Word and conjugacy problem in Baumslag-Solitar groups

Armin Weiß

Universität Stuttgart / Stevens

Les Diablerets, March 8, 2016

Dehn's fundamental problems

Let G be a group, generated by a finite set Σ with $\Sigma = \Sigma^{-1} \subseteq G$. Write $\overline{a}$ for the letter $a^{-1} \in \Sigma$.

- Word problem: Given $w \in \Sigma^*$. Question: Is $w = 1$ in G?
- Conjugacy problem: Given $v, w \in \Sigma^*$. Question: $v \sim w$?

  $$(\exists\, z \in G \text{ such that } zvz^{-1} = w?)$$

Baumslag-Solitar group: $\mathbf{BS}_{p,q} = \left\langle a, t \mid ta^p t^{-1} = a^q \right\rangle$
$= \mathrm{HNN}(\langle a \rangle, t; a^p \mapsto a^q)$

Generalized Baumslag-Solitar group: Fundamental group of graph of groups with infinite cyclic vertex and edge groups

W.l.o.g. $1 \leq p \leq |q|$.

Baumslag-Solitar group:

$$\mathbf{BS}_{p,q} = \left\langle a, t \mid ta^p t^{-1} = a^q \right\rangle$$
$$= \mathrm{HNN}(\langle a \rangle, t; a^p \mapsto a^q)$$

Generalized Baumslag-Solitar group: Fundamental group of graph of groups with infinite cyclic vertex and edge groups

W.l.o.g. $1 \leq p \leq |q|$.

- $\mathbf{BS}_{p,q}$ is solvable $\iff$ $p = 1$, in this case

$$\mathbf{BS}_{1,q} = \mathbb{Z}[1/q] \rtimes \mathbb{Z}$$

- $\mathbf{BS}_{p,q}$ is linear $\iff$ $p = |q|$ or $p = 1$,
- $\mathbf{BS}_{p,q}$ is not linear, otherwise.

Overview: Word and conjugacy problem

- The word problem of $\mathbf{BS}_{p,q}$ is solvable in polynomial time.
- The conjugacy problem is decidable (Anshel, Stebe, 1974).

Overview: Word and conjugacy problem

- The word problem of $\mathbf{BS}_{p,q}$ is solvable in polynomial time.
- The conjugacy problem is decidable (Anshel, Stebe, 1974).

Theorem (Lipton, Zalcstein, 1977 / Simon, 1979)

The word problem of a linear group (in particular, $\mathbf{BS}_{p,\pm p}$ and free groups) can be solved in LOGSPACE.

Overview: Word and conjugacy problem

- The word problem of $\mathbf{BS}_{p,q}$ is solvable in polynomial time.
- The conjugacy problem is decidable (Anshel, Stebe, 1974).

Theorem (Lipton, Zalcstein, 1977 / Simon, 1979)

The word problem of a linear group (in particular, $\mathbf{BS}_{p,\pm p}$ and free groups) can be solved in LOGSPACE.

Theorem (Robinson, 1993 / Diekert, Miasnikov, W., 2014)

The word and conjugacy problem of $\mathbf{BS}_{1,q}$ are uniform TC^0*-complete.*

TC^0 = recognized by a family of circuits of constant depth with unbounded fan-in $\neg$, $\wedge$, $\vee$, and majority gates.

Overview: Word and conjugacy problem

$\mathbf{BS}_{p,q}$ contains a free subgroup $\langle t, ata^{-1} \rangle$ if $|p|, |q| \neq 1$.

$\rightsquigarrow$ word problem is NC^1-hard (Robinson, 1993).

$\rightsquigarrow$ no hope to solve it in TC^0.

Overview: Word and conjugacy problem

$\mathbf{BS}_{p,q}$ contains a free subgroup $\langle t, ata^{-1} \rangle$ if $|p|, |q| \neq 1$.

$\rightsquigarrow$ word problem is NC^1-hard (Robinson, 1993).

$\rightsquigarrow$ no hope to solve it in TC^0.

Theorem (W.)

The word and conjugacy problem of $\mathbf{BS}_{p,q}$ (and generalized Baumslag-Solitar groups) is in LOGSPACE. More precisely,

- *the word problem is uniform-TC^0-many-one-reducible to the word problem of the free group F_2.*
- *the conjugacy problem is uniform-AC^0-Turing-reducible to the word problem of the free group – (CP is in $\mathrm{AC}^0(F_2)$).*

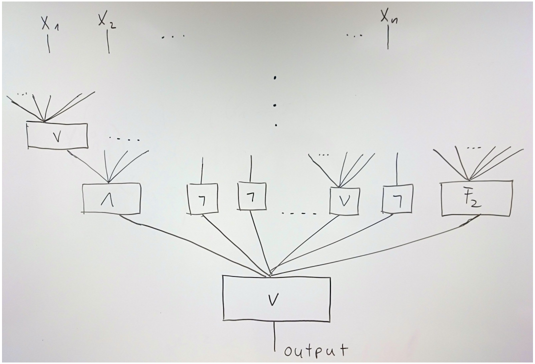
- AC^0 = recognized by a family of circuits of constant depth and polynomial size with unbounded fan-in $\neg$, $\wedge$, $\vee$ gates.

Crash Course Circuit Complexity

- AC^0 = recognized by a family of circuits of constant depth and polynomial size with unbounded fan-in $\neg$, $\wedge$, $\vee$ gates.
- TC^0 allows additionally majority gates.

Crash Course Circuit Complexity

- AC^0 = recognized by a family of circuits of constant depth and polynomial size with unbounded fan-in $\neg$, $\wedge$, $\vee$ gates.
- TC^0 allows additionally majority gates.
- For a group G, $AC^0(G)$ allows additionally oracle gates for the word problem of G.



Crash Course Circuit Complexity

- AC^0 = recognized by a family of circuits of constant depth and polynomial size with unbounded fan-in $\neg$, $\wedge$, $\vee$ gates.
- TC^0 allows additionally majority gates.
- For a group G, $\mathrm{AC}^0(G)$ allows additionally oracle gates for the word problem of G.

Word problem of $\mathbb{Z}$ with generators $\{+1, -1\}$ is in TC^0:

$$\begin{aligned} w = 0 \text{ in } \mathbb{Z} &\iff |w|_{+1} = |w|_{-1} \\ &\iff \neg(\mathrm{Maj}_{+1}(w)) \wedge \neg(\mathrm{Maj}_{-1}(w)) \end{aligned}$$

Crash Course Circuit Complexity

- AC^0 = recognized by a family of circuits of constant depth and polynomial size with unbounded fan-in $\neg$, $\wedge$, $\vee$ gates.
- TC^0 allows additionally majority gates.
- For a group G, $\mathsf{AC}^0(G)$ allows additionally oracle gates for the word problem of G.

