Abstract

The objective of this Thesis is to study the geodesic growth of finitely generated groups.

Firstly, we study direct, free and wreath products of groups. More specifically, we give lower bounds for the minimal geodesic growth rates of abelian groups and upper bounds for the minimal geodesic growth rates of direct products of two groups. Moreover, we give the minimal geodesic growth rate of a free product of the form $C_2 * C_n$, a lower bound for the geodesic growth rate of a free product of two groups, with respect to the standard generating set, and prove that every non trivial free product whose minimal geodesic growth rate is achieved is Hopfian. Also, we study the geodesic growth rate of Lamplighter groups and give the geodesic growth rates of L_2 and L_3 with respect to the standard generating set.

Secondly, we study the geodesic growth rate of some groups acting on regular rooted trees, groups which were known or conjectured to have intermediate spherical growth. We prove, using Schreier graphs, that almost all of these groups have exponential geodesic growth. The exception is the Gupta-Fabrykowski group, for which we show that it is not feasible to prove that the geodesic growth is exponential using Schreier graphs.

Finally, we study the rationality of geodesic growth series for graph products and wreath products. We prove that the free product and direct product of two groups of rational geodesic growth have rational geodesic growth with respect to the standard generating sets. Afterwards we prove that the wreath product $A \wr G$, where A has rational geodesic growth and G is finite and acts on A, has rational geodesic growth, and that the Lamplighter groups L_2 and L_3 have rational geodesic growth. Finally, we give an example of a group which has the h-FFTP property and a non-context-free geodesic language.

Keywords: Geodesic growth, geodesic growth series, intermediate spherical growth, direct product, free product, wreath product, geodesic language, rationality, h-FFTP property.

Résumé

L'objectif de cette thèse est d'étudier la croissance géodésique de groupes finiment générés.

Premièrement, nous étudierons les produits directs, libres et produits en couronne. Plus spécifiquement, nous donnerons des bornes inférieures du taux de croissance géodésique minimal pour des groupes abéliens et des bornes supérieures du taux de croissance géodésique minimal de produits directs. De plus, nous donnerons le taux de croissance géodésique minimal d'un produit libre de la forme $C_2 * C_n$, une borne inférieure du taux de croissance géodésique d'un produit libre, par rapport à l'ensemble générateur standard, et prouverons que chaque produit libre dont le taux minimal géodésique est atteint est Hopfien. Enfin, nous étudierons plus en détails la croissance géodésique des groups de Lamplighter et donnerons le taux de croissance géodésique des groupes L_2 et L_3 par rapport à l'ensemble générateur standard.

Ensuite, nous étudierons le taux de croissance géodésique de groupes agissant sur des arbres enracinés k-réguliers, des groupes connus ou conjecturés pour avoir une croissance sphérique intermédiaire. Nous prouverons, en utilisant les graphes de Schreier, que ces groupes, à l'exception du groupe de Gupta-Fabrykowski dont la croissance géodésique est encore inconnue, ont tous une croissance géodésique exponentielle.

Enfin, nous étudierons la rationalité de la série génératrice de la croissance géodésique pour le cas des produits de graphes et les produits en couronne. Nous prouverons que le produit libre et le produit direct de deux groupes à croissance géodésique rationnelles ont tous deux une croissance géodésique exponentielle par rapport à l'ensemble générateur standard. Par la suite, nous prouverons que le produit en couronne $A \wr G$, où A a une croissance géodésique rationnelle et G est fini et agit sur A, a une croissance géodésique rationnelle. Nous démontrerons aussi que les groupes de Lamplighter L_2 et L_3 ont une croissance géodésique rationnelle par rapport à l'ensemble générateur standard. Finalement, nous donnerons un exemple de groupes qui, par rapport à des systèmes de générateurs bien précis, ont la propriété h-FFTP mais n'ont pas un langage de géodésiques réguliers.

Mots clefs: Croissance géodésique, série génératrice de croissance géodésique, croissance sphérique intermédiaire, produit direct, produit libre, produit en couronne, langage de géodésiques, rationalité, propriété h-FFTP.

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Contents

Introduction

For the last 50 years, the notion of growth in groups and its variations have been a fascinating and successful subject connecting algebra, geometry, analysis and combinatorics to give many ground-breaking and profound results.

Given a finitely generated group, the growth function counts the number of group elements in a ball of a given radius with respect to the word metric. Introduced in the 1950's in the USSR [54], and in the 1960's [44] in the USA, the notion of growth was at first motivated by its connection to volume growth of Riemannian manifolds. One of the major facts about growth is that its asymptotic behaviour is quasi-isometry invariant, and in particular does not depend on the generating set of the group. Another notion of growth which is often considered counts the number of elements on a sphere of a given radius with respect to the word metric. This has the same asymptotic behaviour as counting elements in a ball [43]. The two notions are called volume and spherical growth, respectively.

In this thesis, we study the geodesic growth of finitely generated groups. This function counts the number of geodesic paths of a given length starting from a fixed vertex in the Cayley graph of the group. Motivated by the fact that this is exactly the word growth of the language of geodesics of the group, this notion was first studied in the 1980's by Gromov and Epstein, who established that the language of geodesics of hyperbolic groups is rational, with respect to arbitrary finite generating sets [21, 31]. In contrast to the spherical growth, the asymptotic properties of geodesic growth do depend on the generating set.

Denote by $a_X : \mathbb{N} \to \mathbb{N}$ the growth of a group G with generating set X. We classify the growth into three different types, called polynomial, intermediate and exponential. Many results about growth concern these three types and, more precisely, compute the growth rate $\limsup_{n\to\infty} \sqrt[n]{a_X(n)}$ if a_X is exponential. First, in 1968, Milnor proved that a finitely generated solvable group has exponential growth unless it contains a nilpotent subgroup of finite index [45]. After generalisations of this result, Gromov proved in 1981 that a finitely generated group G has polynomial growth if and only it is virtually nilpotent [30]. This theorem, called the Gromov theorem on polynomial growth, is well known for providing an equivalence between a purely geometric property (the volume growth) and a purely algebraic property (virtual nilpotency). Moreover, its proof introduces new techniques to geometric group theory, such as the Gromov–Hausdorff convergence and asymptotic cones.

Since Gromov's theorem is one of the most important results about growth of finitely generated groups, there were many attempts to prove analogous results for other types of growth. Any group admits a generating set with respect to which the geodesic growth is exponential. Hence, for the geodesic growth, we cannot expect an analogue of Gromov's theorem that holds for all generating sets of a given group. In 2011, Bridson, Burillo, Elder and Šunić [8] proved that given a finitely generated group G, if there exists a group element $x \in G$ whose normal closure is abelian and of finite index, then there exists a finite generating set of G with respect to which the geodesic growth is polynomial. The question to specify exactly which groups have polynomial geodesic growth with respect to some generating set is still open.

A second major result concerning the asymptotic viewpoint of growth comes from another question of Milnor [11]. In 1968, he asked if the growth function of every finitely generated group is equivalent, either to a polynomial function n^d, or to the exponential function 2^n. In 1983, Grigorchuk [27] gave the first example of a group with intermediate growth, i.e. a group which has growth greater than polynomial and less than exponential: the first Grigorchuk group. Later, in 1984, Grigorchuk constructed an entire family of groups of intermediate growth [28]. Since then, many mathematicians studied these groups, trying to find the exact asymptotic behaviour of the growth. Because of that, they are often used as counterexamples to conjectures about growth. The existence of a finitely generated group of intermediate geodesic growth is still open. Since there is at least one geodesic for each element in a group, the geodesic growth is bounded below by the growth. Hence, groups of subexponential spherical growth are possible candidates for groups of intermediate geodesic growth.

The formal viewpoint of growth of groups, which concerns algebraic properties of the growth series $A(z) = \sum_{n\geq 0} a_X(n)z^n$ as a formal power series, has also been studied. The formal viewpoint of geodesic growth, which concerns not only the algebraic properties of the geodesic growth series but also formal properties of the language of geodesics, is interesting because of its links with Dehn's problems. Indeed, the geodesic growth of a group is exactly the growth of its geodesic language. This implies, for example, that if the geodesic language is recursive and the group has a recursively enumerable presentation, then it has solvable word problem [34]. As mentioned before, Gromov and Epstein established rationality for hyperbolic groups with respect to arbitrary finite generating sets based on the regularity of the geodesic language [21, 31]. For non-hyperbolic groups, there are examples of Cannon [47] which show that the regularity of the geodesic language depends completely on the choice of the generating set.

In this thesis, we study the two viewpoints of the geodesic growth of finitely generated groups. In particular, we give geodesic analogues to many results about the growth and prove that many groups of intermediate growth have exponential geodesic growth.

Notation. Let G be a group and X a finite generating set. We denote by $\gamma_X : \mathbb{N} \to \mathbb{N}$ the geodesic growth function of G with respect to X, by $\gamma(G, X) := \limsup_{n\to\infty} \sqrt[n]{\gamma_X(n)}$ the geodesic growth rate of G with respect to X and by $\gamma(G) := \inf_{G=\langle X\rangle} \gamma(G, X)$ the minimal geodesic growth rate of G, where the infimum is taken over finite generating sets. We say that the geodesic growth with respect to the generating set X is exponential if $\gamma(G, X) > 1$.

Presentation of the results

In Chapter 1 of this thesis, we give the basic definitions and notation about groups, presentations and geodesics. Then, we give the definitions of spherical and geodesic growth and basic results about growth.

In Chapter 2 of this thesis, we present results about the geodesic growth of direct products, free products and wreath products of finitely generated groups.

We first show that the geodesic growth rate is additive with respect to direct products, when considering the standard generating set.

Theorem 2.2. *Let H and K be two groups generated by finite sets X and Y, respectively. Then*

$$\gamma(H \times K, X \cup Y) \;=\; \gamma(H,X) \;+\; \gamma(K,Y).$$

Furthermore, we prove that for all $d \geq 2$, the geodesic growth of $\mathbb{Z}^d$ is exponential of rate at least d with respect to every generating set.

Theorem 2.5. *For all $d \geq 2$, $\mathbb{Z}^d$ has exponential geodesic growth with respect to every generating set. Moreover, its minimal geodesic growth rate is d and is achieved by the standard generating set.*

Secondly, we show that the geodesic growth rate of free products is strictly greater than the sum of the geodesic growth rates of the two factors, when considering the standard generating set.

Theorem 2.11. *Let H and K be two groups generated by finite sets X and Y, respectively. Then*

$$\gamma(H * K, X \cup Y) \;>\; \gamma(H,X) \;+\; \gamma(K,Y).$$

In the particular case of free products of cyclic groups, we prove that if the free product is of the form $C_2 * C_n$, where $n \neq 2$ is a power of a prime, then the minimal geodesic growth rate is given by the inverse of the least root of a polynomial with integer coefficients and is achieved with the standard generating set, adapting Talambutsa's proof of the corresponding result for volume growth [55, 56].

Theorem 2.28. *Let G be a free product of the form $C_2 * C_n$, where $n \neq 2$ is a power of a prime p, and $X = \{a, b\}$ its standard generating set. Then the minimal geodesic growth rate $\gamma(G)$ is achieved with the standard generating set X and*

$$\gamma(G) = \gamma(G,X) = \frac{1}{\alpha_n},$$

where α_n is the least positive root of the polynomial

$$\begin{array}{ll} 1 - z - 2z^2 + 2z^{\frac{n+3}{2}} & \textit{for} \quad p \neq 2 \\ 1 - z - 2z^2 + z^{\frac{n+4}{2}} & \textit{for} \quad p = 2. \end{array}$$

In Section 2.2.2, we consider the notion of geodesic entropy. Given a group G finitely generated by a finite set X, the geodesic entropy $GE(G, d_X)$ is defined by $\ln(\gamma(G,X))$. Motivated by the close analogy with the notion of growth rate in Riemannian geometry, the entropy of groups was studied in parallel to growth of groups. In particular, Sambusetti studied the entropy of free products [50, 51]. We say that G, with respect to a generating set X, is geodesic growth tight if $GE(G, d_X)$ is strictly greater than $GE(G/N, d_{\overline{X}})$ for every infinite non-trivial normal subgroup $N \triangleleft G$, where $\overline{X}$ denotes the generating set induced by X on the quotient. We prove that every non-trivial free product different from $C_2 * C_2$ whose minimal geodesic growth rate is achieved is Hopfian, adapting Sambusetti's proof of the corresponding result for spherical growth [50, 51].

Theorem 2.32. *Every non-trivial free product $G = H * K \neq C_2 * C_2$ is geodesic growth tight with respect to any generating set.*

Corollary 2.33. *Every non-trivial free product $G = H * K \neq C_2 * C_2$ whose minimal geodesic growth rate is achieved is Hopfian.*

For any $m \geq 2$, we have $\mathbb{Z} = \langle t \rangle$ and $C_m = \langle a | a^m \rangle$. We define the Lamplighter group L_m by $L_m = C_m \wr \mathbb{Z}$. It is generated by the finite set $X = \{a, t\}$. In Section 2.3, we compute the geodesic growth rates of the L_2 and L_3.

Theorem 2.43. *The geodesic growth rates of L_2 and L_3 with respect to the standard generating set $X = \{a, t\}$ satisfy $\gamma(L_2, X) = 2$ and $\gamma(L_3, X) = \frac{1+\sqrt{17}}{2}$, respectively.*

In Chapter 3, we study groups acting on regular rooted trees. In particular, we study a large family of groups, including the Grigorchuk groups, the Gupta-Sidki p-groups, the Square group and the spinal groups, which are either known or conjectured to have intermediate growth. We show that all of these groups have exponential geodesic growth.

Let $p \geq 2$ and $X = \{0, 1, 2, ..., p-1\}$. Let A, B be finite groups, where A acts faithfully and transitively on X, $|B| > |A|$ and such that the set $Epi(B, A)$ of epimorphisms from B onto A is non empty. Let $\Omega = \{\omega = (\omega_1, \omega_2, \omega_3, \ldots) \mid \omega_n \in Epi(B, A) \ \ \forall n \geq 1 \}$.

By definition, A acts faithfully on X^* as $a(x_1 x_2 \ldots x_k) = a(x_1) x_2 \ldots x_k$ for all $a \in A$, and for each $\omega \in \Omega$ fixed, the faithful action of B on X^* is defined by

$$\begin{aligned} b\Big((p-1)^{n-1}\, 0\, x_{n+1}\, x_{n+2}\, \ldots x_k\Big) &= (p-1)^{n-1}\, 0\, \omega_n(b)(x_{n+1})\, x_{n+2}\, x_{n+3}\, \ldots x_k \\ b(x) &= x \ \text{ for all words } x \text{ not starting with } (p-1)^{n-1}0 \end{aligned}$$

for all $n \geq 1$ and for all $b \in B$. For all $\omega \in \Omega$, the group $\mathcal{G}_\omega$ is defined as the subgroup of $Aut(X^*)$ generated by A and B.

