On the universe for limit groups over free pro-p groups

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Goal

To understand finitely generated models of the universal theory of free groups (in classical categories of groups).

Algebraic characterization: Fully residually H groups

Definition (Fully residually free)

A group G is *fully residually free* if for any finite set of non-trivial elements there is a homomorphism from G to a free group which is injective in the finite set.

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Suppose that a finitely generated group G is fully residually $\mathbb{Z}$, then G is free abelian.
Indeed, if $x, y \in G$, then any homomorphism $\varphi : G \to \mathbb{Z}$ satisfies that $\varphi([x, y]) = [\varphi(x), \varphi(y)] = 1$. Hence $[x, y] = 1$ for all $x, y \in G$. Hence G is f.g. abelian: $G \simeq \mathbb{Z}^n \times T$, where T is the torsion subgroup. Any homomorphism $\varphi : G \to \mathbb{Z}$ satisfies that $\varphi(T) = 1$, hence $T = 1$ and so G is free abelian.

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Example (Finitely generated fully residually $\mathbb{Z}$)

Any finitely generated free abelian group is fully residually $\mathbb{Z}$
Let $n = 2$, $\mathbb{Z}^2 = \langle a, b \rangle$ and $\mathbb{Z} = \langle x \rangle$.
For any finite set $S = \{(a^{r_1}, b^{s_1}), \ldots, (a^{r_k}, b^{s_k})\} \subset \mathbb{Z}^2$, the homomorphism

$$\varphi_S : \begin{cases} a \mapsto x \\ b \mapsto x^n \end{cases}$$

where $n = \sum_{i=1,\ldots,k} (|r_i| + |s_i|)$ is injective in S

Characterization

- **Finitely generated fully residually free group or *limit groups***;
- **Finitely generated models of the universal theory of** F;
- **Coordinate groups of irreducible algebraic sets**;
- Finitely generated subgroups of the ultra-power of F;
- Gromov-Hausdorff limits of free groups in a compact space of marked groups;
- Groups acting on a limit tree obtained from sequences of homomorphisms to F;
- Groups realising atomic types over F...

Universe of limit groups over free groups

Definition

A group G equipped with the action of the ring $\mathbb{Z}[t]$ subject to the following axioms:

1. $g^1 = g$, $g^0 = 1$, $1^\alpha = 1$;
2. $g^{\alpha+\beta} = g^\alpha \cdot g^\beta$, $g^{\alpha\beta} = (g^\alpha)^\beta$;
3. $(h^{-1}gh)^\alpha = h^{-1}g^\alpha h$;
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Let $G = F(a, b)$ be the free group of rank 2.

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Theorem (Lyndon, 61)

The free $\mathbb{Z}[t]$-group is fully residually free.

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Corollary

Any finitely generated subgroup of the free $\mathbb{Z}[t]$-group is fully residually free and so a limit group.

Extension of centralisers

Definition
Let $w \in G$, then $H = \langle G, x = w^t \mid [x, C(w)] = 1 \rangle$ is called the *extension of the centraliser of* w.

Theorem (Miasnikov, Remeslennikov, 94)
The free $\mathbb{Z}[t]$*-group* $F^{\mathbb{Z}[t]}$ *is the direct limit of iterated centraliser extensions.*

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Theorem (Kharlampovich-Miasnikov '98)

- *Any limit group is a subgroup of $F^{\mathbb{Z}[t]}$.*
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One could do amalgamated products instead

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Pro-p groups

- Let $(A_i), i \in \mathbb{N}$ be a family of finite p-groups and
- let $f_{ij} : A_j \to A_i$ for all $i < j$ family of homomorphisms with the following properties:
  - f_{ii} is the identity in A_i,
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- Then

$$P = \varprojlim A_i := \left\{ (a_i) \in \prod_{i \in \mathbb{N}} A_i \;\middle|\; f_{ij}(a_j) = a_i \;\text{ for all }\; i \leq j \right\}$$