Word problem of $\mathbb{Z}$ with generators $\{+1, -1\}$ is TC^0-complete:

Crash Course Circuit Complexity

- AC^0 = recognized by a family of circuits of constant depth and polynomial size with unbounded fan-in $\neg$, $\wedge$, $\vee$ gates.
- TC^0 allows additionally majority gates.
- For a group G, $\mathsf{AC}^0(G)$ allows additionally oracle gates for the word problem of G.

Word problem of $\mathbb{Z}$ with generators $\{+1, -1\}$ is TC^0-complete:

- $\mathsf{TC}^0 \subseteq \mathsf{AC}^0(F_2)$
- $\mathsf{AC}^0(F_2) \subseteq \mathsf{LOGSPACE}$

- AC^0 = recognized by a family of circuits of constant depth and polynomial size with unbounded fan-in $\neg$, $\wedge$, $\vee$ gates.
- TC^0 allows additionally majority gates.
- For a group G, $\mathrm{AC}^0(G)$ allows additionally oracle gates for the word problem of G.

Iterated Addition

- input: n-bit numbers $r_1, \ldots, r_n$,
- compute $\sum_{i=1}^{n} r_i$.

Iterated Addition is in TC^0.

- AC^0 = recognized by a family of circuits of constant depth and polynomial size with unbounded fan-in $\neg$, $\wedge$, $\vee$ gates.
- TC^0 allows additionally majority gates.
- For a group G, $\mathsf{AC}^0(G)$ allows additionally oracle gates for the word problem of G.

Iterated Multiplication

- input: n-bit numbers $r_1, \ldots, r_n$,
- compute $\prod_{i=1}^{n} r_i$.

Crash Course Circuit Complexity

- AC^0 = recognized by a family of circuits of constant depth and polynomial size with unbounded fan-in $\neg$, $\wedge$, $\vee$ gates.
- TC^0 allows additionally majority gates.
- For a group G, $\mathrm{AC}^0(G)$ allows additionally oracle gates for the word problem of G.

Iterated Multiplication

- input: n-bit numbers $r_1, \ldots, r_n$,
- compute $\prod_{i=1}^{n} r_i$.

Integer Division

- input: n-bit numbers a, b,
- compute $\left\lfloor \frac{a}{b} \right\rfloor$.

- AC^0 = recognized by a family of circuits of constant depth and polynomial size with unbounded fan-in $\neg$, $\wedge$, $\vee$ gates.
- TC^0 allows additionally majority gates.
- For a group G, $\mathsf{AC}^0(G)$ allows additionally oracle gates for the word problem of G.

Iterated Multiplication

- input: n-bit numbers $r_1, \ldots, r_n$,
- compute $\prod_{i=1}^n r_i$.

Integer Division

- input: n-bit numbers a, b,
- compute $\left\lfloor \frac{a}{b} \right\rfloor$.

Theorem (Hesse, 2001)

Iterated Multiplication and Integer Division are in TC^0.

$\mathsf{AC}^0 = \mathsf{FO}(+, *)$	$\mathbb{Z}/n\mathbb{Z}$ with one monoid generator
$\mathsf{ACC}^0 = \mathsf{FO}(+, *; \mathrm{Mod})$	finite solvable
$\mathsf{TC}^0 = \mathsf{FO}(+, *; \mathrm{Maj})$	$\mathbb{Z}$, $\mathbf{BS}_{1,q}$, linear solvable (e. g. polycyclic)
$\mathsf{NC}^1 = \mathsf{AC}^0(A_5)$	finite non-solvable, regular languages
$\mathsf{AC}^0(F_2)$	virtually free, $\mathbf{BS}_{p,q}$, GBS groups, RAAGs, free products

Crash Course Circuit Complexity

$\mathsf{AC}^0 = \mathsf{FO}(+,*)$	$\mathbb{Z}/n\mathbb{Z}$ with one monoid generator
$\mathsf{ACC}^0 = \mathsf{FO}(+,*;\mathrm{Mod})$	finite solvable
$\mathsf{TC}^0 = \mathsf{FO}(+,*;\mathrm{Maj})$	$\mathbb{Z}$, $\mathbf{BS}_{1,q}$, linear solvable (e. g. polycyclic)
$\mathsf{NC}^1 = \mathsf{AC}^0(A_5)$	finite non-solvable, regular languages
$\mathsf{AC}^0(F_2)$	virtually free, $\mathbf{BS}_{p,q}$, GBS groups, RAAGs, free products
LOGSPACE	linear groups

$\mathsf{AC}^0 = \mathsf{FO}(+,*)$	$\mathbb{Z}/n\mathbb{Z}$ with one monoid generator
$\mathsf{ACC}^0 = \mathsf{FO}(+,*;\mathrm{Mod})$	finite solvable
$\mathsf{TC}^0 = \mathsf{FO}(+,*;\mathrm{Maj})$	$\mathbb{Z}$, **$\mathbf{BS}_{1,q}$**, linear solvable (e. g. polycyclic)
$\mathsf{NC}^1 = \mathsf{AC}^0(A_5)$	finite non-solvable, regular languages
$\mathsf{AC}^0(F_2)$	virtually free, **$\mathbf{BS}_{p,q}$**, GBS groups, RAAGs, free products
LOGSPACE	linear groups
P polynomial time	

The word problem of $\mathbf{BS}_{p,q}$

Britton's Lemma

$w \in \langle a \rangle = A$ in $\mathbf{BS}_{p,q} \iff w$ can be reduced to some word in $\{a, \overline{a}\}^*$ by Britton reductions

$$t^{\varepsilon} a^{k} t^{-\varepsilon} \to a^{\ell} \qquad (\varepsilon \in \{\pm 1\}), k \in p\mathbb{Z} \text{ (resp. } q\mathbb{Z}).$$

Britton's Lemma

$w \in \langle a \rangle = A$ in $\mathbf{BS}_{p,q} \iff w$ can be reduced to some word in $\{a, \bar{a}\}^*$ by Britton reductions

$$t^{\varepsilon} a^{k} t^{-\varepsilon} \to a^{\ell} \qquad (\varepsilon \in \{\pm 1\}), k \in p\mathbb{Z} \text{ (resp. } q\mathbb{Z}).$$

$\rightsquigarrow$ word problem in P by storing exponents in binary.

The word problem of $\mathbf{BS}_{p,q}$

Britton's Lemma

$w \in \langle a \rangle = A$ in $\mathbf{BS}_{p,q} \iff w$ can be reduced to some word in $\{a, \overline{a}\}^*$ by Britton reductions

$$t^{\varepsilon} a^{k} t^{-\varepsilon} \to a^{\ell} \qquad (\varepsilon \in \{\pm 1\}),\, k \in p\mathbb{Z} \text{ (resp. } q\mathbb{Z}).$$

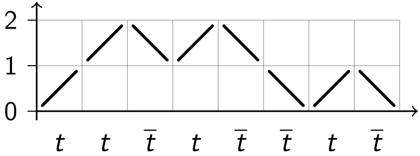
$\rightsquigarrow$ word problem in P by storing exponents in binary.

