

Thank you!