Theorems 3.5, 3.8, 3.10, 3.14. *The Grigorchuk groups, the Gupta-Sidki p-groups, the Square group and the group $\mathcal{G}_\omega$ for any $\omega \in \Omega$ have exponential geodesic growth with respect to their standard generating sets.*

In Chapter 4, we give an introduction to formal grammars and formal languages [24, 25, 35, 36] and study the set of geodesics in a group from a formal language point of view. In particular, we prove that if G is a finite group which acts on a finitely generated group A of rational geodesic growth, then the wreath product $A \wr G$ has rational geodesic growth, adapting Johnson's proof of the corresponding result for growth [37].

Theorem 4.8. *Let G be a finite group which acts on a finitely generated group A of rational geodesic growth. Then $A \wr G$ has rational geodesic growth with respect to its standard generating set.*

Moreover, we use our results in Chapter 2 to show that the Lamplighter groups L_2 and L_3 have rational geodesic growth with respect to their standard generating sets $X = \{a, t\}$.

Theorem 4.9. *The groups L_2 and L_3 have rational geodesic growth with respect to their standard generating sets.*

In Section 4.3, we study the falsification by fellow traveller property and one of its generalisations. A group G with finite generating set X has the FFTP property if there is a constant k such that every non-geodesic word in G with respect to X is k-fellow travelled by a shorter word (not necessarily a geodesic). Neumann and Shapiro introduced this property in order to prove the rationality of the geodesic growth in hyperbolic groups [47]. They proved that the FFTP property implies the regularity of the geodesic language and is dependent of the generating set. Later, Antolín, Ciobanu, Elder and Hermiller were interested to extend this notion [1]. In [19], Elder gave an example of a group which has regular geodesic language but does not have the FFTP property, with respect to a certain generating set. We consider one of the generalisations of the FFTP property given in [1], the $h(n)$-FFTP property, and study the groups $\mathbb{F}_k \times \mathbb{F}_k$. It is already known that $\mathbb{F}_k \times \mathbb{F}_k$, with respect to a specific generating set, does not have a regular geodesic language [21, pp. 81 - 82]. We give other generating sets of $\mathbb{F}_k \times \mathbb{F}_k$ for which these groups have the $h(n)$-FFTP property for $h(n)$ affine. In particular, for $\mathbb{F}_2 \times \mathbb{F}_2$, it is possible to count all geodesics via an algorithm, implemented in C, which counts geodesics having a fixed normal form (c.f. Appendix B).

Theorem 4.17. *The group $\mathbb{F}_k \times \mathbb{F}_k$, with respect to the generating set*

$$X_k = \{a_1, a_2, \ldots, a_k, b_1, b_2, \ldots, b_k, (a_1b_1), (a_2b_2), \ldots, (a_kb_k)\},$$

has the $(2n)$-FFTP property for all $k \geq 2$.

Since all regular languages are context-free, the following result implies that $\mathbb{F}_k \times \mathbb{F}_k$, with respect to X_k, do not have the FFTP property.

Theorem 4.19. *For all $k \geq 2$, the language of geodesics of $\mathbb{F}_k \times \mathbb{F}_k$ with respect to the generating set X_k is not context-free.*

Finally, in Appendix A we give a list of open questions and conjectures, listed by topic, presenting general directions for extending the results in this thesis.

Chapter 1

Preliminaries

In this first chapter we give the basic definitions and notation about groups, presentations and geodesics. We define the spherical growth, the geodesic growth and the types of growth. We give many results on spherical growth of direct products and free products, and give more details on Gromov's theorem and the first Grigorchuk group.

Since there are many connections and analogies that can be made between the spherical and geodesic growth, we start this work by giving a rather lengthy introduction to spherical growth, before delving into geodesic growth.

1.1 Length, geodesics, word metric and presentations

We begin by introducing notation and definitions. Let A be a finite set, called an alphabet, and let A^{-1} be the set of formal inverses of elements in A. Let A^* be the set of words over the alphabet A, that is the set of finite sequences of (positive) letters of A. On A^*, we use $\equiv$ to denote the equality of words and $|\,.\,|_A$ the word length. If w_1, w_2 are two words over A, we denote by $w_1 w_2$ the word obtained by concatenating w_1 and w_2.

Let $G = \langle X \rangle$ be a group generated by a finite set X. Our convention is that the elements of G are to be seen as words over $X \cup X^{-1}$. It is customary in the literature to use the set $A = X \cup X^{-1}$ as a symmetric generating set, but we will only rarely use symmetric generating sets, and then will make it explicit that they are symmetric.

We denote by $\pi_X : (X \cup X^{-1})^* \to G$ the natural projection. For an element g of G, the word length of g with respect to X is given by

$$l_X(g) \;:=\; \min\left\{ n \in N \mid g =_G \pi_X(x_1 \ldots x_n) \text{ for some } x_i \in X \cup X^{-1} \right\}.$$

Then the word $x_1\, x_2 \ldots x_n$ represents the element g and if n is minimal (i.e. $n = l_X(g)$), then this word is called a geodesic representing g.

On G, for any generating set X there is a metric, called the word metric and denoted by d_X which is defined as follows : for all $g, h \in G$, the distance $d_X : G \times G \to \mathbb{R}_{\geq 0}$ between g and h is defined by

$$d_X(g,h) \;:=\; l_X(g^{-1}h).$$

With this distance function, (G, d_X) is a metric space for any generating set X of G.

The rank of G, denoted by $rank(G)$, is given by $\min_{G=\langle X \rangle} |X|$. If it is realized by a generating set X, then X is called a basis of G. A generating set X of G is minimal if $X \setminus \{x\}$ is not a generating set of G for any $x \in X$.

Let X be a generating set of G and $\mathbb{F}_X$ be the free group generated by X. Let R be a set of words on X and $\ll R \gg$ be the normal closure of R, that is, the minimal normal subgroup of $\mathbb{F}_X$ containing R. We say that $\langle X|R\rangle$ is a presentation of G if G is isomorphic to the quotient $\mathbb{F}_X/\ll R \gg$. The elements of R are called the relators. R is said to be minimal if $\ll R\setminus\{r\}\gg\neq\ll R\gg$ for each $r\in R$. A presentation $\langle X|R\rangle$ of G is minimal if X and R are both minimal. Each basis of G is a minimal generating set of G.

For many families of groups, there are "special" presentations called standard presentations. Generally minimal, these presentations are often the ones first used to define a group. In the following examples, we give many standard presentations that we use in this thesis.

Example 1.1 (Standard presentations)**.**

1. The standard presentation of finite cyclic group is given by

$$C_n \quad = \quad \langle\, a \mid a^n = 1 \,\rangle\,.$$

   This is a minimal presentation and the generating set is a basis.

2. The standard presentation of the free abelian group of rank n is given by

$$\mathbb{Z}^n \quad = \quad \langle\, a_1, a_2, ..., a_n \mid [a_i,\, a_j] = 1 \;\; \forall\, i \neq j \,\rangle\,.$$

   This is a minimal presentation and the generating set is a basis.

3. The standard presentation of the free group of rank n is given by

$$\mathbb{F}_n \quad = \quad \langle\, a_1, ..., a_n \mid - \,\rangle\,.$$

   This is a minimal presentation and the generating set is a basis.

4. Let $H = \langle X \mid R\rangle$ and $K = \langle Y \mid S\rangle$ be two groups and assume $X\cap Y = \emptyset$. The standard presentations of the direct product $H\times K$ and the free product $H * K$ are given respectively by

$$H\times K \quad = \quad \langle\, X\cup Y \mid R\cup S\cup\{[x_i,\, y_j]\} \;\; \forall x_i\in X,\; y_j\in Y\}\,\rangle$$

   and

$$H * K \quad = \quad \langle\, X\cup Y \mid R\cup S \,\rangle\,.$$

   They are minimal if $\langle X \mid R\rangle$ and $\langle Y \mid S\rangle$ are both minimal.

1.2 Spherical growth

In this section, we are interested in results on spherical growth. The notation used follows the book *How groups grow* of A. Mann [43].

Let G be a group and X a finite generating set. We define the spherical growth function $a_X : \mathbb{N}\to\mathbb{N}$ by

$$a_X(n) \quad := \quad |\,\{\, g\in G \,|\, l_X(g) = n \,\}\,|\,.$$

Let $\mathcal{C}ay(G, X)$ be the Cayley graph of G with respect to X constructed as:

1. The set of vertices V of $\mathcal{C}ay(G, X)$ is identified with G, that is, there is a bijection between V and G;

2. There is an edge between g, $h \in V$ if and only if $g = hx$ for some $x \in X$.

We note that there is a metric on $\mathcal{C}ay(G, X)$ induced by the word metric d_X, and that the function a_X counts the number of elements in the sphere of radius n centred at the vertex 1_G, for all $n \geq 0$.

Note that by writing a word of length $m + n$ in G as a concatenation of one word of length m and one of length n, we get that $a_X(m+n) \leq a_X(m) \cdot a_X(n)$. Then, by Fekete's Lemma [26], the sequence $(\sqrt[n]{a_X(n)})_{n \geq 0}$ is decreasing and since $a_X(n) \geq 0$ for all n, the limit

$$\omega(G, X) \;\; := \;\; \lim_{n \to \infty} \sqrt[n]{a_X(n)}$$

exists and is finite. This value $\omega(G, X)$ is called the spherical growth rate of G (or exponential spherical growth rate) and is between 1 and $2|X| - 1$.

Since $\omega(G, X)$ depends on X, we denote by

$$\Omega(G) \;\; := \;\; \inf_{G = \langle X \rangle} \; \omega(G, X)$$

the minimal spherical growth rate of G. Denote that $\Omega(G)$ doesn't depend on X. This rate is said to be realised (or achieved) if there is a generating set X_0 of G such that $\Omega(G) = \omega(G, X_0)$.

Since $(a_X(n))_{n \geq 0}$ can be seen as an integer sequence, let $A_X : \mathbb{C} \to \mathbb{C}$ be the generating function of the spherical growth (or the spherical growth series) of $(a_X(n))_{n \geq 0}$, defined by

$$A_X(z) \;\; = \;\; \sum_{n \geq 0} a_X(n) \, z^n.$$

The radius of convergence of A_X is given by $\frac{1}{\omega(G,X)}$.

Many groups (for example $\mathbb{F}_k$ or $\mathbb{Z}^k$) realise their minimal spherical growth rate. Other groups, such as the free product of the Baumslag-Solitar group with a cyclic group defined by

$$\mathbb{B}(2,3) \, * \, C_2 \;\; = \;\; \left\langle \, a, b, c \, | \, a^{-1} b^2 a b^{-3} = 1, \, c^2 = 1 \, \right\rangle,$$

do not realise the minimal spherical growth rate for any generating set [50]. We discuss this example in Chapter 2.

These examples prompt the following questions: what is the minimal spherical growth rate of a direct product $H \times K$ and of a free product $H * K$ of two finitely generated groups $H = \langle X \rangle$ and $K = \langle Y \rangle$?

In [43, pp. 4 - 22], it is proved, only using strong properties of these two products and manipulations of complex series, that

$$\Omega(H \times K) \;\; = \;\; \max \{ \Omega(H) \; , \; \Omega(K) \}$$

and

$$\omega(H * K, X \cup Y) \;\; \geq \;\; \omega(H, X) \; + \; \omega(K, Y).$$

Another question about spherical growth is to determine the type of growth. In fact, there are only three possible types of spherical growth:

1. G has exponential spherical growth with respect to X if $\omega(G, X) > 1$.

2. G has polynomial spherical growth with respect to X if there exist numbers c, s such that $a_X(n) \leq cn^s$ for all n. Its degree is thus defined by
$$d(G) \quad := \quad \inf \{ \, s \, | \, \exists \, c, s \text{ such that } a_X(n) \leq cn^s \} \, .$$

3. G has intermediate spherical growth with respect to X if its spherical growth function a_X is neither exponential nor polynomial.

We say that a group G has subexponential spherical growth with respect to X if $\omega(G, X) = 1$. Clearly, groups of polynomial or intermediate spherical growth have subexponential spherical growth.

It is a classical result (see [43, p. 19]), that the type of growth of G, i.e. exponential, intermediate or polynomial, does not depend on the choice of generators. More precisely, if the growth is polynomial, then the degree of the polynomial does not depend on the generators. It implies that the exact value of $\omega(G, X)$ depends on X, but that the fact that whether $\omega(G, X)$ is equal to 1 or not does not depend on X.

Examples 1.2.

1. If $G = \mathbb{Z}^d = \langle a_1, ..., a_d | a_i a_j = a_j a_i \, \forall i \neq j \rangle$ then it has polynomial spherical growth of degree $d - 1$.

2. If G is the free product of $2d$ groups $H_i = \langle x_i | x_i^2 = 1 \rangle$, $1 \leq i \leq d$, and G is provided with its standard generating set $X = \{x_1, x_2, ..., x_{2d}\}$, then each element of G is represented by a unique word of the form $x_{i_1} \, x_{i_2} \ldots x_{i_r}$, where $x_{i_k} \neq x_{i_{k+1}}$ for all $k = 1, ..., r - 1$. It implies that $a_X(n) = 2d(2d - 1)^{n-1}$, thus $\omega(G, X) = 2d - 1$. In particular, the infinite dihedral group $D_\infty = C_2 * C_2$ has polynomial growth of degree 0.

3. If G is the free product of $d > 0$ copies of $\mathbb{Z} = \langle a \, | \, \_ \rangle$ and of $e > 0$ copies of $C_2 = \langle b \, | \, b^2 = 1 \rangle$ and G has standard generating set X, then $a_X(n) = k(k - 1)^{n-1}$ where $k = 2d + e$. It implies that $\omega(G, X) = 2d + e - 1$ and G has exponential spherical growth.

4. If $G = \langle X | R \rangle$ has a hyperbolic Cayley graph, it is called a hyperbolic group. Koubi proved in [38] that if G is hyperbolic, then there is a constant $c_G > 1$ depending on G such that every non-trivial subgroup H of G of finite index satisfies $\Omega(H) \geq c_G$.

A major theorem about spherical growth was given by M. Gromov in 1981 [30]. In this theorem, he characterized all the groups of polynomial spherical growth.

Theorem 1.3 (Gromov, [30])**.** *A finitely generated group has polynomial spherical growth if and only if it contains a nilpotent subgroup of finite index.*

In 1968, J. Wolf proved that all the virtually nilpotent groups have polynomial spherical growth [57]. The reciprocal was proved by Gromov [30], and in 2010 further proofs employing distinct approaches were given by B. Kleiner, T.Tao and Y.Shalom [13, 52].