Two aspects:

- Word problem of solvable Baumslag-Solitar groups.
- Word problem of the free group F_2.

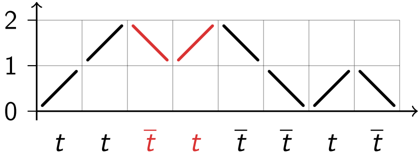
The word problem of $\mathbf{BS}_{p,q}$

$$w = t\,a\,t\,a\,\overline{t}\,aaa\,t\,a\,\overline{t}\,a\,\overline{t}\,t\,aa\,\overline{t} \in \mathbf{BS}_{2,3}$$



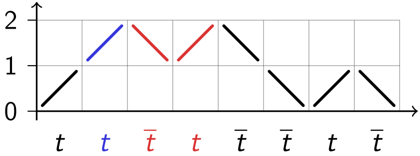
Idea: assign different colors to letters t which cannot cancel:

$$w = t\,a\,t\,a\,\bar{t}\,aaa\,t\,a\,\bar{t}\,a\,\bar{t}\,t\,aa\,\bar{t} \in \mathbf{BS}_{2,3}$$



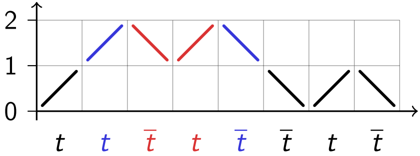
Idea: assign different colors to letters t which cannot cancel:

$$w = t\,a\,t\,a\,\bar{t}\,aaa\,t\,a\,\bar{t}\,a\,\bar{t}\,t\,aa\,\bar{t} \in \mathbf{BS}_{2,3}$$



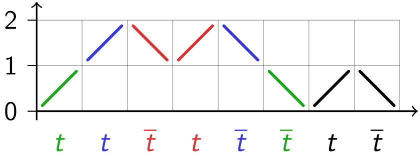
Idea: assign different colors to letters t which cannot cancel:

$$w = t\,a\,t\,a\,\bar{t}\,aaa\,t\,a\,\bar{t}\,a\,\bar{t}\,t\,aa\,\bar{t} \in \mathbf{BS}_{2,3}$$



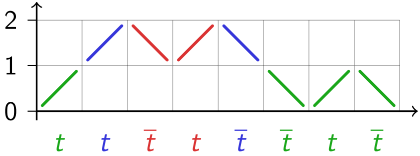
Idea: assign different colors to letters t which cannot cancel:

$$w = t\,a\,t\,a\,\bar{t}\,aaa\,t\,a\,\bar{t}\,a\,\bar{t}\,t\,aa\,\bar{t} \in \mathbf{BS}_{2,3}$$



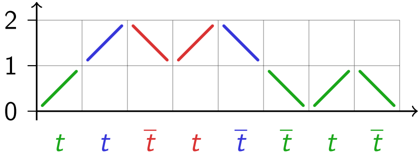
Idea: assign different colors to letters t which cannot cancel:

$$w = t\,a\,t\,a\,\bar{t}\,aaa\,t\,a\,\bar{t}\,a\,\bar{t}\,t\,aa\,\bar{t} \in \mathbf{BS}_{2,3}$$



Idea: assign different colors to letters t which cannot cancel:

$$w = t\,a\,t\,a\,\bar{t}\,aaa\,t\,a\,\bar{t}\,a\,\bar{t}\,t\,aa\,\bar{t} \in \mathbf{BS}_{2,3}$$

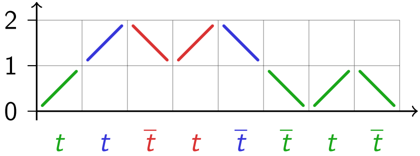


$\rightsquigarrow w \in \langle a \rangle = A$
Forget about letters a

$$\tilde{w} = t\,t\,\bar{t}\,t\,\bar{t}\,\bar{t}\,t\,\bar{t} \in$$

Idea: assign different colors to letters t which cannot cancel:

$$w = t\,a\,t\,a\,\bar{t}\,aaa\,t\,a\,\bar{t}\,a\,\bar{t}\,t\,aa\,\bar{t} \in \mathbf{BS}_{2,3}$$



$\rightsquigarrow w \in \langle a \rangle = A$

Forget about letters a

$$\tilde{w} = t\,t\,\bar{t}\,t\,\bar{t}\,\bar{t}\,t\,\bar{t} \in F(t, t, t)$$

$\rightsquigarrow$ word problem of the free group

The word problem of $\mathbf{BS}_{p,q}$

How to compute the color?

How to compute the color?

$$w = a^{k_0} t^{\varepsilon_1} a^{k_1} \cdots t^{\varepsilon_i} \underbrace{a^{k_i} t^{\varepsilon_{i+1}} a^{k_{i+1}} \cdots t^{\varepsilon_j} a^{k_j}} t^{\varepsilon_{j+1}} a^{k_{j+1}} \cdots t^{\varepsilon_n} a^{k_n}$$

with $\varepsilon_\mu \in \{ \pm 1 \}$, $k_\mu \in \mathbb{Z}$. Define

$$w_{i,j} = a^{k_i} t^{\varepsilon_{i+1}} a^{k_{i+1}} \cdots t^{\varepsilon_j} a^{k_j}$$

$$k_{i,j} = \sum_{\nu=i}^{j} k_\nu \cdot \prod_{\mu=i+1}^{\nu} \left(\frac{q}{p}\right)^{\varepsilon_\mu} \qquad \in \mathbb{Z}[1/pq]$$

How to compute the color?

$$w = a^{k_0} t^{\varepsilon_1} a^{k_1} \cdots t^{\varepsilon_i} \underbrace{a^{k_i} t^{\varepsilon_{i+1}} a^{k_{i+1}} \cdots t^{\varepsilon_j} a^{k_j}} t^{\varepsilon_{j+1}} a^{k_{j+1}} \cdots t^{\varepsilon_n} a^{k_n}$$

with $\varepsilon_\mu \in \{\pm 1\}$, $k_\mu \in \mathbb{Z}$. Define

$$w_{i,j} = a^{k_i} t^{\varepsilon_{i+1}} a^{k_{i+1}} \cdots t^{\varepsilon_j} a^{k_j}$$

$$k_{i,j} = \sum_{\nu=i}^{j} k_\nu \cdot \prod_{\mu=i+1}^{\nu} \left(\frac{q}{p}\right)^{\varepsilon_\mu} \quad \in \mathbb{Z}[1/pq]$$

Numbers $k_{i,j}$ can be computed in TC^0.