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- Consider $\mathbb{Z}$ and all its p-quotients $\mathbb{Z}/p^n\mathbb{Z}$ there are natural maps from $f_{ij} : \mathbb{Z}/p^j\mathbb{Z} \to \mathbb{Z}/p^i\mathbb{Z}$, $i < j$.
- We take the inverse limit and obtain $\mathbb{Z}_p$.
- The free group is residually nilpotent. The quotient of the free group by the $c + 1$-th member of the lower central is the free nilpotent group of class c
- Make it into a p-group by imposing all generators to have order p^n.
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Elementary theory of free pro-p groups

Analogue of Tarski's problem for free pro-p groups:

- Elementary equivalence of free pro-p groups is trivial;
- More generally, two f.g. pro-p groups are elementarily equivalent iff they are isomorphic (Jarden-Lubotzky, 2011).
- Decidability of the first-order theory of the free pro-p group.
- Axiomatisation of the free pro-p group and description of (abstract) groups elementarily equivalent to it. abstract groups.

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Algebraic geometry over pro-p groups

Pro-p groups form a category and all usual definitions from algebraic geometry carry over to the pro-p setting, e.g. $G[[X]] = G *_p \mathbb{F}(X)$.

A pro-p group G is fully residually free pro-p if for any $g_1, g_2, \ldots, g_n$ from G there exists a surjective continuous $\phi : G \to \mathbb{F}$, so that $\phi(g_i) \neq 1$.

Theorem (Kochloukova, Zalesskii, 2010)

Every orientable surface pro-p group

$$G_d = \langle x_1, x_2, \ldots, x_{2d} \mid [x_1, x_2] \cdots [x_{2d-1}, x_{2d}] = 1 \rangle$$

is fully residually free pro-p, d even.

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What are limit groups over pro-p groups?

Theorem (Kharlampovich-Miasnikov '98)

- *Any limit group is a subgroup of* $F^{\mathbb{Z}[t]}$.
- *Every limit group is a subgroup of a finite iterated extension of centralisers of the free group.*

- Let $\mathcal{G}_0$ be the class of all free pro-p groups of finite rank.
- Define inductively $\mathcal{G}_n$, where $G_n \in \mathcal{G}_n$ is a free pro-p amalgamated product $G_{n-1} \amalg_C A$, where $G_{n-1} \in \mathcal{G}_{n-1}$, C is any self-centralised procyclic pro-p subgroup of G_{n-1}, A is any finite rank free abelian pro-p group such that C is a direct summand of A.
- *Define* limit groups to be (topologically) f.g. subgroups of extensions of centralisers.

Problem (Kochloukova, Zalesskii, 2011)

Is every pro-p "limit group" (fully) residually free pro-p?

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- Define inductively $\mathcal{G}_n$, where $G_n \in \mathcal{G}_n$ is a free pro-p amalgamated product $G_{n-1} \sqcup_C A$, where $G_{n-1} \in \mathcal{G}_{n-1}$, C is any self-centralised procyclic pro-p subgroup of G_{n-1}, A is any finite rank free abelian pro-p group such that C is a direct summand of A.
- *Define* limit groups to be (topologically) f.g. subgroups of extensions of centralisers.

Problem (Kochloukova, Zalesskii, 2011)

Is every pro-p "limit group" (fully) residually free pro-p?

Binomial rings and closures

Definition

A domain R of characteristic 0 is called *binomial* if for any $\lambda \in R$ and $n \in \mathbb{N}$, the ring R contains the binomial coefficients:

$$C_\lambda^n = \frac{\lambda(\lambda-1)(\lambda-2)\cdots(\lambda-n+1)}{n!}.$$

Binomial rings and closures

Definition

Define R^{bin} recursively as follows. Let $R = R_0 < L$, suppose that $R_i < L$ is already defined and define R_{i+1} as follows

$$R_{i+1} = \langle R_i, C_n^{\alpha} \mid \alpha \in R_i \setminus R_{i-1} \rangle < L.$$

Set $\varinjlim R_i = R^{bin}$.

Proposition (Casals-Ruiz, K, Remeslennikov)

1. R^{bin} is a binomial ring containing R;
2. For any binomial ring S and any ϕ there exists a unique $\phi' : R^{bin} \to S$ s.t.:
$$\begin{tikzcd} R \arrow[rr, "\phi"] \arrow[dr] & & S \\ & R^{bin} \arrow[ur, "\phi'"'] & \end{tikzcd}$$
3. R^{bin} is discriminated by R

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Nilpotent groups and R-groups

When considering non-commutative groups, it is natural to attempt to extend the idea of a module to the noncommutative case - a group admitting exponents in some ring R.