One other important result on spherical growth is about intermediate spherical growth. In 1968, J. Milnor posed the question whether there are finitely generated groups of intermediate spherical growth. In 1980, R. Grigorchuk built the first example of a group of intermediate spherical growth. This group, called the First Grigorchuk group and denoted by $\mathcal{G}$, is finitely generated but not finitely presented [42].

In 1985, Lysenok defined in [42] the standard presentation of $\mathcal{G}$ by
$$\mathcal{G} \;=\; \left\langle \, a, b, c, d \, | \, a^2 = b^2 = c^2 = d^2 = bcd = w_n^4 = (w_n w_{n+1})^4 = 1 \; \forall \, n \geq 0 \, \right\rangle ,$$

where $w_n \in \mathbb{F}_{\{a,b,c,d\}}$ are recursively given by

$$w_0 = ad, \qquad w_{n+1} = \sigma(w_n)\ \forall\, n \geq 0$$

and $\sigma : \mathbb{F}_{\{a,b,c,d\}} \to \mathbb{F}_{\{a,b,c,d\}}$ is a homomorphism where $\sigma(a) = aca$, $\sigma(b) = d$, $\sigma(c) = b$ and $\sigma(d) = c$.

This generating set is not minimal because $\mathcal{G} = \langle a, b, c\rangle$, but it is often used as the standard generating set since it gives nice properties about geodesics.

In 1984, Grigorchuk proved that $\mathcal{G}$ has intermediate spherical growth [28]. More precisely, he proved the following theorem.

Theorem 1.4 (Grigorchuk's bounds, [28])**.** *There exist two constants $c_1, c_2 > 0$ such that $\mathcal{G}$ has its spherical growth bounded by*

$$c_1\, e^{\sqrt{n}} \;\leq\; a_{\{a,b,c,d\}}(n) \;\leq\; c_2\, e^{n^{\log_{32}(31)}}$$

for all n big enough.

Since 1985, many results improving the intermediate growth bounds for $\mathcal{G}$ have been found. For example, L. Bartholdi proved in 2000 that there exists a constant $c_2 > 0$ such that

$$a_{\{a,b,c,d,\}}(n) \;\leq\; c_2\, e^{n^{\alpha}},$$

where $\alpha = \frac{\log(2)}{\log\left(\frac{2}{\eta}\right)} \simeq 0.767...$, η is the unique real root of $x^3 + x^2 + x - 2$ [3].

J. Brieussel, on the other hand, proved in 2008 that there exists a constant $c_1 > 0$ such that

$$c_1\, e^{n^{0.5207}} \;\leq\; a_{\{a,b,c,d\}}(n)$$

for all n big enough [9].

Since 1980, Grigorchuk and other mathematicians have found new groups of intermediate spherical growth. In Chapter 3 we prove that many of these groups have exponential geodesic growth.

1.3 Geodesic growth

Let $G = \langle X \mid R\rangle$ be a finitely generated group. We define the geodesic growth function $\gamma_X : \mathbb{N} \to \mathbb{N}$ with respect to X by

$$\gamma_X(n) \;:=\; \left|\, \left\{ w \in (X \cup X^{-1})^* \;:\; |w|_{X\cup X^{-1}} \;=\; l_X(\pi_X(w)) \;=\; n \right\} \right|.$$

By definition, this function counts the number of geodesics representing an element in the sphere of radius n centered in 1_G in the Cayley graph $\mathcal{C}ay(G, X)$ for $n \geq 0$.

In the same way as for the spherical growth a_X, we define the generating function of the geodesic growth with respect to X (or the geodesic growth series with respect to X) of the sequence $(\gamma_X(n))_{n\geq 0}$ by $\Gamma_X : \mathbb{C} \to \mathbb{C}$, where

$$\Gamma_X(z) \;=\; \sum_{n=0}^{\infty} \gamma_X(n) \cdot z^n.$$

The geodesic growth rate with respect to X is then defined by

$$\gamma(G,X) \;:=\; \limsup_{n\to\infty} \sqrt[n]{\gamma_X(n)}$$

and the minimal geodesic growth rate by

$$\gamma(G) \;=\; \inf_{G=\langle X\rangle} \gamma(G,X).$$

Notice that the radius of convergence of $\Gamma_X(z)$ is given by $\frac{1}{\gamma(G,X)}$. Furthermore, as there is at least one (not necessarily unique) geodesic which represents each element in the sphere of radius n centered in 1_G, $a_X(n) \leq \gamma_X(n)$ for all $n \geq 0$.

There are many groups verifying $a_X(n) = \gamma_X(n)$ for all $n \geq 0$. A basic example is the group $\mathbb{Z}$ with respect to the standard generating set X.

Example 1.5. Let $\mathbb{Z}$ finitely generated by the standard generating set X. We have $\gamma_X(0) = a_X(0) = 1$ and $\gamma_X(n) = a_X(n) = 2$ for all $n \geq 1$, so $\gamma(G,X) = \omega(G,X) = 1$.

The inequality $a_X(n) \leq \gamma_X(n)$ implies that

$$1 \;\leq\; \omega(G,X) \;\leq\; \gamma(G,X) \;\leq\; \gamma(\mathbb{F}_X,X) \;=\; 2|X| - 1. \tag{1.1}$$

From these inequalities and from the fact that all prefixes of a geodesic are geodesics, many results found for the spherical growth have an analogue for geodesic growth.

Fact 1. Let G be a group. Then $\gamma(G,X) = 0$ for a particular presentation $\langle X \mid R\rangle$ if and only if G is finite.
Furthermore, if G is finite, then $\gamma(G,X) = 0$ for all presentations $\langle X \mid R\rangle$ of G.

Fact 2. Let $G = \langle X \mid R\rangle$ be a group. The geodesic growth, seen as an integer sequence, is submultiplicative. That is, $\gamma_X(m+n) \leq \gamma_X(n)\cdot\gamma_X(m)$ for all $m,n \geq 0$.

The second fact implies that the limit $\lim_{n\to\infty} \sqrt[n]{\gamma_X(n)}$ exists and is bounded between 1 and $2|X|-1$. Then, by Fekete's Lemma [26],

$$\gamma(G,X) \;=\; \lim_{n\to\infty} \sqrt[n]{\gamma_X(n)}.$$

Notice that in many articles, the geodesic growth function is cumulative and defined as

$$\tilde{\gamma}_X(n) \;=\; \Big|\Big\{ w \in (X\cup X^{-1})^* \;:\; |w|_{X\cup X^{-1}} \;=\; l_X(\pi_X(w)) \;\leq\; n \Big\}\Big|.$$

Similarly, we can define $\tilde{\Gamma}_X(z)$, $\tilde{\gamma}(G,X)$ and $\tilde{\gamma}(G)$.

Proposition 1.6. *If G is a group generated by a finite set X 968such that the geodesic growth is exponential, then*

$$\gamma(G,X) \;=\; \tilde{\gamma}(G,X).$$

Proof. By definition, we have that

$$1 \;<\; \gamma(G,X) \;\leq\; \tilde{\gamma}(G,X) \tag{1.2}$$

and

$$\Gamma_X(z) \;=\; (1-z)\,\tilde{\Gamma}_X(z). \tag{1.3}$$

By (1.2), we know that the radius of convergence of $\Gamma_X(z)$ is at least the radius of convergence of $\tilde{\Gamma}_X(z)$ and they are both strictly smaller than 1. Then, by (1.3), the two radii are equal and the result is proved. □

This proposition implies that it is equivalent to study γ_X or $\tilde{\gamma}_X$. Thus, we interchange between the two, depending of the context.

Proposition 1.7. *Let $G = \langle X \mid R \rangle$ be a finitely generated group with geodesic growth rate $\alpha \geq 0$. Then we have*

- *For all $n > 0$, $\gamma_X(n) \geq \alpha^n$;*
- *$\gamma_X(n) = \alpha^n$ for all $n > 0$ if and only if G is trivial and $\alpha = 0$.*

Proof. The first assertion is a direct implication of the fact that if a sequence converges, then every subsequence converges to the same limit. Suppose there is $n_0 \geq 1$ such that $\gamma_X(n_0) < \alpha^{n_0}$. Then for all $m \geq 1$

$$\sqrt[n_0 * m]{\gamma_X(n_0 * m)} \quad \leq \quad \sqrt[n_0]{\gamma_X(n_0)} \quad < \quad \alpha,$$

which gives a contradiction for Fekete's Lemma [26]. Furthermore, if G is trivial then $\gamma_X(n) = 0$ for all $n > 0$.

Then we need to prove that if $\alpha > 0$, then $\gamma_X(n) \neq \alpha^n$ for all $n > 0$.
Suppose that there exists a number $\alpha > 0$ and a non-trivial, finitely generated, group $G = \langle X \rangle$ such that $\gamma_X(n) = \alpha^n$ for all $n > 0$. In particular, $\gamma_X(1) = \alpha$. It implies that $\alpha = 2|X|$. But we know from the sequence of inequalities (1.1) that if X is a generating set of G, then

$$\gamma_X \quad \leq \quad 2|X| - 1 \quad = \quad \alpha - 1 \quad = \quad \lim_{n\to\infty} \sqrt[n]{\gamma_X(n)} - 1 \quad = \quad \gamma_X \ - \ 1.$$

□

Similar to the case of the spherical growth, we can define three types of geodesic growth:

1. G has <u>exponential geodesic growth with respect to X</u> if $\gamma(G, X) > 1$.
2. G has <u>polynomial geodesic growth with respect to X</u> if there exist numbers c, s such that $\gamma_X(n) \leq cn^s$ for all n. Its <u>degree</u> is thus defined by

$$d(G) \quad := \quad \inf \{\, s \mid \exists\, c, s \text{ such that } \gamma_X(n) \leq cn^s \} \,.$$

3. G has <u>intermediate geodesic growth with respect to X</u> if its geodesic growth w.r.t X is neither exponential nor polynomial.

We say that a group G has <u>subexponential geodesic growth</u> if $\gamma(G, X) = 1$. Clearly, groups of polynomial or intermediate geodesic growth have subexponential geodesic growth.

We note that in these definitions we took into account the generating set X. In fact, compared to the spherical growth [43, p. 9], geodesic growth depends not only on G, but also on the choice of generators. Also note that the existence of a group with intermediate geodesic growth is unknown.

Example 1.8. Let $G = \mathbb{Z} \times C_2 = \langle t, a \mid a^2 = 1,\, at = ta \rangle$. Then $\gamma_{\{a,t\}}(n) = 2n + 2$ for all $n > 0$, which is a polynomial of degree 1. If G is presented by $\langle t, c \mid c^2 = t^2,\, ct = tc \rangle$, where $c = at$, then $\gamma_{\{c,t\}}(n) \geq 2^n$ for all $n \geq 0$, which is exponential of rate (at least) 2.

From this example, the question of minimality of the geodesic growth rate is interesting to study. A first observation is the following.

Corollary 1.9. *Let G be a group. If there is a generating set X of G such that $\gamma_X(n)$ is polynomial, then G is virtually nilpotent.*

Proof. If $\gamma_X(n)$ is polynomial, then $a_X(n)$ is polynomial too. Then by Theorem 1.3, G is virtually nilpotent. □

In [8], Bridson, Burillo, Elder and Šunić provided a partial converse to Corollary 1.9.

Theorem 1.10 ([8])**.** *Let G be a finitely generated group. If there exists an element $g \in G$ whose normal closure is abelian and of finite index, then there exists a finite generating set for G with respect to which the geodesic growth of G is polynomial.*

Furthermore, since we do not know if one can obtain upper and lower bounds of the same polynomial degree for Theorem 1.10, Bridson, Burillo, Elder and Šunić gave in the same article a result in the case of virtually cyclic groups.

Theorem 1.11 (Bridson, Burillo, Elder and Šunić, [8])**.** *Let G be a virtually cyclic group generated by a finite symmetric set X. The geodesic growth function γ_X is either bounded above and below by an exponential function, or else is bounded above and below by polynomials of the same degree.*

To prove this theorem, they used the following lemma on groups epimorphisms.

Lemma 1.12 (Bridson, Burillo, Elder and Šunić, [8])**.** *Let $G = \langle X \rangle$ be a group with a finite symmetric generating set X. Let $\phi : G \to G'$ be an epimorphism of groups and take $X' = \phi(X)$ as a generating set for G'. The geodesic growth functions of G and G' satisfy the following inequality: for $n \geq 0$,*

$$\gamma_{G,X}(n) \quad \geq \quad \gamma_{G',X'}(n).$$

An interesting example of an epimorphism is conjugation by a fixed element. As this is an automorphism, the geodesic growth of G doesn't change in this case.

Other interesting epimorphisms are the four Tietze Transformations, called R_+, X_+, R_-, X_- which can be seen as presentation transformations. For any group G with presentation $\langle X|R\rangle$, R_+, X_+, R_- and X_- are defined as follows:

I) Adding a relator: Let $r \in \overline{R} \setminus R$ fixed. Then

$$R_+ : \quad \langle X \,|\, R\rangle \;\to\; \langle X \,|\, R \cup \{r\}\,\rangle$$

II) Adding a generator: Let $y \notin X \cup X^{-1}$, $w \in F(X)$ fixed. Then

$$X_+ : \quad \langle X \,|\, R\rangle \;\to\; \langle X \cup \{y\} \,|\, R \cup \{y^{-1}w\}\,\rangle$$

III) Removing a relator: Let $r \in R \cap \overline{R \setminus \{r\}}$ fixed. Then

$$R_- : \quad \langle X \,|\, R\rangle \;\to\; \langle X \,|\, R \setminus \{r\}\,\rangle$$

IV) Removing a generator: Let $y \in X$ and $w \in < X \setminus \{y\} >$. Then

$$X_- : \quad \langle X \,|\, R\rangle \;\to\; \langle X \setminus \{y\} \,|\, R \setminus \{y^{-1}w\}\,\rangle$$

If we apply Lemma 1.12 to the Tietze Transformations, we have

Proposition 1.13. *Let G be a group with a presentation $P =< X|R >$. Then*

$$\gamma_{X_-(P)}(n) \;\leq\; \gamma_P(n) \;=\; \gamma_{R_-(P)}(n) \;=\; \gamma_{R_+(P)}(n) \;\leq\; \gamma_{X_+(P)}(n)$$

for all $n \geq 0$. In other words, the geodesic growth decreases if we apply X_- to P, does not change if we apply R_- or R_+ and increases if we apply X_+.

Proof. The proof is in three steps:

If we apply R_- or R_+, the equalities

$$\gamma_P(n) \quad = \quad \gamma_{R_-(P)}(n) \quad = \quad \gamma_{R_+(P)}(n)$$

are trivial since these two transformations don't change the geodesics.