The word problem of $\mathbf{BS}_{p,q}$

How to compute the color?

$$w = a^{k_0} t^{\varepsilon_1} a^{k_1} \cdots t^{\varepsilon_i} \underbrace{a^{k_i} t^{\varepsilon_{i+1}} a^{k_{i+1}} \cdots t^{\varepsilon_j} a^{k_j}} t^{\varepsilon_{j+1}} a^{k_{j+1}} \cdots t^{\varepsilon_n} a^{k_n}$$

with $\varepsilon_\mu \in \{\pm 1\}$, $k_\mu \in \mathbb{Z}$. Define

$$w_{i,j} = a^{k_i} t^{\varepsilon_{i+1}} a^{k_{i+1}} \cdots t^{\varepsilon_j} a^{k_j}$$

$$k_{i,j} = \sum_{\nu=i}^{j} k_\nu \cdot \prod_{\mu=i+1}^{\nu} \left(\frac{q}{p}\right)^{\varepsilon_\mu} \qquad \in \mathbb{Z}[1/pq]$$

Numbers $k_{i,j}$ can be computed in TC^0.

Lemma 1

$$w_{i,j} \in \langle a \rangle \iff w_{i,j} = a^{k_{i,j}} \text{ in } \mathbf{BS}_{p,q}$$

Proof.

Induction: by Britton's Lemma, $w = a^{k_0} t^{\varepsilon_1} w' t^{-\varepsilon_1} w''$. $\square$

The word problem of $\mathbf{BS}_{p,q}$

Define a relation $\sim_{\mathcal{C}} \subseteq \{1, \ldots, n\} \times \{1, \ldots, n\}$:
For $i < j$:

$$i \sim_{\mathcal{C}} j \iff \varepsilon_i = -\varepsilon_j \text{ and } \sum_{\ell=i+1}^{j-1} \varepsilon_\ell = 0 \quad \text{(same level)}$$

$$\text{and } k_{i,j-1} \in \begin{cases} p\mathbb{Z} & \text{if } \varepsilon_i = 1 \\ q\mathbb{Z} & \text{if } \varepsilon_i = -1. \end{cases}$$

For $i > j$: $i \sim_{\mathcal{C}} j \iff j \sim_{\mathcal{C}} i$.

$\rightsquigarrow i \sim_{\mathcal{C}} j \iff t^{\varepsilon_i}$ and t^{ε_j} are on the same level and cancel if everything in between cancels.

Define a relation $\sim_{\mathcal{C}} \subseteq \{1,\ldots,n\} \times \{1,\ldots,n\}$:
For $i < j$:

$$i \sim_{\mathcal{C}} j \iff \varepsilon_i = -\varepsilon_j \text{ and } \sum_{\ell=i+1}^{j-1} \varepsilon_\ell = 0 \quad \text{(same level)}$$

$$\text{and } k_{i,j-1} \in \begin{cases} p\mathbb{Z} & \text{if } \varepsilon_i = 1 \\ q\mathbb{Z} & \text{if } \varepsilon_i = -1. \end{cases}$$

For $i > j$: $i \sim_{\mathcal{C}} j \iff j \sim_{\mathcal{C}} i$.

$\rightsquigarrow i \sim_{\mathcal{C}} j \iff t^{\varepsilon_i}$ and t^{ε_j} are on the same level and cancel if everything in between cancels.

$\approx =$ reflexive and transitive closure of $\sim_{\mathcal{C}}$

Define a relation $\sim_{\mathcal{C}} \subseteq \{1, \ldots, n\} \times \{1, \ldots, n\}$:
For $i < j$:

$$i \sim_{\mathcal{C}} j \iff \varepsilon_i = -\varepsilon_j \text{ and } \sum_{\ell=i+1}^{j-1} \varepsilon_\ell = 0 \quad \text{(same level)}$$

$$\text{and } k_{i,j-1} \in \begin{cases} p\mathbb{Z} & \text{if } \varepsilon_i = 1 \\ q\mathbb{Z} & \text{if } \varepsilon_i = -1. \end{cases}$$

For $i > j$: $i \sim_{\mathcal{C}} j \iff j \sim_{\mathcal{C}} i$.

$\rightsquigarrow i \sim_{\mathcal{C}} j \iff t^{\varepsilon_i}$ and t^{ε_j} are on the same level and cancel if everything in between cancels.

$\approx =$ reflexive and transitive closure of $\sim_{\mathcal{C}}$

Lemma 2

If $i \approx j$ and $\varepsilon_i = -\varepsilon_j$, then $i \sim_{\mathcal{C}} j$.

How to compute the color? Color $=$ $\approx$-class.

$$w = a^{k_0} t^{\varepsilon_1} a^{k_1} \cdots t^{\varepsilon_n} a^{k_n} \in \mathbf{BS}_{p,q}$$

Let $\Sigma_w = \left\{ t_{[i]}, \overline{t}_{[i]} \;\middle|\; i \in \{1, \dots, n\} \right\}$ be a new set of generators:

$$\widetilde{w} := t_{[1]}^{\varepsilon_1} \cdots t_{[n]}^{\varepsilon_n}$$

How to compute the color? Color $=\approx$-class.

$$w = a^{k_0} t^{\varepsilon_1} a^{k_1} \cdots t^{\varepsilon_n} a^{k_n} \in \mathbf{BS}_{p,q}$$

Let $\Sigma_w = \{\, t_{[i]}, \overline{t}_{[i]} \;\mid\; i \in \{1, \ldots, n\} \,\}$ be a new set of generators:

$$\widetilde{w} := t_{[1]}^{\varepsilon_1} \cdots t_{[n]}^{\varepsilon_n}$$

Example

$$w = t\,a\,t\,a\,\overline{t}\,aaa\,t\,a\,\overline{t}\,a\,\overline{t}\,t\,aa\,\overline{t} \;\mapsto\; \widetilde{w} = t_{[1]}\,t_{[2]}\,\overline{t}_{[3]}\,t_{[3]}\,\overline{t}_{[2]}\,\overline{t}_{[1]}\,t_{[1]}\,\overline{t}_{[1]}$$

How to compute the color? Color = $\approx$-class.

$$w = a^{k_0} t^{\varepsilon_1} a^{k_1} \cdots t^{\varepsilon_n} a^{k_n} \in \mathbf{BS}_{p,q}$$

Let $\Sigma_w = \left\{ t_{[i]}, \overline{t}_{[i]} \;\middle|\; i \in \{1, \ldots, n\} \right\}$ be a new set of generators:

$$\widetilde{w} := t_{[1]}^{\varepsilon_1} \cdots t_{[n]}^{\varepsilon_n}$$

Example

$$w = t\, a\, t\, a\, \overline{t}\, aaa\, t\, a\, \overline{t}\, a\, \overline{t}\, t\, aa\, \overline{t} \;\mapsto\; \widetilde{w} = t_{[1]}\, t_{[2]}\, \overline{t}_{[3]}\, t_{[3]}\, \overline{t}_{[2]}\, \overline{t}_{[1]}\, t_{[1]}\, \overline{t}_{[1]}$$

Proposition

$$w = 1 \textit{ in } \mathbf{BS}_{p,q} \iff \widetilde{w} = 1 \textit{ in } F(\Sigma_w) \textit{ and } k_{0,n} = 0.$$

How to compute the color?