The chief difficulty lies in attempting to replace the rule $r(x+y) = rx + ry$ (define an action of the ring).

1. If N is a group which is complete and Hausdorff in its p-adic topology, then for any $x \in N$, the homomorphism of $\mathbb{Z}$ into N taking n to x^n extends naturally to a homomorphism of the groups $\mathbb{Z}_p$ of p-adic integers into N. We make N into a group admitting exponents in the ring of p-adic integers.
2. If K is any field of characteristic zero, then an exponent can be defined on $UT_n(K)$:
$$(1+x)^r = 1 + rx + C_r^2 x^2 + \ldots$$
3. If K is the field of real numbers, then $UT_n(K)$ is a nilpotent Lie group, and for any $g \in UT_n(K)$, the set of elements of the form g^r defined in this way, is exactly the one-parameter subgroup generated by g.

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Hall R-groups

Definition

Let R be a *binomial* ring. A nilpotent group G of a class m is called a Hall R-group if for all $x, y, x_1, \ldots, x_n \in G$ and any $\lambda, \mu \in R$ one has:

- G is a nilpotent R-group of class m;
- $(y^{-1}xy)^{\lambda} = (y^{-1}xy)^{\lambda}$;
- $x_1^{\lambda} \cdots x_n^{\lambda} = (x_1 \cdots x_n)^{\lambda} \tau_2(x)^{C_2^{\lambda}} \cdots \tau_m(x)^{C_m^{\lambda}}$, where $\tau_i(x) \in \Gamma_{i-1}(F(x))$ is the i-th Petrescu word defined in the free group $F(x)$ by
$$x_1^i \cdots x_n^i = \tau_1(x)^{C_1^i} \tau_2(x)^{C_2^i} \cdots \tau_i(x)^{C_i^i}.$$

Free pro-Hall R-group

- Let $F(X, R, c)$ be the free Hall R-group on X of class c.
- For all $c \geq 1$, there is a natural homomorphism $\psi_c : F(X, R, c+1) \to F(X, R, c)$.
- Define the free pro-Hall R-group as $\varprojlim F(X, R, c) = \mathbb{F}(X, R)$.

1. If $R = \mathbb{Z}_p$, then any torsion-free Hall R-group is a pro-p.
2. The pro-Hall $\mathbb{Z}_p$-group $\mathbb{F}(X, \mathbb{Z}_p)$ is the free pro-p group with base X.

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Free pro-Hall R-group as the universe

Theorem (Casals-Ruiz, K, Remeslennikov)

Let R be a ring from the universal class of $\mathbb{Z}_p$. Then $\mathbb{F}(A, R^{bin})$ is fully residually $\mathbb{F}$.

Note that even, the even if $R \equiv_\forall \mathbb{Z}_p$ the free pro-Hall R-group need not be a pro-p group.

Theorem

Let $h_1, \ldots, h_m \in \mathbb{F}(A, R^{bin})$, then there exists a natural pro-p subgroup $H_\mathbb{F}$ of $\mathbb{F}(A, R^{bin})$ containing/generated by the elements $\{h_1, \ldots, h_m\}$.

The group $H_\mathbb{F}$ is fully residually $\mathbb{F}$ and hence is a coordinate group of an irreducible algebraic set over $\mathbb{F}$.

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Free pro-Hall R-group as the universe

Conjecture

1. Let G be a finitely generated fully residually $\mathbb{F}$ pro-p group, then there exists a finitely generated binomial $\mathbb{Z}_p$-algebra $R = R(G)$ from the universal class of $\mathbb{Z}_p$ so that G embeds into the free pro-Hall R-group $\mathbb{F}(A, R)$.
2. The universal class of $\mathbb{Z}_p$ contains a $\mathbb{Z}_p$-ring S so that any finitely generated fully residually $\mathbb{F}$ pro-p group embeds into $\mathbb{F}(A, S)$.

THANK YOU!

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