If we apply X_-:
Let $\langle S \,|\, R \rangle$ be a presentation of G and fix $S = \{x_1, ..., x_n, y\}$. Let $\phi : S \to G$ be the set morphism defined by $x_i \mapsto x_i$ for all $1 \le i \le n$ and $y \mapsto 1_G$. Then X_- is exactly the morphism from $\langle S \,|\, R \rangle$ to $\langle S \setminus \{y\} \,|\, R \setminus \{y^{-1}w\} \rangle$ defined as an extension of ϕ. By Lemma 1.12 we are done.

If we apply X_+:
Let $\langle S \,|\, R \rangle$ be a presentation of G. Denote by $\langle S, \{y\} \,|\, R \cup \{y^{-1}w\} \rangle$ the presentation of G after we apply X_+, which means adding y. As this morphism is the inverse of the Tietze transformation X_- from $\langle S, \{y\} \,|\, R \cup \{y^{-1}w\} \rangle$ to $\langle S \,|\, R \rangle$ where we delete y, then by the last point we are done. □

Proposition 1.13 implies that the geodesic growth depends on the generating set only and not on the relators. But a classical result states that two finite presentations define the same group if and only if there is a finite sequence of Tietze Transformations which goes from one to the other [41, p. 91]. This implies that from a presentation $\langle X|R \rangle$ of a group G, there is a minimal presentation $\langle M \,|\, T \rangle$ of G verifying that $M \subseteq X$, $T \subseteq R$ and $\gamma_M(n) \;\le\; \gamma_X(n)$ for all $n \ge 0$. Searching for the minimal geodesic growth rate could then be restricted to the minimal presentations:

$$\begin{aligned}\gamma(G) \quad &= \quad \inf_{\langle X \rangle = G} \gamma(G, X) \\ &= \quad \inf_{\langle X \rangle = G;\, X minimal} \gamma(G, X).\end{aligned}$$

Since a basis of G is minimal, the natural question would be to know whether the minimal geodesic growth rate is obtained on a basis. Yet the problem is that changing from a minimal generating set to another involves applying X_+ and X_- and they have opposite effects on the geodesic growth.

To finish this chapter, we focus on finite cyclic groups. Indeed, there are several points of views on geodesics of finite cyclic groups C_n. In this thesis, we consider two different cases: if $n \ge 3$ and $n = 2$.

If $n \ge 3$ is even and $C_n = \langle a \,|\, a^n = 1 \rangle$, then $n = 2k$ and $a^k = a^{-k}$. It implies that there are exactly two geodesics of length m for all $1 \le m \le k$ in C_n. Since C_{n+1} with standard presentation $\langle a \,|\, a^{n+1} = 1 \rangle$ has unique geodesics, there are exactly two geodesics of length m for all $1 \le m \le k$ in C_{n+1}. This implies:

Proposition 1.14. *For all $k \ge 2$, we have*

$$\Gamma_{C_{2k}, \langle a \rangle}(z) \quad = \quad \Gamma_{C_{2k+1}, \langle a \rangle}(z) \quad = \quad A_{C_{2k+1}, \langle a \rangle}(z) \quad = \quad 1 + \sum_{i=1}^{k} 2z^i$$

and, in particular,

$$\Gamma_{C_3 = \langle a | a^3 = 1 \rangle}(z) \quad = \quad 1 + 2z.$$

The case of $C_2 = \langle a \,|a^2 = 1\rangle$ is seen differently. Since $a = a^{-1}$ is not only a double path but a double edge in the Cayley graph of C_2, we could count it twice or count it once. If we count it twice then many groups, like the First Grigorchuk group or many free products, would easily have exponential geodesic growth. Because of these implications on growth, we consider in this thesis that C_2, with standard generating set, has geodesic growth given by the generating function $\Gamma_{C_2}(z) \ = 1 \ + \ z$, i.e. that we count only one time the double edge from 1 to a in the Cayley graph.

Chapter 2

Minimal growth rate in products of groups

Let H and K be two groups generated by finite sets X and Y, respectively. We always assume $X \cap Y = \emptyset$. The following results for the spherical growth of direct and free products are well-known (see Mann [43]).

$$\Omega(H \times K) \;=\; \max\{\Omega(H)\,,\;\Omega(K)\}$$

and

$$\omega(H * K,\, X \cup Y) \;\geq\; \omega(H, X) \;+\; \omega(K, Y).$$

In [39], Loeffler, Meier and Worthington computed the generating function of the geodesic growth of these two products with respect to their standard generating sets. We use this proposition to study the attainability of the geodesic growth rate and the minimal geodesic growth rate of these two products.

At the end of the chapter, we study the wreath product of two groups, and focus on the Lamplighter groups L_m, with $m \geq 2$. We give the geodesic growth rate of L_2 and L_3 with respect to their standard generating sets and give a conjecture for the geodesic growth rate of L_m, $m \geq 2$.

2.1 Direct product

Let X be an alphabet and w_1, w_2 be two words on X. The shuffle product of w_1 and w_2, denoted by $w_1 \sqcup\!\sqcup\, w_2$, is a formal sum over the $\binom{n+m}{n}$ ways of interleaving the two words w_1 and w_2. For example,

$$\begin{aligned} ab \sqcup\!\sqcup\, xy &= abxy + axby + xaby + axyb + xayb + xyab \\ aaa \sqcup\!\sqcup\, aa &= 10\, aaaaa. \end{aligned}$$

Introduced in 1953 by Eilenberg and Mac Lane, this product has many interesting properties. For example, it is commutative and associative [16, 40].

Let H and K be two groups finitely generated by X and Y, respectively. Let $(a_n)_{n\geq 0}$ and $(b_n)_{n\geq 0}$ be the integer sequences defined by $a_n = \gamma_X(n)$ and $b_n = \gamma_Y(n)$ for all $n \geq 0$, where γ_X, γ_Y are the geodesic growth functions of H and K, respectively. The sequence $(c_n)_{n\geq 0}$ defined by

$$c_n \;=\; \sum_{k=0}^{n} \binom{n}{k} a_k b_{n-k}$$

for all $n \geq 0$ counts all the possibilities of shuffling geodesic words of length k on X and geodesic words of length $n-k$ on Y for all k. With an abuse of language, this product is called the shuffle product of the two sequences $(a_n)_{n\geq 0}$ and $(b_n)_{n\geq 0}$.

Loeffler, Meier and Worthington proved the following proposition.

Proposition 2.1 (Loeffler, Meier and Worthington, [39, p. 753])**.** *Let H and K be two groups generated by finite sets X and Y, respectively. Then the geodesic growth series for the direct product $H \times K$, with respect to the generating set $X \cup Y$, is given by*

$$\Gamma_{H\times K}(z) \;=\; \sum_{n=0}^{\infty} c_n z^n, \qquad \textit{where} \quad c_n = \sum_{k=0}^{n} \binom{n}{k} a_k b_{n-k}.$$

Proof. In the direct product, all geodesic words of length n are obtained by taking a geodesic word of length $k \leq n$ in H and a geodesic word of length $(n-k)$ in K and then shuffling them together in all possible sequences. That is, in the combined word of length n there is complete freedom in choosing the k places for the letters from the word in H. We have then the formula

$$c_n = \sum_{k=0}^{n} \binom{n}{k} a_k b_{n-k}$$

and have completed the proof. □

Another property of the shuffle product implies the following theorem.

Theorem 2.2 (Nicaud, [48])**.** *Let H and K be two groups generated by finite sets X and Y, respectively. Then*

$$\gamma(H \times K, X \cup Y) \;=\; \gamma(H, X) \;+\; \gamma(K, Y).$$

Proof. Let $\Gamma_X(z) = \sum_{n=0}^{\infty} a_n z^n$ and $\Gamma_Y(z) = \sum_{n=0}^{\infty} b_n z^n$ be the geodesic growth series of H and K with respect to the generating sets X and Y, respectively. Denote $\alpha := \lim_{n\to\infty} \sqrt[n]{a_n}$ and $\beta := \lim_{n\to\infty} \sqrt[n]{b_n}$.

Let $(c_n)_{n\geq 0}$ be the sequence defined by

$$c_n = \sum_{k=0}^{n} \binom{n}{k} a_k b_{n-k}$$

for all $n \geq 0$. We prove that $\lim\limits_{n\to\infty} \sqrt[n]{c_n} \;=\; \alpha \;+\; \beta$: For all $\epsilon > 0$ there is an integer $n_0 > 0$ such that for all $n > n_0$ we have

$$|\, \sqrt[n]{a_n} \;-\; \alpha\,| \;\leq\; \epsilon \quad \text{and} \quad \left|\, \sqrt[n]{b_n} \;-\; \beta \right| \;\leq\; \epsilon.$$

Then for all $n > n_0$,

$$(\alpha - \epsilon)^n \;\leq\; a_n \;\leq\; (\alpha + \epsilon)^n \quad \text{and} \quad (\beta - \epsilon)^n \;\leq\; b_n \;\leq\; (\beta + \epsilon)^n.$$

Furthermore, if $n > 2n_0 + 1$ then

$$c_n \;=\; \sum_{k=0}^{n_0} \binom{n}{k} a_k b_{n-k} \;+\; \sum_{k=n_0+1}^{n-n_0-1} \binom{n}{k} a_k b_{n-k} \;+\; \sum_{k=n-n_0}^{n} \binom{n}{k} a_k b_{n-k}.$$

Since

$$\begin{aligned}\sum_{k=0}^{n_0}\binom{n}{k}a_k b_{n-k} &\le n^{n_0}\cdot \max_{k\in\{0,1,\ldots,n_0\}}\{a_k\}\cdot\sum_{k=0}^{n} b_{n-k}\\ &\le n^{n_0}\cdot \max_{k\in\{0,1,\ldots,n_0\}}\{a_k\}\cdot(\beta+\epsilon)^n\cdot\sum_{k=0}^{n_0}(\beta+\epsilon)^{-k},\end{aligned}$$

Therefore, there is a constant $M_1>0$ depending on ϵ and n_0 such that

$$\sum_{k=0}^{n_0}\binom{n}{k}a_k b_{n-k} \le M_1\, n^{n_0}\,(\beta+\epsilon)^n. \tag{2.1}$$

Similarly, there is a constant $M_2>0$ depending on ϵ and n_0 such that

$$\sum_{k=n-n_0}^{n}\binom{n}{k}a_k b_{n-k} \le M_2\, n^{n_0}\,(\alpha+\epsilon)^n. \tag{2.2}$$

At last,

$$\begin{aligned}\sum_{k=n_0+1}^{n-n_0-1}\binom{n}{k}a_k b_{n-k} &\le \sum_{k=n_0+1}^{n-n_0-1}\binom{n}{k}(\alpha+\epsilon)^k\,(\beta+\epsilon)^{n-k}\\ &\le \sum_{k=0}^{n}\binom{n}{k}(\alpha+\epsilon)^k\,(\beta+\epsilon)^{n-k}\\ &\le (\alpha+\beta+2\epsilon)^n.\end{aligned}$$

The upper bound of $\lim \sqrt[n]{c_n}$ is then, by (2.1) and (2.2), given by

$$\begin{aligned}c_n &\le (\alpha+\beta+2\epsilon)^n + M\,n^{n_0}\left((\alpha+\epsilon)^n+(\beta+\epsilon)^n\right)\\ &= (\alpha+\beta+2\epsilon)^n\left[1+M\,n^{n_0}\left(\frac{\alpha+\epsilon}{\alpha+\beta+2\epsilon}\right)^n + M\,n^{n_0}\left(\frac{\beta+\epsilon}{\alpha+\beta+2\epsilon}\right)^n\right]\end{aligned}$$

where $M=\max\{M_1,M_2\}$. As

$$\lim_{n\to\infty} M\,n^{n_0}\left(\frac{\alpha+\epsilon}{\alpha+\beta+2\epsilon}\right)^n = \lim_{n\to\infty} M\,n^{n_0}\left(\frac{\beta+\epsilon}{\alpha+\beta+2\epsilon}\right)^n = 0,$$

we have that for n big enough

$$c_n \le (\alpha+\beta+2\epsilon)^n,$$

i.e we have the upper bound

$$\sqrt[n]{c_n} \le \alpha+\beta+2\epsilon. \tag{2.3}$$

for all $\epsilon>0$ and n big enough.

We look now at the lower bound of $\lim \sqrt[n]{c_n}$: For all $0<\epsilon<\frac{1}{2}\min\{\alpha,\beta\}$ and n_0 fixed there are two positive constants M_3 and M_4 such that

$$\begin{aligned}\sum_{k=0}^{n_0}\binom{n}{k}(\alpha-\epsilon)^k\,(\beta-\epsilon)^{n-k} &\le n_0^n\,\max\{1;\,(\alpha-\epsilon)^{n_0}\}\,(\beta-\epsilon)^n\,\sum_{k=0}^{n_0}(\beta-\epsilon)^{-k}\\ &\le M_3\,n^{n_0}\,(\beta-\epsilon)^n\end{aligned}$$

and similarly

$$\sum_{k=n-n_0}^{n}\binom{n}{k}(\alpha-\epsilon)^k\,(\beta-\epsilon)^{n-k} \le M_4\,n^{n_0}\,(\alpha-\epsilon)^n.$$

Then

$$\begin{aligned}
\sum_{k=n_0+1}^{n-n_0-1} \binom{n}{k} a_k b_{n-k} &\geq \sum_{k=n_0+1}^{n-n_0-1} \binom{n}{k} (\alpha-\epsilon)^k (\beta-\epsilon)^{n-k} \\
&= (\alpha+\beta-2\epsilon)^n - \sum_{k=0}^{n_0} \binom{n}{k} (\alpha-\epsilon)^k (\beta-\epsilon)^{n-k} \\
&\quad - \sum_{k=n-n_0}^{n} \binom{n}{k} (\alpha-\epsilon)^k (\beta-\epsilon)^{n-k} \\
&\geq (\alpha+\beta-2\epsilon)^n - N\, n^{n_0} \left((\alpha-\epsilon)^n + (\beta-\epsilon)^n\right) \\
&= (\alpha+\beta-2\epsilon)^n \left[1 - N\, n^{n_0} \left(\left(\frac{\alpha-\epsilon}{\alpha+\beta-2\epsilon}\right)^n + \left(\frac{\beta-\epsilon}{\alpha+\beta-2\epsilon}\right)^n\right)\right]
\end{aligned}$$

where $N = \max\{M_3; M_4\}$. But as

$$\lim_{n\to\infty} N\, n^{n_0} \left(\frac{\alpha-\epsilon}{\alpha+\beta-2\epsilon}\right)^n = \lim_{n\to\infty} N\, n^{n_0} \left(\frac{\beta-\epsilon}{\alpha+\beta-2\epsilon}\right)^n = 0,$$

the lower bound is given by

$$c_n \geq \sum_{k=n_0+1}^{n-n_0-1} \binom{n}{k} a_k b_{n-k} \geq (\alpha+\beta-2\epsilon)^n$$

for all n big enough and for all $0 < \epsilon < \frac{1}{2}\min\{\alpha; \beta\}$.
Then, for all $0 < \epsilon < \frac{1}{2}\min\{\alpha, \beta\}$ there is an integer n_0 big enough such that for all $n \geq n_0$,

$$\alpha+\beta-2\epsilon \leq \sqrt[n]{c_n} \leq \alpha+\beta+2\epsilon.$$

Thus $\lim_{n\to\infty} \sqrt[n]{c_n} = \alpha+\beta$. □

Theorem 2.2 implies that to find the geodesic growth rate of a direct product on the standard generating set, it suffices to find the geodesic growth rates of the two factors.