On input w, compute $\widetilde{w}$:

How to compute the color?

On input w, compute $\widetilde{w}$:

- For every index i compute the smallest j with $i \approx j$ as representative of $[i]$: by Lemma 2, two steps of $\sim_{\mathcal{C}}$ suffice.
- $i \sim_{\mathcal{C}} j$ can be checked in TC^0:

The word problem of $\mathbf{BS}_{p,q}$

How to compute the color?

On input w, compute $\widetilde{w}$:

- For every index i compute the smallest j with $i \approx j$ as representative of $[i]$: by Lemma 2, two steps of $\sim_{\mathcal{C}}$ suffice.
- $i \sim_{\mathcal{C}} j$ can be checked in TC^0:
  - check whether $\varepsilon_i = -\varepsilon_j$

How to compute the color?

On input w, compute $\widetilde{w}$:

- For every index i compute the smallest j with $i \approx j$ as representative of $[i]$: by Lemma 2, two steps of $\sim_{\mathcal{C}}$ suffice.
- $i \sim_{\mathcal{C}} j$ can be checked in TC^0:
  - check whether $\varepsilon_i = -\varepsilon_j$
  - compute $\sum_{\ell=i+1}^{j-1} \varepsilon_\ell$

How to compute the color?

On input w, compute $\widetilde{w}$:

- For every index i compute the smallest j with $i \approx j$ as representative of $[i]$: by Lemma 2, two steps of $\sim_{\mathcal{C}}$ suffice.
- $i \sim_{\mathcal{C}} j$ can be checked in TC^0:
  - check whether $\varepsilon_i = -\varepsilon_j$
  - compute $\sum_{\ell=i+1}^{j-1} \varepsilon_\ell$
  - compute $k_{i,j-1}$

How to compute the color?

On input w, compute $\widetilde{w}$:

- For every index i compute the smallest j with $i \approx j$ as representative of $[i]$: by Lemma 2, two steps of $\sim_{\mathcal{C}}$ suffice.
- $i \sim_{\mathcal{C}} j$ can be checked in TC^0:
  - check whether $\varepsilon_i = -\varepsilon_j$
  - compute $\sum_{\ell=i+1}^{j-1} \varepsilon_\ell$
  - compute $k_{i,j-1}$
  - check whether $q \mid k_{i,j-1}$ (resp. $p \mid k_{i,j-1}$)

How to compute the color?

On input w, compute $\widetilde{w}$:

- For every index i compute the smallest j with $i \approx j$ as representative of $[i]$: by Lemma 2, two steps of $\sim_{\mathcal{C}}$ suffice.
- $i \sim_{\mathcal{C}} j$ can be checked in TC^0:
  - check whether $\varepsilon_i = -\varepsilon_j$
  - compute $\sum_{\ell=i+1}^{j-1} \varepsilon_\ell$
  - compute $k_{i,j-1}$
  - check whether $q \mid k_{i,j-1}$ (resp. $p \mid k_{i,j-1}$)

$\rightsquigarrow$ TC^0-reduction to the word problem of $F(\Sigma_w) \leq F_2$.

The word problem of $\mathbf{BS}_{p,q}$

How to compute the color?

On input w, compute $\widetilde{w}$:

- For every index i compute the smallest j with $i \approx j$ as representative of $[i]$: by Lemma 2, two steps of $\sim_{\mathcal{C}}$ suffice.
- $i \sim_{\mathcal{C}} j$ can be checked in TC^0:
  - check whether $\varepsilon_i = -\varepsilon_j$
  - compute $\sum_{\ell=i+1}^{j-1} \varepsilon_\ell$
  - compute $k_{i,j-1}$
  - check whether $q \mid k_{i,j-1}$ (resp. $p \mid k_{i,j-1}$)

$\rightsquigarrow$ TC^0-reduction to the word problem of $F(\Sigma_w) \leq F_2$.

Corollary

The computation of Britton reduced words is in $\mathrm{AC}^0(F_2)$.

Let $g = a^k \in \langle a \rangle$. Then

$$
\begin{aligned}
aga^{-1} &= g, \\
tgt^{-1} &= ta^k t^{-1} \begin{cases} = a^{\frac{q}{p}k} & \text{if } p \mid k \\ \text{is Britton reduced} & \text{otherwise.} \end{cases}
\end{aligned}
$$

Solving the conjugacy problem of $\mathbf{BS}_{p,q}$

Let $g = a^k \in \langle a \rangle$. Then

$$aga^{-1} = g,$$
$$tgt^{-1} = ta^k t^{-1} \begin{cases} = a^{\frac{q}{p}k} & \text{if } p \mid k \\ \text{is Britton reduced} & \text{otherwise.} \end{cases}$$

Thus, for $k \neq \ell$:

$$a^k \sim a^\ell \iff \exists j \in \mathbb{Z} \text{ such that } k \cdot \left(\frac{q}{p}\right)^j = \ell$$
$$\text{and } \begin{cases} k \in p\mathbb{Z},\, \ell \in q\mathbb{Z}, & \text{if } j > 0, \\ k \in q\mathbb{Z},\, \ell \in p\mathbb{Z}, & \text{otherwise.} \end{cases}$$

There are only polynomially many possibilities for j
$\rightsquigarrow$ check them all in parallel.

Let $g = a^k \in \langle a \rangle$. Then

$$aga^{-1} = g,$$
$$tgt^{-1} = ta^k t^{-1} \begin{cases} = a^{\frac{q}{p}k} & \text{if } p \mid k \\ \text{is Britton reduced} & \text{otherwise.} \end{cases}$$

Thus, for $k \neq \ell$:

$$a^k \sim a^\ell \iff \exists j \in \mathbb{Z} \text{ such that } k \cdot \left(\frac{q}{p}\right)^j = \ell$$
$$\text{and } \begin{cases} k \in p\mathbb{Z},\ \ell \in q\mathbb{Z}, & \text{if } j > 0, \\ k \in q\mathbb{Z},\ \ell \in p\mathbb{Z}, & \text{otherwise.} \end{cases}$$

There are only polynomially many possibilities for j
$\rightsquigarrow$ check them all in parallel.

Corollary

It can be checked in TC^0 *whether* $a^k \sim a^\ell$.

