Algebraic subgroups of acylindrically hyperbolic groups

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Equations and formal languages in algebra
Les Diablerets
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Equations

Notation:
For each $w(x) \in G * \langle x \rangle$ and each $g \in G$, let $w(g)$ denote the image of $w(x)$ under the homomorphism $G * \langle x \rangle \to G$ given by taking $id : G \to G$ and sending $x \mapsto g$.

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Definition

Write $w(x) = 1$ to represent an ***equation*** in the single variable x with coefficients in G whose ***solution set*** is

$$\{g \in G \,|\, w(g) = 1\}.$$

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- Zariski-closed sets are closed in every T_0 group topology, and, in the case of countable groups, the Zariski-closed sets are the only such sets (Markov, 1944).

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- Centralizers of subsets of G
- Exercise: If G is a torsion-free abelian group, then the only algebraic subgroups of G are G and $\{1\}$. (Hint: WLOG, each equation is of the form $gx^n = 1$, where $g \in G$, $n \in \mathbb{Z}$.)

Algebraic subgroups of G

Goal: structural result for algebraic subgroups in the case where G is an acylindrically hyperbolic group

Next: definition of acylindrically hyperbolic group, some examples

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$$d(x,y) \geq R \implies |\{g \in G \mid d(x,gx) \leq \varepsilon \text{ and } d(y,gy) \leq \varepsilon\}| \leq N.$$

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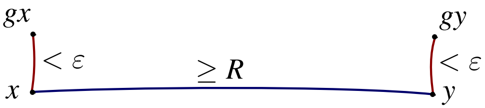

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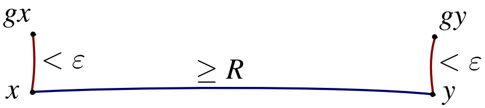

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Definition

A group G is called ***acylindrically hyperbolic*** if it admits a non-elementary acylindrical action on a hyperbolic space.

Acylindrically hyperbolic groups

Examples:

- non-elementary hyperbolic and relatively hyperbolic groups
- infinite mapping class groups of punctured closed surfaces (Mazur-Minsky, Bowdich)
- $Out(F_n)$ for $n \geq 2$ (Bestvina-Feign, Dahmani-Guirardel-Osin)
- directly indecomposable non-cyclic right angled Artin groups (Sisto, Caprace-Sageev, Osin)
- most 3-manifold groups (Minasyan-Osin)
- groups of deficiency ≥ 2 (Osin)

Acylindrically hyperbolic groups

Definition

Given an acylindrically hyperbolic group G, a subgroup $H \leq G$ is called ***non-elementary*** if for *some* acylindrical action of G on a hyperbolic space S, the action of H on S is non-elementary.

Main Result

Theorem (J., 2015)

Suppose that G is an acylindrically hyperbolic group and that $H \leq G$ is non-elementary. Then H is algebraic if and only if H is a virtual centralizer of some finite subgroup of G.

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Theorem (J., 2015)

Suppose that G is an acylindrically hyperbolic group and H is a non-elementary subgroup of G. Then the Zariski closure of H contains $C_G(E_G(H))$ where $E_G(H)$ is the unique maximal finite subgroup of G normalized by H.

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($E_G(H)$ exists by Hull (2013) or Antolin-Minasyan-Sisto (2013).)

Classification of acylindrical actions on hyperbolic spaces

Theorem (Osin, 2013)

If G acts acylyndrically on a hyperbolic space, then exactly one of the following holds:

1. *The action of G is non-elementary.*
2. *G has bounded orbits.*
3. *G is virtually cyclic and contains a loxodromic element.*

Corollaries

Corollary (Free products)

Let A and B be nontrivial groups, and let H be an algebraic subgroup of $A * B$. Then at least one of the following holds:

(a) *H is either infinite cyclic or isomorphic to D_∞, the infinite dihedral group.*

(b) *H is conjugate to a subgroup of either A or B.*

(c) *$H = A * B$.*

Corollaries

Corollary (Torsion-free relatively hyperbolic groups)

Let G be a torsion-free relatively hyperbolic group with peripheral subgroups $\{H_\lambda\}_{\lambda \in \Lambda}$, and let $H \leq G$ be an algebraic subgroup. Then at least one of the following holds:

(a) $H = G$.

(b) H *is cyclic.*

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(a) $H = G$.

(b) H *is cyclic.*

(c) H *is conjugate to a subgroup of some* H_λ.

Furthermore, if (c) holds for an abelian H_λ, then either $H = \{1\}$ or H is conjugate to H_λ.

Corollaries

Corollary (Ascending chains of algebraic subgroups)

Let G be an acylindrically hyperbolic group and let $H_1 \leq H_2 \leq H_3 \leq \ldots$ be an ascending chain of algebraic subgroups of G. Then either

(a) *for each acylindrical action of G on a hyperbolic space S, the subgroup $\cup_{i \in \mathbb{N}} H_i$ acts on S with bounded orbits (in particular, each H_i is elliptic), or*

(b) *the chain stabilizes.*

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(b) *the chain stabilizes.*

Note: in general, it is actually possible for an acylindrically hyperbolic group to have an ascending chain of (elliptic) algebraic subgroups which does not stabilize.

Ascending chain example

Consider a group with presentation $\langle X|R\rangle$ that contains an infinite ascending chain of subgroups

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Let

$$H = \langle X, a, t_1, t_2, \ldots \mid R, [X, a] = 1, [K_1, t_1] = 1, [K_2, t_2] = 1, \ldots \rangle$$

Then $G = H * \mathbb{Z}$ is acylindrically hyperbolic.

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Then $G = H * \mathbb{Z}$ is acylindrically hyperbolic. Furthermore,

$$K_i = \{x \in H * \mathbb{Z} \mid [x, t_i] = 1\} \cap \{x \in H * \mathbb{Z} \mid [x, a] = 1\}$$

so that $K_1 \leq K_2 \leq \dots$ is an ascending chain of algebraic subgroups of G that does not stabilize.

Thank you!