The following proposition was proved by Shapiro and gives us a more precise result about the number of geodesics in a direct product, with respect to the standard generating set.

Proposition 2.3 (Shapiro, [53])**.** *Let H and K be two groups generated by finite sets X and Y, respectively. Then the direct product $H \times K$ is generated by the finite set $X \cup Y$ and*

$$\rho_{X\cup Y}\left((a,b)\right) = \binom{l_X(a)+l_Y(b)}{l_X(a)} \cdot \rho_X(a)\rho_Y(b),$$

where

$$\rho_S(g) = \#\{\text{ geodesics from } 1_G \text{ to } g \in G\ \}.$$

The proposition above and Example 1.5 allow us to study the free abelian groups $\mathbb{Z}^d$, where $d > 1$. From Theorem 2.2, we know that $\mathbb{Z}^d$ with standard generating set has exponential geodesic growth rate d. The following proposition gives a more precise formula for geodesic growth.

Proposition 2.4. *Let $d > 1$ be a fixed integer. Then for the group $\mathbb{Z}^d$ with standard generating set S_d we have*

$$\gamma_{S_d}(n) = 2^d\, d^n - 2^d.$$

Proof. Let $\mathbb{Z}^d$ have S_d as its standard generating set. For all elements $x = (x_1, x_2, ..., x_d) \in \mathbb{Z}^d$, the geodesic length is given by $l_{S_d}(x) = \sum_{i=1}^{d} |x_i|$. From Proposition 2.3 we have

$$\begin{aligned} \rho_{S_d}(x) &= \prod_{i=2}^{d} \binom{\sum_{k=1}^{i} |x_k|}{|x_i|} \\ &= \prod_{i=2}^{d} \binom{\sum_{k=1}^{i} |x_k|}{\sum_{k=1}^{i-1} |x_k|} = \frac{(\sum_{i=1}^{n} |x_i|)!}{\prod_{i=1}^{n} (x_i!)} \end{aligned}$$

Then for all $n > 0$:

$$\begin{aligned} \gamma_{S_d}(n) &= 2^d \cdot \sum_{\substack{i_1,...,i_{d-1}=1 \\ \sum_{k=1}^{d-1} i_k \leq n}}^{n} \rho_s \left(i_1, ..., i_{d-1}, n - \sum_{k=1}^{d-1} i_k \right) \\ &= 2^d \cdot \left[\sum_{\substack{i_1,...,i_{d-1}=0 \\ \sum_{k=1}^{d-1} i_k \leq n}}^{n} \left(\prod_{l=2}^{d} \binom{\sum_{k=1}^{l} i_k}{i_l} \right) - 1 \right] \\ &= 2^d \cdot [(1 + 1 + ... + 1)^n - 1] = 2^d d^n - 2^d \end{aligned}$$

□

Thus

$$\gamma(\mathbb{Z}^d, S_d) = \lim_{n\to\infty} \sqrt[n]{2^d \cdot d^n - 2^d} = d.$$

The next question we ask is: what is the minimal geodesic growth rate of a finitely generated free abelian group over all its generating sets? The following Theorem provides the answer and is a generalisation of a theorem given by Bridson, Burillo, Elder and Šunić [8].

Theorem 2.5. *For all $d \geq 2$, $\mathbb{Z}^d$ has exponential geodesic growth with respect to every generating set. Moreover, its minimal geodesic growth rate is d and is achieved by the standard generating set.*

Proof. Let $X = \{x_1, ..., x_n\}$ be a generating set of $\mathbb{Z}^d$ with $n \geq d$ and let m be a non-zero integer. We see $\mathbb{Z}^d$ as the subset of points in $\mathbb{R}^d$ with integer coordinates, and we define $x_i = (x_i^1, ..., x_i^d) \in \mathbb{R}^d$ for all $x_i \in X$.
The polytope associated to $X \cup X^{-1}$, defined as the intersection of all convex subsets of $\mathbb{R}^d$ containing $X \cup X^{-1}$, is then given by

$$P = \left\{ \sum_{i=1}^{n} a_i x_i \mid \sum_{i=1}^{n} |a_i| \leq 1 \right\}.$$

P is a symetric polyhedron with center $O = \{0, ..., 0\}$.

In [49, Theorem 19.1, p. 171], Rockafellar proved that a convex set is a polyhedron if and only if it is closed and has only finitely many faces, i.e it can be expressed as the intersection of finitely many closed half spaces of $\mathbb{R}^d$ of the form

$$H_{b,\beta}^{+} = \left\{ x \in \mathbb{R}^d \mid < x, b > \geq \beta \right\} \quad \text{or} \quad H_{b,\beta}^{-} = \left\{ x \in \mathbb{R}^d \mid < x, b > \leq \beta \right\},$$

where $b \in \mathbb{R}^d$ is a vector orthogonal to H of length one and $\beta := dist(H; 0) \in R_{\geq 0}$. These half spaces depend only on the hyperplane

$$H_{b,\beta} = \left\{ x \in \mathbb{R}^d \mid < x, b >= \beta \right\},$$

and each hyperplane $H_{b,\beta}$ is a linear subspace of $\mathbb{R}^d$ of rank $d-1$. That is, there are d points $P_1, P_2, ..., P_d \in \mathbb{R}^d$ which determine $H_{b,\beta}$ in the sense that $H_{b,\beta}$ is given by

$$H_{b,\beta} = \left\{ P \in \mathbb{R}^d \mid \overrightarrow{OP} = \overrightarrow{OP_1} + \sum_{i=2}^{d} \lambda_i \overrightarrow{P_1 P_i},\ \lambda_i \in \mathbb{R},\ 2 \leq i \leq d \right\}.$$

Let $x_{i_1}, ..., x_{i_d}$ be d elements of X which represent d points on ∂P and determine one of these hyperplanes.

Define now λP to be the image of P under the dilation $v \mapsto \lambda v$ on $\mathbb{R}^d$ with $\lambda \in \mathbb{R}$. By definition of P,

$$\lambda P = \left\{ \sum_{i=1}^{n} a_i x_i \mid \sum_{i=1}^{n} |a_i| \leq \lambda \right\}.$$

Since for all $k \geq 1$ each geodesic of length $k+1$ could be written as wx_i, where w is a geodesic of length k and $x_i \in X \cup X^{-1}$, then each geodesic in $\mathbb{Z}^d$ of length less than or equal to k represents an element in kP.

By the basic property of affine subspaces, the point $mx_{i_1} + ... + mx_{i_d}$ in $\mathbb{Z}^d$ is in $(dm)P$, more particularly in $(dm)P \setminus (dm-1)P$. Thus the word $x_{i_1}^m ... x_{i_d}^m$ is a geodesic of length dm in $\mathbb{Z}^d =< X >$.

As $\mathbb{Z}^d$ is abelian, there are exactly $\frac{(dm)!}{(m!)^d}$ permutations of the letters $x_{i_1}, ..., x_{i_d}$ in the word $x_{i_1}^m ... x_{i_d}^m$ to create geodesics which represent the same element. By Stirling's Formula, we have then

$$\begin{aligned} \gamma_X(dm) &= \#\{\text{ geodesics of length } dm\} \geq \frac{(dm)!}{(m!)^d} \\ &\simeq \frac{\sqrt{2d\pi m}\left(\frac{dm}{e}\right)^{dm}}{(\sqrt{2\pi m})^d \left(\frac{m}{e}\right)^{dm}} = \frac{\sqrt{d}\, d^{dm}}{(2\pi m)^{\frac{d-1}{2}}}. \end{aligned}$$

Then

$$\gamma(\mathbb{Z}^d, X) \geq \lim_{m \to \infty} \sqrt[dm]{\frac{\sqrt{d}\, d^{dm}}{(2\pi m)^{\frac{d-1}{2}}}} = d.$$

Thus $\mathbb{Z}^d$ has exponential geodesic growth with respect to every generating set, with geodesic growth rate at least d.

Finally, from Proposition 2.4, we have that the geodesic growth rate of $\mathbb{Z}^d$ with respect to the standard generating set is d, which gives us the minimal geodesic growth rate. □

Recall the fundamental theorem of finitely generated abelian groups.

Theorem 2.6 (Fundamental Theorem of finitely generated abelian groups). *Let G be a finitely generated abelian group. Then G is isomorphic to a direct sum*

$$C_{a_1} \times ... \times C_{a_k} \times \mathbb{Z}^d,$$

where the rank $d \geq 0$, and the numbers $a_1, \ldots, a_k$ are powers of (not necessarily distinct) prime numbers. In particular, G is finite if and only if $d = 0$.

Hence, with Lemma 1.12 about epimorphic images, we have

$$\gamma(G) \geq \max(\gamma(H); \gamma(K)),$$

for a direct product $G = H \times K$ of two finitely generated groups H and K. Then we have the following corollary.

Corollary 2.7. *Let $G \simeq C_{a_1} \times ... \times C_{a_k} \times \mathbb{Z}^d$ be a finitely generated abelian group. Then the minimal geodesic growth rate is d and achieved by the standard generating set.*

We know that for all $\epsilon > 0$, there are a generating set X_ϵ of H and a generating set Y_ϵ of K such that

$$\gamma(H, X_\epsilon) \; < \gamma(H) + \frac{\epsilon}{2} \qquad \text{and} \qquad \gamma(K, Y_\epsilon) \; < \gamma(K) + \frac{\epsilon}{2}.$$

It implies that for all $\epsilon > 0$ there is a standard generating set $X_\epsilon \cup Y_\epsilon$ of the direct product such that

$$\max(\gamma(H); \gamma(K)) \;\leq\; \gamma(H \times K, X_\epsilon \cup Y_\epsilon) \;\leq\; \gamma(H) + \gamma(K) + \epsilon.$$

Then

$$\max(\gamma(H); \gamma(K)) \;\leq\; \gamma(H \times K) \;\leq\; \gamma(H) + \gamma(K).$$

The explicit formula for the minimal geodesic growth rate of a direct product is still open. We have the following conjecture.

Conjecture 2.8. *Let H and K be two finitely generated groups. Then*

$$\gamma(H \times K) \;=\; \gamma(H) + \gamma(K).$$

However, a direct application of Lemma 1.12 and Corollary 2.7 is the following.

Theorem 2.9. *Let G be a finitely generated group, $ab(G)$ be the abelianization of G and $Tor(ab(G))$ the torsion subgroup of $ab(G)$. Then*

$$\gamma(G) \;\geq\; rank\left(ab(G) \middle/ Tor\,(ab(G))\right).$$

2.2 Free product

In the same article [39], Loeffler, Meier and Worthington proved the following proposition.

Proposition 2.10 (Loeffler, Meier and Worthington, [39, p.753])**.** *Let H and K be two groups generated by finite sets X and Y, respectively. Then*

$$\frac{1}{\Gamma_{H*K, X\cup Y}(z)} \;=\; \frac{1}{\Gamma_{H,X}(z)} + \frac{1}{\Gamma_{K,Y}(z)} - 1.$$

Proof. Let $G = H * K$. Since G is a free product of groups, each word representing an element of G could be written in its normal form. More precisely, for all $g \in G$ there is an integer $r \geq 1$ and elements $a_0, .., a_{r-1} \in H$ and $b_1, .., b_r \in K$ non-trivial (except perhaps a_0 and b_r) such that

$$g \;=\; a_0\, b_1\, a_1\, b_2 \ldots a_{r-1}\, b_r.$$

But the length of such a word is given by $\sum_{i=1}^{r} (l_X(a_{i-1}) + l_Y(b_i))$.

Fix an integer n. The number of geodesics in G of length n in this form is then given by the sum of the number of possible geodesics for a_1, multiplied by the number of the possibilities for the $2r-2$ elements b_1 to b_{r-1} and a_1 to a_{r-1}, finally multiplied by the number of possibilities for b_r. Thus we have the formula

$$\sum \gamma_X(s_0)\gamma_Y(t_1)\gamma_X(s_1)\gamma_Y(t_2) \ldots \gamma_X(s_{r-1})\gamma_Y(t_r),$$

where $s_{i-1} = l_X(a_{i-1})$ and $t_i = l_Y(b_i)$ for all $1 \le i \le r$ and where the sum has the following constraints: $\sum_{i=1}^{r} s_{i-1} + t_i = n$, $s_i \ge 1$ for all $i \ge 1$ and $t_i \ge 1$ for all $i < r$.

But this result is exactly the n-th coefficient of the series

$$\Gamma_X(z)\,\Gamma_Y(z)\,\left((\Gamma_X(z)-1)\,(\Gamma_Y(z)-1)\right)^{r-1}.$$

If we sum over all positive integers r, we have the formula

$$\begin{aligned}\Gamma_{H*K,X\cup Y}(z) &= \frac{\Gamma_{H,X}(z)\,\Gamma_{K,Y}(z)}{1-(\Gamma_{H,X}(z)-1)\,(\Gamma_{K,Y}(z)-1)} \\ &= \frac{\Gamma_{H,X}(z)\,\Gamma_{K,Y}(z)}{\Gamma_{H,X}(z)+\Gamma_{K,Y}(z)-\Gamma_{H,X}(z)\Gamma_{K,Y}(z)}.\end{aligned}$$

Taking inverses we get the formula in the statement. □

In the next subsection, we study in more detail the geodesic growth rate of $H * K$ with respect to the generating set $X \cup Y$.

2.2.1 Bounds

From Lemma 1.12, we have that the geodesic growth rate of $H * K$, with respect to the standard generating set $X \cup Y$, verifies the inequality

$$\gamma_{X\cup Y} \;\ge\; \max\{\gamma_X,\, \gamma_Y\}.$$

But a better lower bound is given by the following theorem.

Theorem 2.11. *Let H and K be two non-trivial groups generated by finite sets X and Y, respectively. Then*

$$\gamma(H * K, X \cup Y) \;>\; \gamma(H,X) + \gamma(K,Y).$$

Proof. The proof is separated into three cases: if both groups are infinite, if only one is infinite, and if both are finite. In the case when both H and K are finite, the free product $H * K$ is infinite. It implies that

$$\gamma_{H*K,X\cup Y} \;\ge\; 1 \;>\; \gamma_{H,X} + \gamma_{K,Y} \;=\; 0.$$

If at least one of the two groups is infinite, we have the following.