Solving the conjugacy problem of $\mathbf{BS}_{p,q}$

- After a cyclic permutation we may assume that

$$w = a^{k_0} t^{\varepsilon_1} a^{k_1} \cdots t^{\varepsilon_n} a^{k_n} \in \mathbf{BS}_{p,q},$$
$$v = a^{\ell_0} t^{\varepsilon_1} a^{\ell_1} \cdots t^{\varepsilon_n} a^{\ell_n} \in \mathbf{BS}_{p,q}.$$

and $v \sim w$ if and only if there is an integral solution $x, y_1, \ldots, y_n$ for the system of equations

$$y_i = \frac{1}{\alpha_i} \left(x \cdot \prod_{\mu=1}^{i-1} \left(\frac{p}{q}\right)^{\varepsilon_\mu} + \sum_{\nu=1}^{i-1} (k_\nu - \ell_\nu) \cdot \prod_{\mu=\nu+1}^{i-1} \left(\frac{p}{q}\right)^{\varepsilon_\mu} \right),$$

$$x = k_n - \ell_n + x \cdot \prod_{\mu=1}^{n} \left(\frac{p}{q}\right)^{\varepsilon_\mu} + \sum_{\nu=1}^{n-1} (k_\nu - \ell_\nu) \cdot \prod_{\mu=\nu+1}^{n} \left(\frac{p}{q}\right)^{\varepsilon_\mu}.$$

- Can be done in TC^0.

Theorem

The conjugacy problem of any Baumslag-Solitar group $\mathbf{BS}_{p,q}$ *is in* $\mathrm{AC}^0(F_2)$.

Generalized Baumslag-Solitar groups

A generalized Baumslag-Solitar group (GBS group) is a

- fundamental group of a finite graph of groups
- with infinite cyclic vertex and edge groups.

A GBS group G is given by a graph of groups $\mathcal{G}$:

- an undirected graph (V, E) (with involution $\bar{\cdot} : E \to E$, $\iota(t)$ the initial, $\tau(t)$ the terminal vertex of $t \in E$),
- $\alpha_t, \beta_t \in \mathbb{Z} \setminus \{ 0 \}$ for $t \in E$ such that $\alpha_t = \beta_{\bar{t}}$.

A generalized Baumslag-Solitar group (GBS group) is a

- fundamental group of a finite graph of groups
- with infinite cyclic vertex and edge groups.

A GBS group G is given by a graph of groups $\mathcal{G}$:

- an undirected graph (V, E)
  (with involution $\bar{\cdot} : E \to E$, $\iota(t)$ the initial, $\tau(t)$ the terminal vertex of $t \in E$),
- $\alpha_t, \beta_t \in \mathbb{Z} \setminus \{0\}$ for $t \in E$ such that $\alpha_t = \beta_{\bar{t}}$.

$$F(\mathcal{G}) = \left\langle V, E \;\middle|\; \bar{t}t = 1, tb^{\beta_t}\bar{t} = a^{\alpha_t} \text{ for } t \in E, a = \iota(t), b = \tau(t) \right\rangle$$

A generalized Baumslag-Solitar group (GBS group) is a

- fundamental group of a finite graph of groups
- with infinite cyclic vertex and edge groups.

A GBS group G is given by a graph of groups $\mathcal{G}$:

- an undirected graph (V, E)
  (with involution $\overline{\cdot} : E \to E$, $\iota(t)$ the initial, $\tau(t)$ the terminal vertex of $t \in E$),
- $\alpha_t, \beta_t \in \mathbb{Z} \setminus \{ 0 \}$ for $t \in E$ such that $\alpha_t = \beta_{\overline{t}}$.

$$F(\mathcal{G}) = \left\langle V, E \;\middle|\; \overline{t}t = 1, tb^{\beta_t}\overline{t} = a^{\alpha_t} \text{ for } t \in E, a = \iota(t), b = \tau(t) \right\rangle$$

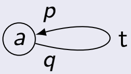
Fix a vertex $a \in V$: $G = \pi_1(\mathcal{G}, a) \leq F(\mathcal{G})$

$$G = \{ a_0 t_1 a_1 \cdots t_n a_n \mid t_i \in E, a_i = \tau(t_i) = \iota(t_{i+1}), a_0 = a_n = a \}$$
$=$ "all closed paths starting at a."

Generalized Baumslag-Solitar groups

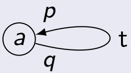
Example

$\mathbf{BS}_{p,q}$

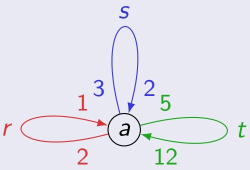


Example

$\mathbf{BS}_{p,q}$



Example



$$G = F(\mathcal{G}) = \left\langle a, r, s, t \;\middle|\; ra\overline{r} = a^2, sa^2\overline{s} = a^3, ta^{12}\overline{t} = a^5 \right\rangle$$

- Word problem
- Britton reductions
- Conjugacy for cyclically reduced words $u, v \notin \langle a \rangle$

work all as for ordinary Baumslag-Solitar groups.
$\rightsquigarrow$ everything in $\mathrm{AC}^0(F_2)$

Word and Conjugacy Problem in GBS groups

- Word problem
- Britton reductions
- Conjugacy for cyclically reduced words $u, v \notin \langle a \rangle$

work all as for ordinary Baumslag-Solitar groups.
$\rightsquigarrow$ everything in $\mathrm{AC}^0(F_2)$

- But: Conjugacy for cyclically reduced words $u, v \in \langle a \rangle$ does not work as for ordinary Baumslag-Solitar groups.

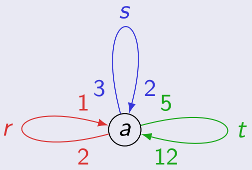
Remember:

$$a^k \sim a^\ell \text{ in } \mathbf{BS}_{p,q} \iff \exists j \in \mathbb{Z} \text{ with } k \cdot \left(\frac{q}{p}\right)^j = \ell \text{ and...}$$

Now: more than polynomially many potential conjugating elements.

Example

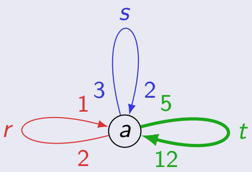
$$G = F(\mathcal{G}) = \left\langle a, r, s, t \;\middle|\; ra\overline{r} = a^2, sa^2\overline{s} = a^3, ta^{12}\overline{t} = a^5 \right\rangle$$



$$a^{15} \sim a^{16}?$$

Example

$$G = F(\mathcal{G}) = \left\langle a, r, s, t \;\middle|\; ra\overline{r} = a^2, sa^2\overline{s} = a^3, ta^{12}\overline{t} = a^5 \right\rangle$$



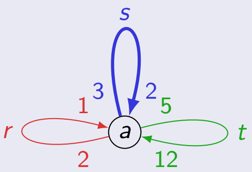
$$a^{15} \sim a^{16}?$$

$$\overline{t}\, a^{15}\, t \quad = \quad a^{36}$$

Word and Conjugacy Problem in GBS groups

Example

$$G = F(\mathcal{G}) = \left\langle a, r, s, t \;\middle|\; r a \bar{r} = a^2, s a^2 \bar{s} = a^3, t a^{12} \bar{t} = a^5 \right\rangle$$

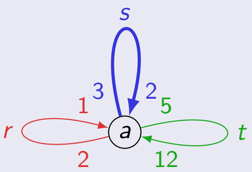


$$a^{15} \sim a^{16}?$$

$$\begin{aligned} \bar{s}\bar{t}\, a^{15}\, ts &= \bar{s}\, a^{36}\, s \\ &= a^{24} \end{aligned}$$