The generating growth functions of H and K are $\Gamma_X(z) = \sum_{n=0}^{\infty} \gamma_X(n)\, z^n$ and $\Gamma_Y(z) = \sum_{n=0}^{\infty} \gamma_Y(n)\, z^n$, respectively.
Let $B : \mathbb{R}^+ \to \mathbb{R}^+$ and $C : \mathbb{R}^+ \to \mathbb{R}^+$ be two functions defined by

$$B(z) = \begin{cases} \sum_{n=0}^{\infty} \alpha^n z^n & \text{if } H \text{ is infinite} \\ 1+z & \text{otherwise} \end{cases}$$

and

$$C(z) = \begin{cases} \sum_{n=0}^{\infty} \beta^n z^n & \text{if } K \text{ is infinite} \\ 1+z & \text{otherwise} \end{cases},$$

where $\alpha = \gamma(H,X)$ and $\beta = \gamma(K,Y)$.

By Proposition 1.7, we have $B(z) < \Gamma_X(z)$ and $C(z) < \Gamma_Y(z)$ for all $z \in \mathbb{R}^+$. Moreover, if H is infinite, the radius of convergence of B is $\frac{1}{\alpha}$ and for all $z < \frac{1}{\alpha}$ we have $B(z) = \frac{1}{1-\alpha z}$.

In the same way, if K is infinite, the radius of convergence of C is $\frac{1}{\beta}$ and for all $z < \frac{1}{\beta}$, $C(z) = \frac{1}{1-\beta z}$.
Then let $A : \mathbb{R}^+ \to \mathbb{R}^+$ be defined by

$$A(z) = \frac{B(z)\,C(z)}{B(z) + C(z) - B(z)C(z)}.$$

Since on $\mathbb{R}^+$ we have $B(z) < \Gamma_X(z)$ and $C(z) < \Gamma_Y(z)$, we have

$$B(z)\,C(z) < \Gamma_X(z)\,\Gamma_Y(z)$$

and

$$\begin{aligned} B(z) + C(z) - B(z)C(z) &= 1 - (1 - B(z))(1 - C(z)) \\ &> 1 - (1 - \Gamma_X(z))\,(1 - \Gamma_Y(z))\,. \end{aligned}$$

Then we have $\Gamma_{X\cup Y}(z) > A(z)$ for all $z \in \mathbb{R}^+$. This implies that the radius of convergence of $\Gamma_{X\cup Y}$ must be strictly smaller than the radius of convergence of $A(z)$. Then we have three cases:

1. If H and K are infinite, we have for all $z < \min\left\{\frac{1}{\alpha}\,;\,\frac{1}{\beta}\right\}$, that

$$A(z) = \frac{\frac{1}{1-\alpha z}\cdot\frac{1}{1-\beta z}}{\frac{1}{1-\alpha z} + \frac{1}{1-\beta z} - \frac{1}{1-\alpha z}\cdot\frac{1}{1-\beta z}} = \frac{1}{1 - (\alpha+\beta)z}.$$

   Its radius of convergence is $\frac{1}{\alpha+\beta}$. Then we have

$$\gamma(H * K, X\cup Y) > \alpha + \beta = \gamma(H,X) + \gamma(K,Y).$$

2. If H is infinite and K finite, we have for all $z < \frac{1}{\alpha}$ that

$$A(z) = \frac{\frac{1+z}{1-\alpha z}}{(1+z) + \frac{1}{1-\alpha z} - \frac{1+z}{1-\alpha z}} = \frac{1+z}{(1+z)(1-\alpha z) - z}.$$

   Its radius of convergence is

$$\frac{1}{2}\left(\sqrt{1+\frac{4}{\alpha}} - 1\right) = \frac{2}{\alpha\left(\sqrt{1+\frac{4}{\alpha}}+1\right)}.$$

   Then we have $\gamma(H * K, X\cup Y) > \alpha = \gamma(H,X) + \gamma(K,Y)$.

3. If H is finite and K infinite, we have for all $z < \frac{1}{\beta}$ that

$$A(z) = \frac{\frac{1+z}{1-\beta z}}{(1+z) + \frac{1}{1-\beta z} - \frac{1+z}{1-\beta z}} = \frac{1+z}{(1+z)(1-\beta z) - z}.$$

   Its radius of convergence is

$$\frac{1}{2}\left(\sqrt{1+\frac{4}{\beta}} - 1\right) = \frac{2}{\beta\left(\sqrt{1+\frac{4}{\beta}}+1\right)}.$$

   Then we have $\gamma(H * K, X\cup Y) > \beta = \gamma(H,X) + \gamma(K,Y)$.

□

Now look at some particular cases of free products.

Examples 2.12.

1. If $H = \mathbb{F}_k = \langle X|\_\rangle$ and $K = \mathbb{F}_l = \langle Y|\_\rangle$ with $|X| = k$ and $|Y| = l$, then we have $G = H * K = \mathbb{F}_{k+l} = \langle X \cup Y|\_\rangle$. Moreover, since for each element $h \in \mathbb{F}_k$ there is an unique geodesic representative we have for all $n \geq 1$

$$\gamma_X(n) = a_X(n) = 2k(2k-1)^{n-1}.$$

In the same way, we have for all $n \geq 1$

$$\gamma_Y(n) = a_Y(n) = 2l(2l-1)^{n-1}.$$

Then

$$\gamma_{X\cup Y} = 2(k+l) - 1 = \gamma_X + \gamma_Y + 1.$$

2. Let $(H_i)_{i=1}^n$ be a family of groups defined by $H_i = \langle a_i|\_\rangle \cong \mathbb{Z}$ for all $i = 1, 2, ..., n$. Then we have

$$G := *_{i=1}^n H_i = \langle a_1, ..., a_n|\_\rangle \cong \mathbb{F}_n$$

and its geodesic growth rate with respect to the generating set $X := \bigcup_{i=1}^n X_i$ is

$$\gamma_X = 2n - 1 > n = \sum_{i=1}^n 1 = \sum_{i=1}^n \gamma_{X_i}.$$

3. Let $H_1, ..., H_n$ and $K_1, ..., K_m$ be defined by $H_i = \langle a_i|\_\rangle \cong \mathbb{Z}$ for all $1 \leq i \leq n$ and $K_j = \langle b_j|b_j^2 = 1\rangle$ for all $1 \leq j \leq m$. Let $G = *_{i=1}^n H_i * *_{j=1}^m K_j$ be the free product with generating set $S = \{a_1, a_2, ..., a_n, b_1, b_2, ..., b_m\}$. Then each element in G has an unique geodesic representative. It implies that $\gamma_S(r) = a_S(r)$ for all $r \geq 1$. Since each element $g \in G$ has an unique expression as a product

$$s_{i_1} s_{i_2} \dots s_{i_l},$$

where $s_{i_k} \neq s_{i_{k+1}}$ for all $k = 1, ..., l-1$, we get that $a_S(n) = 2N(2N-1)^{n-1}$, where $N = 2n + m$. Then

$$\gamma(G, S) = \omega(G, S) = 2N - 1,$$

which is strictly bigger than the sum of the geodesic growth rates of its factors.

4. If $G = \mathbb{F}_n * C_p$, where p is a prime number big enough, then the geodesic growth rate with respect to the standard generating set is $2n$.

We remark that in each of these examples, the geodesic growth rate of the free product is at least the sum of the geodesic growth rate of each factor plus one. An open question is then to prove that this is always the case for all free product.

Conjecture 2.13. *Let H and K be two non-trivial groups generated by finite sets X and Y, respectively. Then*

$$\gamma(H * K, X \cup Y) \geq \gamma(H, X) + \gamma(K, Y) + 1.$$

Now if we study more precisely the free products of finite groups, we observe that the lower bound given in Theorem 2.11 can be improved for this particular case. Firstly, observe the following.

Consider the groups $C_2 = \langle a|a^2 = 1\rangle$ and $C_3 = \langle b|b^3 = 1\rangle$. Then the geodesic growth rates are $\gamma(C_2, \langle a\rangle) = \gamma(C_3, \langle b\rangle) = 0$ and each element of these two groups has an unique geodesic representative. Then the geodesic growth is equal to the spherical growth for the two groups and in particular

$$\Gamma_a(z) \;=\; 1+z \qquad \Gamma_b(z) \;=\; 1+2z.$$

From Proposition 2.10, the geodesic growth series of $C_2 * C_2$, $C_2 * C_3$ and $C_3 * C_3$ are

$$\frac{1+z}{1-z}, \quad \frac{(1+z)(1+2z)}{(1+\sqrt{2}z)(1-\sqrt{2}z)}, \quad \frac{1+2z}{1-2z},$$

respectively. It implies that the geodesic growth rates of these products are 1, $\sqrt{2}$, and 2, respectively.

Since the geodesic growth series of the finite cyclic groups C_{2n} and C_{2n+1} are identical to the spherical growth series of C_{2n} for all $n \geq 2$, it is possible to prove many results analogue to the results on spherical growth of A. Talambutsa in [56].

In [10], M. Bucher and A. Talambutsa proved that $\Omega(G) \geq \frac{1+\sqrt{5}}{2}$ if G is not isomorphic to $C_2 * C_2$ or $C_2 * C_3$. Then we have the following corollary.

Corollary 2.14. *Let $G = H * K$ be a free product. If G is not isomorphic to $C_2 * C_2$ or $C_2 * C_3$, we have*

$$\gamma(G) \geq \frac{1+\sqrt{5}}{2}.$$

Let G be the free product of H, K, where H, K are finite. Let S be a generating set of G. We define $\gamma_k(G)$ as

$$\gamma_k(G) \;:=\; \inf_{|S|=k} \gamma(G,S).$$

Let $S = \{x, y\}$ be a generating set of a free product $G = C_n * C_m$. S is said to be of the first type if at least one of its elements has finite order and of the second type otherwise.

Example 2.15.
We know that the geodesic growth series $\Gamma_S^{(n)}(z)$ of C_n, with respect to the standard generating set S, is related to the spherical growth series $A_m(z)$ for some $m \geq 2$. In particular,

$$\Gamma_S^{(n)}(z) \;=\; \begin{cases} 1+z & \text{if} \quad n=2 & = \; A_2(z) \\ 1+2\sum_{k=1}^{m-1} z^k & \text{if} \quad n=2m-1 & = \; A_{2m-1}(z) \\ 1+2\sum_{k=1}^{m} z^k & \text{if} \quad n=2m & = \; A_{2m+1}(z) \end{cases}$$

Moreover, we know that the formula for the geodesic growth series of a free product is exactly the same as that of the spherical growth series. Thus we have

1. If $G = C_2 * C_{2n-1}$, where $n \geq 2$, then $\gamma(G,S) = \omega(G,S) \leq 2$.
2. If $G = C_2 * C_{2n}$, where $n \geq 2$, then $\gamma(G,S) = \omega(C_2 * C_{2n+1}, S) \leq 2$.
3. If $G = C_3 * C_3$ then $\gamma(G,S) = \omega(G,S) = 2$.
4. If $G = C_m * C_n$, where $m, n \geq 3$ then $\gamma(G,S) \geq 2$.

Let $G = C_n * C_m = \langle a, b | a^n = b^m = 1 \rangle$ be a free product. If $g \in G$ has finite order, then all conjugates of g have finite order too. Let $[g]$ be the conjugacy class of g. In this class there exists an unique element $\tilde{g} \in [g]$ of one of the three forms

$$\tilde{g} = \begin{cases} a^k & 1 \le k \le m-1, \\ b^l & 1 \le l \le n, \\ a^{k_1} b^{l_1} ... a^{k_r} b^{l_r} & r \ge 1, \ k_1, l_r \neq 0, \ k_i, l_i \ge 0 \quad \forall 1 \le i \le r. \end{cases} \tag{2.4}$$

Proposition 2.16. *Let $G = C_n * C_m$ be a free product. Then an element of G has finite order if and only if it is conjugate to some power of a or b.*

The three next Lemmas have analogue proofs to those for spherical growth of the Lemmas 3,4 and 5 in the article [56] of Talambutsa.

Lemma 2.17 (Talambutsa, [56, p. 292])**.** *Let $G = C_m * C_n$ when $n \ge 2$ and $m \ge 3$. If $S = \{x, y\}$ is a set of generators of the second type for G, then $\gamma(G, S) \ge 2$.*

Lemma 2.18 (Talambutsa, [56, p. 292])**.** *Let $G = C_m * C_n = \langle a, b | a^m, b^n \rangle$ and $u \in G$ be such that its normal form is given by $b^{i_1} a^{j_1} ... b^{i_N}$, where $N \ge 2$. Then $G \neq \langle a, u \rangle$.*

Lemma 2.19 (Talambutsa, [56, p. 292])**.** *If S is a set of generators of the first type of the free product $G = C_m * C_n$, $n \ge 2$ et $m \ge 3$, then there exists a set $T = \{x, y\}$ of two elements which generates G and such that $\gamma(G, S) = \gamma(G, T)$ and either*

1. *$x = a$ and $y = ba^r$ for $0 \le r \le m/2$, or*
2. *$x = b$ and $y = ab^r$ for $0 \le r \le n/2$.*

Suppose that $m, n \ge 3$. It follows from Lemma 2.19 that the set Γ_1 of geodesic growth rates of the free product $G = C_m * C_n$ with respect to various generators of the first type is finite and consists of rates $\gamma(G, S_r)$ and $\gamma(G, T_r)$, where $S_r = \{a, ba^r\}$ and $T_r = \{b, ab^r\}$. To prove this assertion, we define a few auxiliary sets of words.

Let $\mathcal{F}_A$ be the set of all reduced words in the alphabet $A = \{a, a^{-1}, d, d^{-1}\}$. Denote by $\mathcal{G}_{m,n,r}$ the set of all words in $\mathcal{F}_A$ that do not contain subwords of the form

$$a^{f(m)}, \ a^{-f(m)}, \ (da^r)^{f(n)-1} d, \ \left((da^r)^{f(n)-1} d\right)^{-1},$$

where the function f is defined as follows:

$$f(n) = \begin{cases} 1 + \frac{n}{2} & \text{if } n \text{ is even}, \\ \frac{n+1}{2} & \text{if } n \text{ is odd}. \end{cases}$$

Remark 2.20. Let $G = C_m * C_n = \langle a, d | a^m = d^n = 1 \rangle$ be a free product, where $m, n \ge 3$. Then $\mathcal{G}_{m,n,0}$ is the set of geodesics of G. In particular, if for any set of words $\mathcal{W}$ we denote by $\mathcal{W}(k)$ the set of all words in $\mathcal{W}$ whose length is smaller or equal than k, we have that

$$\mathcal{G}_{m,n,0}(k) = \sum_{i=0}^{k} \gamma_{S_0}(i).$$

Lemma 2.21. *Let $m, n \ge 3$ and $0 \le r \le f(m)$. Then the following inequality holds for any integer $k \ge 0$:*

$$\#\mathcal{G}_{m,n,0}(k) \le \#\mathcal{G}_{m,n,r}(k).$$

Proof. Let $\mathcal{G}_m$ be the set of all reduced words in $\mathcal{A}$ that don't contain subwords of the form $a^{\pm f(m)}$. Then $\mathcal{G}_{m,n,r}, \mathcal{G}_{m,n,0} \subseteq \mathcal{G}_m$.

Denote by $\mathcal{C}_{m,n,r}$ and $\mathcal{C}_{m,n,0}$ the complements of $\mathcal{G}_{m,n,r}$ and $\mathcal{G}_{m,n,0}$, respectively. Then the assertion of the lemma is equivalent to the inequality

$$\#\mathcal{C}_{m,n,0}(k) \quad \geq \quad \#\mathcal{C}_{m,n,r}(k)$$

and we prove it similarly to the proof of the Lemma 6 in the article [56].

□

Lemma 2.22. *Let $G = C_m * C_n =< a, b|a^m, b^n >$, where $m, n \geq 3$, and $d = ba^r$, where $1 \leq r \leq f(m)$. If $U(a, d)$ and $V(a, d)$ are two different words in $\mathcal{G}_{m,n,r}$, then $U \neq V$ in the group G.*

Proof. Since $U \in \mathcal{G}_{m,n,r}$, it doesn't contain subwords $a^{\pm f(m)}$ and can be expressed as

$$U \quad = \quad a^{x_1} d^{\epsilon_1} a^{x_2} d^{\epsilon_2} \ldots a^{x_l} d^{\epsilon_l} a^{x_{l+1}},$$

where $\epsilon_i \in \{\pm 1\}$ and $|x_i| < f(m)$ for all i.

Similarly to the proof of Lemma 7 in [56], we then represent U as an element of G in terms of the generating set $\{a, b\}$.

□

Lemma 2.23. *Is S is a set of generators of the first type of the free product $G = C_m * C_n = \langle a, b|a^m, b^n\rangle$, then the inequality $\gamma(G, S) \geq \gamma(G, \{a, b\})$ holds.*

Proof. By Lemma 2.19, we can find a set of generators $T = \{x, y\}$ such that $\gamma(G, S) = \gamma(G, T)$ and either

1. $x = a$ and $y = ba^r$, $0 \leq r \leq m/2$, or
2. $x = b$ and $y = ab^r$, $0 \leq r \leq n/2$.

Then it suffices to prove the inequality in both cases. In the first case, we know from Lemmas 2.21 and 2.22 and Remark 2.20 that the cummulative geodesic growth rate, defined in page 12, verifies

$$\tilde{\gamma}(G, \{a, b\}) \quad \leq \quad \tilde{\gamma}(G, T).$$

Then the inequality follows from Proposition 1.6. In the second case, we need the same arguments, but now for the symmetrically presented group

$$G = C_n * C_m = \langle a, b|a^n = b^m = 1\rangle.$$

□

Lemma 2.24. *Let G be a free product of the form $C_2 * C_n$, $n \geq 3$, or $C_3 * C_3$, and $X = \{a, b\}$ its standard generating set. Then*

$$\gamma_2(G) = \gamma(G, X).$$

Proof. Let S a generating set of G such that $|S| = 2$. There are two different cases:

1. If S is of the first type, then $\gamma(G, S) \geq \gamma(G, st)$ by Lemma 2.23.
2. If S is of the second type, then $\gamma(G, S) \geq 2$ by Lemma 2.17 and Example 2.15.

Then $\gamma(G, S) \geq \gamma(G, X)$ for all generating sets S of G such that $|S| = 2$. Hence, we obtain the desired inequality. □

To prove the main theorem, it remains to establish the equality $\gamma_2(G) = \gamma(G)$: consider separately the case when the orders of factors of the free product are powers of the same prime number. The proof of the next lemma is analogue to the proof for spherical growth of Lemma 10 in [56]. We allow ourselves to write the complete proof since it is an interesting result.

Lemma 2.25. *If $G = C_{p^{i_1}} * C_{p^{i_2}} * ... * C_{p^{i_k}} * \mathbb{F}^r$, where $k \geq 2$, $r \geq 0$ and p is prime, then $\gamma(G) = \gamma_{k+r}(G)$.*

Proof. Assume that G is defined in the standard way

$$G \;=\; \left\langle\, u_1, u_2, ..., u_k, v_1, ..., v_r \mid u_1^{p^{i_1}} = 1, ..., u_k^{p^{i_k}} = 1 \,\right\rangle.$$

Let $N := [G;G]G^p$, $H := {}^{G}/_{N}$ and φ be the natural epimorphism $G \twoheadrightarrow H$.
By the definition of N, H is isomorph to the $(k+r)$-direct sum $C_p \oplus C_p \oplus ... \oplus C_p$. Consider this direct sum as a vector space of dimension $k+r$ over a finite field of p elements.
Now suppose that G is generated by a set $S = \{a_1, a_2, ..., a_m\}$, where $m \geq k+r$. Since φ is an epimorphism, H is generated by the system of vectors $\varphi(a_1), ..., \varphi(a_m)$.
In this system there exists a linearly independant subsystem $\{w_i\}_{i=1}^{k+r}$ of $k+r$ vectors. Let $T = \{b_1, ..., b_{k+r}\}$ be the subset of S such that each element b_i is projected onto the vector w_i of this subsystem.
Since the set $\{w_i\}_{i=1}^{k+r}$ is linearly independent and H has rank $k+r$, $\{w_i\}_{i=1}^{k+r}$ generates H. It implies that the subgroup Γ of G generated by T has rank $k+r$. By the Kurosh theorem of subgroups of free products, Γ must be one of the following:

$$\mathbb{F}_{k+r},\;\; \mathbb{F}_{k+r-1} * C_{p^{d_1}},\;\; ....\,,\;\; \mathbb{F}_1 * C_{p^{d_1}} * ... * C_{p^{d_{k+r-1}}},\;\; C_{p^{d_1}} * ... * C_{p^{d_{k+r}}}.$$

It also follows from the Kurosh theorem that the generators f_j of the finite components of Γ are conjugate to powers of generators u_i of G. Those possible powers are not multiples of p, because multiples of p lie in the kernel of φ. Therefore, the order of f_i is exactly equal to the power of the correspondent element u_i.
Moreover, different elements f_i cannot be conjugate to powers of the same element u_i because, else, they would be expressible in terms of each other in $\varphi(\Gamma)$. It implies that there are at most k finite-order generators f_j and each of them corresponds to a certain element u_i and has the same order. In particular, Γ has a decomposition of the form

$$\mathbb{F}_{r+n} * C_{p^{l_1}} * ... * C_{p^{l_{k-n}}}$$

for some $n \geq 0$ and G is isomorph to its quotient group.
Then there is an epimorphism $\psi : \Gamma \twoheadrightarrow \tilde{G} \simeq G$. In particular,

$$\gamma(\Gamma, T) \geq \gamma(G, \psi(T)).$$

Moreover, the application

$$\begin{aligned} \phi : G &\rightarrow \Gamma \\ b_i &\mapsto b_i \\ a_j &\mapsto 1 \end{aligned}$$

where $b_i \in T$ and $a_j \in S \setminus T$, is an epimorphism. Then

$$\gamma(G, S) \;\geq\; \gamma(\Gamma, T) \;\geq\; \gamma(G, \psi(T))$$

and $|\psi(T)| = k + r$. □

Corollary 2.26. *If $G = C_{p^{i_1}} * C_{p^{i_2}}$, where $k \geq 2$ and p is prime, then $\gamma(G) = \gamma_2$.*

Corollary 2.27. *Let G be a free product of the form $C_2 * C_{2^k}$, $k \geq 2$, or $C_3 * C_3$, and $X = \{a, b\}$ its standard generating set. Then $\gamma(G) = \gamma(G, X)$.*

Proof. By Lemma 2.25 we have that $\gamma(G) = \gamma_2(G)$. Then, by Lemma 2.24, we obtain the desired equality. □

We have then the following result. It is analogue to Theorem 1 in [56], but the polynomials are different.

Theorem 2.28. *Let G be a free product of the form $C_2 * C_n$, where n is a power of a prime number p and $n \geq 3$, or $C_3 * C_3$, and $X = \{a, b\}$ its standard generating set. Then*

$$\gamma(G) = \gamma(G, X) = \frac{1}{\alpha_n},$$

where α_n is the least positive root of the polynomial

$$\begin{array}{ll} 1 - z - 2z^2 + 2z^{\frac{n+3}{2}} & for \quad p \neq 2 \\ 1 - z - 2z^2 + z^{\frac{n+4}{2}} & for \quad p = 2. \end{array}$$

Proof. Suppose that $p = 2$. By Corollary 2.27, we have that $\gamma(G) = \gamma(G, X)$. To find α_n, since $n > 2$, it follows that n is divisible by 4. We know that

$$\Gamma_2(z) \quad = \quad 1 + z$$

and

$$\Gamma_n(z) \quad = \quad 1 + 2z + 2z^2 + ... + 2z^{n/2}.$$

Then, by Proposition 2.10, we have

$$\begin{aligned} \Gamma_{C_2 * C_n}(z) &= \frac{\Gamma_2(z) \cdot \Gamma_n(z)}{\Gamma_2(z) + \Gamma_n(z) - \Gamma_2(z) \cdot \Gamma_n(z)} \\ &= \frac{(1+z)(1+2z+2z^2+...+2z^{n/2})}{1 - 2z^2 - 2z^3 - ... - 2z^{\frac{n}{2}+1}}. \end{aligned}$$

Let $P(z)$ be the numerator and $Q(z)$ the denominator of this fraction. We know that $z = -1$ is a root of P but not of Q. Then the common roots of P and Q are precisely the common roots of $P'(z) = 1 + 2z + 2z^2 + ... + 2z^{n/2}$ and Q.
Since $GCD(P'(z), Q(z)) = GCD\left((P' + Q)(z), Q(z)\right)$ and $(P'+Q)(z) = 2\left(1 + z - z^{\frac{n}{2}+1}\right)$, then all common roots of P and Q cannot be in $]0, 1]$ because $1 + z > 1$ and $z^{\frac{n}{2}+1} \leq 1$. In the same way, all common roots of P and Q cannot be in $[-1, 0[$ because $1 - z^{\frac{n}{2}+1} > 1$ and $z < 1$.
Since P and Q have no common roots with modulus less than 1 and $Q(0) = 1$ and $Q(1) < 0$, then the smallest root of $Q(z)$ is strictly smaller than 1 and is exactly the same that the smallest root of

$$(1 - z) \cdot Q(z) = 1 - z - 2z^2 + 2z^{\frac{n+4}{2}}.$$

Suppose now that $p \geq 3$. By Theorem 1 in the article of A. Talambutsa (c.f. [56], page 297), we have that the minimal spherical growth rate $\Omega(C_2 * C_n)$ is attained with the standard generating set. By Example 2.15 we know, if C_2 and C_n is generated by the standard generating set, that

$$\begin{aligned} \Gamma_2(z) &= A_2(z) = 1 + z \\ \Gamma_n(z) &= A_n(z) = 1 + 2z + 2z^2 + ... + 2z^{\frac{n-1}{2}}. \end{aligned}$$

Then $\omega(C_2, X) = \gamma(C_2, X)$ and $\omega(C_n, X) = \gamma(C_n, X)$. In particular,

$$\gamma(C_2 * C_n) \geq \omega(C_2 * C_n) = \omega(C_2 * C_n, X) = \gamma(C_2 * C_n, X).$$

Then the minimal geodesic growth rate is attainable with the standard generating set. To find α_n, we use the same arguments as for $p = 2$;

$$\begin{aligned} \Gamma_{C_2 * C_n}(z) &= \frac{\Gamma_2(z) \cdot \Gamma_n(z)}{\Gamma_2(z) + \Gamma_n(z) - \Gamma_2(z) \cdot \Gamma_n(z)} \\ &= \frac{(1+z)(1+2z+2z^2+\ldots.+2z^{\frac{n-1}{2}})}{1 - 2z^2 - 2z^3 - \ldots - 2z^{\frac{n+1}{2}}}. \end{aligned}$$

Let $P(z)$ be the numerator and $Q(z)$ be the denominator of this fraction. We know that $z = -1$ is a root of P but not of Q. Then the common roots of P and Q are precisely the common roots of $P'(z) = 1 + 2z + 2z^2 + \ldots + 2z^{\frac{n-1}{2}}$ and Q.
Since $GCD(P'(z), Q(z)) = GCD\left((P' + Q)(z), Q(z)\right)$ and $(P'+Q)(z) = 2\left(1 + z - z^{\frac{n+1}{2}}\right)$, then all common roots of P and Q cannot be in $]0, 1]$ because $1 + z > 1$ and $z^{\frac{n+1}{2}} \leq 1$. In the same way, all common roots of P and Q cannot be in $[-1, 0[$ because $1 - z^{\frac{n+1}{2}} > 1$ and $z < 1$.

Since P and Q have no common roots with modulus less than 1 and $Q(0) = 1$ and $Q(1) < 0$, then the smallest root of $Q(z)$ is strictly smaller than 1 and is exactly the same that the smallest root of

$$(1 - z) \cdot Q(z) = 1 - z - 2z^2 + 2z^{\frac{n+3}{2}}.$$

□

Theorem 2.29. *The minimal growth rate of the free product $C_3 * C_3$ is equal to 2 and is attained on the standard set of generators.*

Proof. By Example 2.15, the equality $\gamma(G, X) = 2$ holds. By Lemmas 2.24 and 2.25, the attainability is proved. □

2.2.2 Attainability

Let G be an infinite group generated by a finite set X. We denote by d_X the word metric on G with respect to X. The geodesic entropy of G with respect to X is defined by

$$GE(G, d_X) := \ln(\gamma(G, X)) = \ln(\tilde{\gamma}(G, X)),$$

whre $\tilde{\gamma}$ is defined page 12.

The (minimal) algebraic geodesic entropy of G is defined by

$$AGE(G) = \inf_X GE(G, d_X).$$

Notice that $GE(G, d_X) = \lim_{n\to\infty} \frac{\ln(\gamma_X(n))}{n}$. Thus $GE(G, d_X) \geq 0$ and if G, with respect to X, has exponential geodesic growth, then $GE(G, d_X) > 0$. Moreover, $AGE(G)$ is achieved if and only if $\gamma(G)$ is achieved.