Example

$$G = F(\mathcal{G}) = \left\langle a, r, s, t \;\middle|\; ra\overline{r} = a^2, sa^2\overline{s} = a^3, ta^{12}\overline{t} = a^5 \right\rangle$$



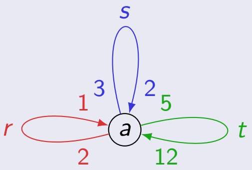
$$a^{15} \sim a^{16}?$$

$$\overline{s}\,\overline{s}\,\overline{t}\; a^{15}\; tss = \overline{s}\,\overline{s}\; a^{36}\; ss$$

$$= a^{16}$$

Example

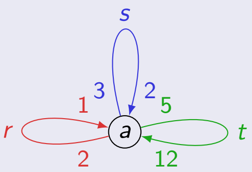
$$G = F(\mathcal{G}) = \left\langle a, r, s, t \;\middle|\; ra\overline{r} = a^2, sa^2\overline{s} = a^3, ta^{12}\overline{t} = a^5 \right\rangle$$



$a^{15} \sim a^{17}$?

Example

$$G = F(\mathcal{G}) = \left\langle a, r, s, t \;\middle|\; ra\overline{r} = a^2, sa^2\overline{s} = a^3, ta^{12}\overline{t} = a^5 \right\rangle$$

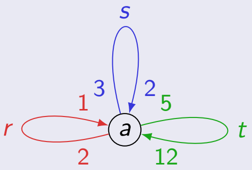


$a^{15} \sim a^{17}$?

no, cannot "create" a prime factor 17

Example

$$G = F(\mathcal{G}) = \left\langle a, r, s, t \;\middle|\; ra\overline{r} = a^2, sa^2\overline{s} = a^3, ta^{12}\overline{t} = a^5 \right\rangle$$



$a^{15} \sim a^{17}$?

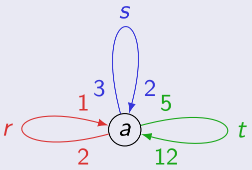
no, cannot "create" a prime factor 17

Question: $a^k \sim a^\ell$? Write $k = r_k \cdot 2^c \cdot 3^d \cdot 5^e$,

$$aa^k\overline{a} = a^k,$$

Example

$$G = F(\mathcal{G}) = \left\langle a, r, s, t \;\middle|\; ra\overline{r} = a^2, sa^2\overline{s} = a^3, ta^{12}\overline{t} = a^5 \right\rangle$$



$a^{15} \sim a^{17}$?

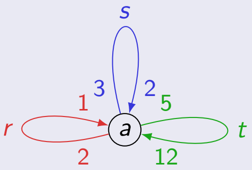
no, cannot "create" a prime factor 17

Question: $a^k \sim a^\ell$? Write $k = r_k \cdot 2^c \cdot 3^d \cdot 5^e$,

$aa^k\overline{a} = a^k$, $\qquad ra^k\overline{r} = a^{r_k \cdot 2^{c+1} \cdot 3^d \cdot 5^e}$,

Example

$$G = F(\mathcal{G}) = \left\langle a, r, s, t \;\middle|\; ra\overline{r} = a^2, sa^2\overline{s} = a^3, ta^{12}\overline{t} = a^5 \right\rangle$$



$a^{15} \sim a^{17}$?

no, cannot "create" a prime factor 17

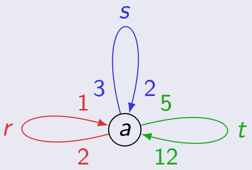
Question: $a^k \sim a^\ell$? Write $k = r_k \cdot 2^c \cdot 3^d \cdot 5^e$,

$$aa^k\overline{a} = a^k, \qquad ra^k\overline{r} = a^{r_k \cdot 2^{c+1} \cdot 3^d \cdot 5^e},$$

$$sa^k\overline{s} = a^{r_k \cdot 2^{c-1} \cdot 3^{d+1} \cdot 5^e},$$

Example

$$G = F(\mathcal{G}) = \left\langle a, r, s, t \;\middle|\; ra\overline{r} = a^2, sa^2\overline{s} = a^3, ta^{12}\overline{t} = a^5 \right\rangle$$



$a^{15} \sim a^{17}$?

no, cannot "create" a prime factor 17

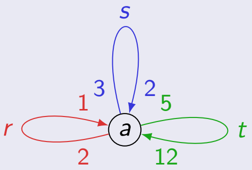
Question: $a^k \sim a^\ell$? Write $k = r_k \cdot 2^c \cdot 3^d \cdot 5^e$,

$$aa^k\overline{a} = a^k, \qquad ra^k\overline{r} = a^{r_k \cdot 2^{c+1} \cdot 3^d \cdot 5^e},$$

$$sa^k\overline{s} = a^{r_k \cdot 2^{c-1} \cdot 3^{d+1} \cdot 5^e}, \qquad ta^k\overline{t} = a^{r_k \cdot 2^{c-2} \cdot 3^{d-1} \cdot 5^{e+1}}.$$

Example

$$G = F(\mathcal{G}) = \left\langle a, r, s, t \;\middle|\; ra\overline{r} = a^2, sa^2\overline{s} = a^3, ta^{12}\overline{t} = a^5 \right\rangle$$



$a^{15} \sim a^{17}$?

no, cannot "create" a prime factor 17

Question: $a^k \sim a^\ell$? Write $k = r_k \cdot 2^c \cdot 3^d \cdot 5^e$,

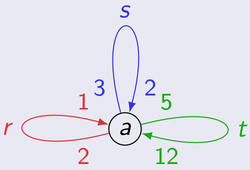
$$aa^k\overline{a} = a^k, \qquad ra^k\overline{r} = a^{r_k \cdot 2^{c+1} \cdot 3^d \cdot 5^e},$$

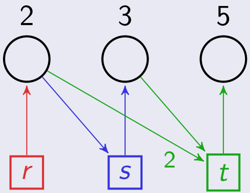
$$sa^k\overline{s} = a^{r_k \cdot 2^{c-1} \cdot 3^{d+1} \cdot 5^e}, \qquad ta^k\overline{t} = a^{r_k \cdot 2^{c-2} \cdot 3^{d-1} \cdot 5^{e+1}}.$$

$\rightsquigarrow$ suffices to consider (c, d, e).

Example

$$G = F(\mathcal{G}) = \left\langle a, r, s, t \;\middle|\; ra\overline{r} = a^2, sa^2\overline{s} = a^3, ta^{12}\overline{t} = a^5 \right\rangle$$

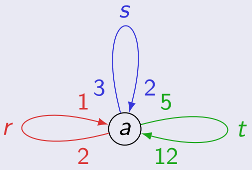


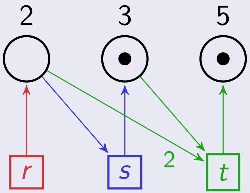


+ inverse transitions

Example

$$G = F(\mathcal{G}) = \left\langle a, r, s, t \;\middle|\; ra\overline{r} = a^2, sa^2\overline{s} = a^3, ta^{12}\overline{t} = a^5 \right\rangle$$



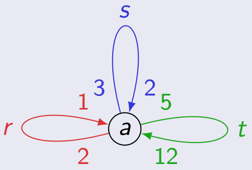


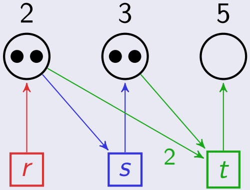
$+$ inverse transitions

a^{15} $(0, 1, 1)$

Example

$$G = F(\mathcal{G}) = \left\langle a, r, s, t \;\middle|\; ra\overline{r} = a^2, sa^2\overline{s} = a^3, ta^{12}\overline{t} = a^5 \right\rangle$$



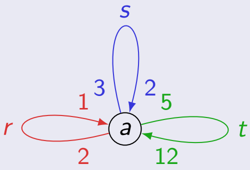


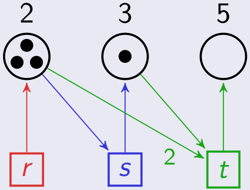
$+$ inverse transitions

a^{15}	$(0, 1, 1)$
$\overline{t}\, a^{15}\, t$	$(2, 2, 0)$

Example

$$G = F(\mathcal{G}) = \left\langle a, r, s, t \;\middle|\; ra\bar{r} = a^2, sa^2\bar{s} = a^3, ta^{12}\bar{t} = a^5 \right\rangle$$



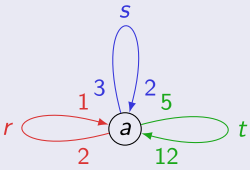


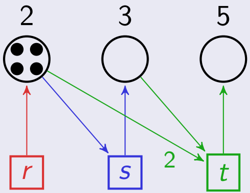
+ inverse transitions

a^{15}	$(0,1,1)$
$\bar{t}\, a^{15}\, t$	$(2,2,0)$
$\bar{s}\bar{t}\, a^{15}\, ts$	$(3,1,0)$

Example

$$G = F(\mathcal{G}) = \left\langle a, r, s, t \;\middle|\; ra\overline{r} = a^2, sa^2\overline{s} = a^3, ta^{12}\overline{t} = a^5 \right\rangle$$





$+$ inverse transitions

$$\begin{array}{rl} a^{15} & (0,1,1) \\ \overline{t}\, a^{15}\, t & (2,2,0) \\ \overline{s}\overline{t}\, a^{15}\, ts & (3,1,0) \\ a^{16} = \overline{s}\overline{s}\overline{t}\, a^{15}\, tss & (4,0,0) \end{array}$$

Word and Conjugacy Problem in GBS groups

Question: $a^k \sim a^\ell$?

Let $\mathcal{P} = \{$primes occurring in α_t, β_t for $t \in E\}$.

$$k = r_k \cdot \prod_{p \in \mathcal{P}} p^{e_p(k)}, \qquad \ell = r_\ell \cdot \prod_{p \in \mathcal{P}} p^{e_p(\ell)}.$$

If $r_k \neq r_\ell$, then $a^k \not\sim a^\ell$. Otherwise,

$$a^k \sim a^\ell \iff (e_p(k))_{p \in \mathcal{P}} \approx (e_p(\ell))_{p \in \mathcal{P}}$$

$\approx$ = congruence on $\mathbb{N}^{\mathcal{P}}$ generated by $(e_p(\alpha_t))_{p \in \mathcal{P}} \approx (e_p(\beta_t))_{p \in \mathcal{P}}$

Word and Conjugacy Problem in GBS groups

Question: $a^k \sim a^\ell$?
Let $\mathcal{P} = \{$primes occurring in α_t, β_t for $t \in E\}$.

$$k = r_k \cdot \prod_{p\in\mathcal{P}} p^{e_p(k)}, \qquad \ell = r_\ell \cdot \prod_{p\in\mathcal{P}} p^{e_p(\ell)}.$$

If $r_k \neq r_\ell$, then $a^k \not\sim a^\ell$. Otherwise,

$$a^k \sim a^\ell \iff (e_p(k))_{p\in\mathcal{P}} \approx (e_p(\ell))_{p\in\mathcal{P}}$$

$\approx$ = congruence on $\mathbb{N}^{\mathcal{P}}$ generated by $(e_p(\alpha_t))_{p\in\mathcal{P}} \approx (e_p(\beta_t))_{p\in\mathcal{P}}$

Theorem (Eilenberg, Schützenberger, 1969)

Congruences are semi-linear subsets of $\mathbb{N}^{\mathcal{P}} \times \mathbb{N}^{\mathcal{P}}$.

Theorem (Ibarra, Jiang, Chang, Ravikumar, 1991)

Membership in a semi-linear set can be testet in NC^1.

$\rightsquigarrow$ conjugacy is in $\mathsf{AC}^0(F_2)$.

Input:

- a finite graph of groups $\mathcal{G}$ consisting of
  - (V, E),
  - $\alpha_t, \beta_t \in \mathbb{Z} \setminus \{ 0 \}$ for $t \in E$ given in binary,
- two words $v, w \in \pi_1(\mathcal{G}, a)$

Question: $v \sim w$ in $\pi_1(\mathcal{G})$.

Input:

- a finite graph of groups $\mathcal{G}$ consisting of
  - (V, E),
  - $\alpha_t, \beta_t \in \mathbb{Z} \setminus \{0\}$ for $t \in E$ given in binary,
- two words $v, w \in \pi_1(\mathcal{G}, a)$

Question: $v \sim w$ in $\pi_1(\mathcal{G})$.

Theorem (W.)

The uniform conjugacy problem for GBS groups is EXPSPACE*-complete.*

Proof.

The uniform reachability problem for symmetric Petri nets
(= uniform word problem for commutative monoids)
is EXPSPACE-complete (Mayr, Meyer, 1982). □

More General

Fundamental groups of finite graphs of groups with free abelian vertex and edge groups:

Conjecture

- Word problem in LOGSPACE.
- If all edge groups have rank at most two, in $\mathsf{AC}^0(F_2)$.

Theorem (Bogopolski, Martino, Ventura, 2010)

Conjugacy problem is undecidable in general.

Theorem (Diekert, Miasnikov, W., 2015)

Conjugacy problem is strongly generically in P.

More General

Fundamental groups of finite graphs of groups with free abelian vertex and edge groups:

Conjecture

- Word problem in LOGSPACE.
- If all edge groups have rank at most two, in $\mathrm{AC}^0(F_2)$.

Theorem (Bogopolski, Martino, Ventura, 2010)

Conjugacy problem is undecidable in general.

Theorem (Diekert, Miasnikov, W., 2015)

Conjugacy problem is strongly generically in P*.*

Thank you!