For example, let $\mathbb{F}_k$ be the free group of rank $k \geq 2$, and let X be its standard generating set. Then $GE(\mathbb{F}_k, d_X) = \ln(2k - 1)$ and, furthermore, if G is a group generated by X, then $GE(G, d_X) \leq \ln(2k - 1)$.

From these definitions we can state an analogue, in Corollary 2.33, to the results of Sambusetti in [51] and [50] for free products.

Let $C_2 = \{1, a\}$ be the cyclic group of order 2 and let $d_l : C_2 \times C_2 \to \mathbb{R}_{\geq 0}$ be a map defined by

$$\begin{aligned} d_l(1,1) &= d_l(a,a) = 0, \\ d_l(1,a) &= d_l(a,1) = l, \end{aligned}$$

where $l \in \mathbb{N}^*$ is fixed. Then d_l is a metric on C_2 and we say that a has weight l.

Consider the group $G * C_2$ generated by the set $X \cup \{a\}$. For all $g \in G * C_2$, there is an integer $n \geq 1$ and non-trivial elements $g_1, g_2, ..., g_{n+1} \in G$, except perhaps g_1 and g_{n+1}, and $a \in C_2$ such that

$$g = g_1\, a\, g_2\, a\, \ldots\, a\, g_{n+1}.$$

The product metric $d_X * d_l$ is the left invariant distance associated to the norm

$$\|g_1\, a\, g_2\, a\, \ldots\, a\, g_{n+1}\|_{d_X * d_l} = \sum_{i=1}^{n+1} \|g_i\|_{d_X} + nl.$$

One can see it as a word length (or weight) where each generator $x \in X$ has weight 1 and a has weight l. If $a = b^l$ for some letter b and $\mathcal{L}$ is the set of non empty geodesics in G, we can extend the definition of the geodesic entropy seen on page 32 to

$$GE(G * C_2,\, d_X * d_l) := \lim_{n \to \infty} \frac{\ln(s(n))}{n},$$

where $s(n)$ is the number of words in

$$\mathbb{L} := (\mathcal{L}\, b^l)^* \cup b^l(\mathcal{L}\, b^l)^* \cup (\mathcal{L}\, b^l)^* \mathcal{L} \cup (b^l\, \mathcal{L})^* \tag{2.5}$$

of length n with respect to the letters in $X \cup X^{-1} \cup \{b\}$.

Proposition 2.30. *Let G be a group with generating set X and word metric d_X. Suppose that $GE(G, d_X) = h > 0$. Then, for all $l \in \mathbb{N}^*$, we have*

$$GE(G * C_2,\, d_X * d_l) \geq h + \frac{\ln\left(1 + e^{-hl}\right)}{4l}$$

Proof. Denote by $\mathcal{L}(R)$ the subset of $\mathcal{L}$ of words over X of length $\leq R$. Let $A(R) = \mathcal{L}(R) \setminus \mathcal{L}(R-1)$. Then $\gamma_X(R) = |A(R)|$. Let moreover $\mathbb{L}(R)$ denotes the set of words of length $\leq R$ in $\mathbb{L}$, and let $\mathbb{L}_n(R)$ be its subset of elements of the form $g_1\, b^l\, g_2\, b^l\, \ldots\, g_n\, b^l$ with g_i non-trivial for each i. Finally, set $R_i = (4k_i - 1)l$ for $k_i \in \mathbb{N}^*$.

If $g = g_1\, b^l\, g_2\, b^l\, \ldots\, g_n\, b^l$ belongs to the subset $A(R_1)\, b^l\, A(R_2)\, b^l\, \ldots\, A(R_n)\, b^l$ and $\sum_{i=1}^{n} k_i = N$, then g has length $\leq 4lN$. Therefore we have the decomposition

$$\mathbb{L}(4lN) \supset \bigcup_{k \geq 1} \mathbb{L}_k(4lN) \supset \bigcup_{n \geq 1} \bigcup_{\substack{k_1, \ldots, k_n \geq 1 \\ \sum k_i = N}} A(R_1)\, b^l\, A(R_2)\, b^l\, \ldots\, A(R_n)\, b^l.$$

Since $R_i > 0$ and $A(R_i) \cap A(R_j) = \emptyset$ if $i \neq j$, these unions are disjoint.

By the definition of geodesic entropy, the geodesic growth rate of G is e^h. From Proposition 1.7, $\gamma_X(R) \geq e^{hR}$ for all $R \in \mathbb{N}^*$. Then

$$\begin{aligned} |\mathbb{L}(4lN)| &\geq \sum_{k=1}^{N} \sum_{\substack{k_1,\dots,k_n \geq 1 \\ \sum k_i = N}} \gamma(R_1) \dots \gamma(R_n) \geq \sum_{k=1}^{N} \sum_{\substack{k_1,\dots,k_n \geq 1 \\ \sum k_i = N}} e^{R_1 h} \dots e^{R_n h} \\ &= \sum_{k=1}^{N} \sum_{\substack{k_1,\dots,k_n \geq 1 \\ \sum k_i = N}} e^{4k_1 hl} \dots e^{4k_n hl} \cdot e^{-hln} \\ &= \sum_{k=1}^{N} \sum_{\substack{k_1,\dots,k_n \geq 1 \\ \sum k_i = N}} e^{4hlN} \cdot e^{-hln} \\ &= e^{4hlN} \sum_{n=1}^{N} \binom{N-1}{n-1} \cdot e^{-hln} = e^{4hlN - hl} \cdot \left(1 + e^{-hl}\right)^{N-1} \end{aligned}$$

and, therefore,

$$GE\,(G * C_2,\, d_X * d_l) \geq \lim_{N\to\infty} \frac{\ln(\,|\mathbb{L}(4lN)|\,)}{4lN} = h + \frac{\ln\left(1 + e^{-hl}\right)}{4l}.$$

□

When N is a subgroup of G we can give the left coset space G/N the quotient metric, that is, the G-invariant distance

$$\overline{d_X}\,(\,gN,\, g'N) := \inf_{h,h' \in N} d_X\,(\,gh,\, g'h') = d_X\,(\,N,\, g^{-1}g'N).$$

Notice that if N is normal, then $\overline{d_X}$ is the word metric $d_{\overline{X}}$, where $\overline{X}$ denotes the generating set induced by X on the quotient.

Then we have the following proposition given by A. Sambusetti in [51].

Proposition 2.31 (Sambusetti, [51, Proposition 2.4])**.** *Let $G = H*K \neq C_2*C_2$ be a non-trivial free product, X a generating set of G and d_X the associated word metric. For any non-trivial normal subgroup $N \triangleleft G$, there exists an injective map $\phi : (G/N * C_2, d_{\overline{X}} * d_l) \hookrightarrow (G, d_X)$ (not necessarily an homomorphism) such that $d_X(\phi(x), \phi(y)) \leq (d_{\overline{X}} * d_l)(x, y)$ for all $x, y \in G/N * C_2$, for $l \in \mathbb{N}^*$ big enough.*

Propositions 2.30 and 2.31 imply the geodesic growth tightness of free products. We say that G, with respect to X, is geodesic growth tight if for every infinite normal subgroup $N \triangleleft G$ one has

$$GE\,(G, d_X) > GE\,(G/N,\, d_{\overline{X}}).$$

Theorem 2.32. *Every non-trivial free product $G = H * K \neq C_2 * C_2$ is geodesic growth tight with respect to any generating set.*

Proof. Let $G = H * K \neq C_2 * C_2$ be a non-trivial free product with generating set X and d_X its associated word metric. Let $N \triangleleft G$ be any non-trivial normal subgroup of G. Finally, let $C_2 = \{1, a\}$ be the cyclic group of order 2 with the metric d_l defined before where $l \gg 0$.

Since $G = H * K$ is different from $C_2 * C_2$, G has exponential spherical growth of rate at least $\sqrt{2}$ ([43], page 167, Theorem 16.12), and the same holds for the geodesic growth.

We may assume that $GE(G/N, d_{\overline{X}}) = h > 0$, otherwise the inequality $GE\,(G, d_X) > GE\,(G/N,\, d_{\overline{X}})$ is trivial.

From Proposition 2.30, we have

$$GE(G/N * C_2,\, d_{\overline{X}} * d_l) \;\geq\; h + \frac{\ln\left(1 + e^{-hl}\right)}{4l} \;>\; GE(G/N, \overline{d_X}). \qquad (2.6)$$

From Proposition 2.31 there is an injective map ϕ from $(G/N * C_2, d_{\overline{X}} * d_l)$ to (G, d_X) such that $d_X(\phi(x), \phi(y)) \leq (d_{\overline{X}} * d_l)(x, y)$ for all $x, y \in G/N * C_2$, for $l \in \mathbb{N}^*$ big enough. Then the subset of geodesics of length $\leq R$ in G, with respect to the generating set X, contains more elements than the number of words of length $\leq R$, with respect to the letters in $X \cup X^{-1} \cup \{b\}$, in $\mathbb{L}$ defined in (2.5), where $\mathcal{L}$ is the set of non empty geodesics in G/N. Then

$$GE(G, d_X) \;\geq\; GE(G/N * C_2, d_{\overline{X}} * d_l), \qquad (2.7)$$

which together with equation (2.6) shows that G is geodesic growth tight. □

An important remark about Theorem 2.32 is that the proof of

$$GE(G/N * C_2,\, d_{\overline{X}} * d_l) \;>\; GE(G/N, d_{\overline{X}})$$

could be generalised to all groups G of exponential geodesic growth, which is not the case for the inequality (2.7). Indeed, in the proof of Proposition 2.31 in the article [51], Sambusetti uses a normal form of words in a free product which leads to the definition of two words which "match well". This definition cannot be generalised to all groups of exponential geodesic growth, so we cannot generalise Theorem 2.32 to all of these groups.

However, we obtain an interesting corollary. A group G is Hopfian if every epimorphism $G \to G$ is an isomorphism, or equivalently, if G is not isomorphic to a proper quotient of itself.

Corollary 2.33. *Every non-trivial free product $G = H * K \neq C_2 * C_2$ whose minimal geodesic growth rate is achieved is Hopfian.*

Proof. Let $G = H * K \neq C_2 * C_2$ be a non-trivial free product and X a generating set of G such that $AGE(G) = GE(G, X)$. If G is not Hopfian there is an isomorphism $\varphi : G \to G/N$ with $N \lhd G$ non-trivial. By geodesic growth tightness,

$$GE(G, \varphi(X)) \;=\; GE(G/N, \overline{X}) \;<\; GE(G, X),$$

which is a contradiction. □

A consequence of this corollary is that the free product

$$\mathbb{B}(2,3) \,*\, C_2 \;=\; \left\langle a, b, c \,|\, a^{-1}b^2a = b^3,\, c^2 = 1 \right\rangle$$

does not realise the minimal geodesic growth rate since the Baumslag-Solitar group $\mathbb{B}(2,3)$ is non Hopfian. More precisely, let $\phi : G \to G$ be the homomorphism such that $\phi(a) = a$, $\phi(b) = b^{-1}a^{-1}ba$, $\phi(c) = c$. By geodesic growth tightness,

$$GE(G, \phi(X)) \;=\; GE(G/Ker(\phi), \overline{X}) \;<\; GE(G, X),$$

for $X = \{a, b, c\}$. In particular, defining by recursion $X_n := \phi(X_{n-1})$ for all $n \geq 1$ and $X_0 := X$, the geodesic entropies $GE(G, X_n)$ form a strictly decreasing sequence.

2.3 Wreath Product

We start by introducing semidirect products. Let $G = \langle X|R\rangle$ and $A = \langle Y|S\rangle$ be two finitely generated groups such that G acts on A via the homomorphism $\alpha : G \to Aut(A)$.

The semi-direct product of A and G with respect to α is defined by

$$A \rtimes_\alpha G \;=\; \Big\langle\, X, Y \mid R, S,\ xyx^{-1} = \alpha_x(y)\ \ \forall\, x \in X, y \in Y \,\Big\rangle.$$

For example, suppose that $A = \mathbb{Z} = \langle a|\_\rangle$ and $G = \mathbb{Z} = \langle b|\_\rangle$. There exist only 2 automorphisms of $\mathbb{Z}$: the identity $id : x \mapsto x$ and the inverse $inv : x \mapsto x^{-1}$. This implies that there exist only two semi-direct products of $\mathbb{Z}$ and $\mathbb{Z}$, defined by

$$S_1 \;=\; \mathbb{Z} \rtimes_{id} \mathbb{Z} \;=\; \Big\langle\, a, b \mid bab^{-1} = a \,\Big\rangle \;=\; \mathbb{Z}^2$$

and

$$S_2 = \mathbb{Z} \rtimes_{inv} \mathbb{Z} = \Big\langle\, a, b \mid bab^{-1} = a^{-1} \,\Big\rangle.$$

We note that A is abelian if and only if the inverse $inv : A \to A$, defined by $inv(x) = x^{-1}$ for all $x \in A$, is an automorphism. Then, if A is abelian, $A \rtimes_{id} G$ and $A \rtimes_{inv} G$ are two possible semi-direct products of A and G.

Theorem 2.34. *Let G and A be two groups generated by finite and symmetric sets X and Y, respectively. Suppose that there exists a homomorphism $\alpha : G \to Aut(A)$ defined by $x \mapsto \alpha_x$, where $\alpha_x : A \to A$ is the extension of a bijection $Y \to Y$, for all $x \in X$. Then*

$$\gamma(A \rtimes_\alpha G,\ X \cup Y\,) \;=\; \gamma(A \times G,\ X \cup Y\,) \;=\; \gamma(G, X) \;+\; \gamma(A, Y).$$

Proof. For an element $h \in A \rtimes_\alpha G$ of length n, there exist a geodesic $a \in A$ of length k and a geodesic $g \in G$ of length $n - k$ such that $h = ag$.

If $\alpha_x : A \to A$ is the unique homomorphism which is the extension of a bijection $\sigma_x : Y \to Y$ for $x \in X$, then the standard presentation of the semi-direct product $A \rtimes_\alpha G$ is of the form

$$\Big\langle\, X, Y \mid R,\ S,\ x^{-1}y_1x = y_2\ \ \forall x \in X,\ y_1 \in Y \,\Big\rangle,$$

where $Y \ni y_2 = \sigma_x(y_1)$ for all $y_1 \in Y$, $x \in X$. Then all the geodesics which represent h are of the form

$$a_1\ g_1\ a_2\ \ldots\ g_{r-1}\ a_r\ g_r$$

with $\min\{k, n-k\} \geq r > 0$, where $g_1\, g_2 \ldots g_r \;=\; g,\ \sum_{i=1}^{r} l_X(g_i) \;=\; n - k\ ,\ \sum_{i=1}^{r} l_Y(a_i) \;=\; k$ and

$$a_1 \cdot \alpha_{g_1}^{-1}(a_2) \cdot \alpha_{g_1}^{-1}(\alpha_{g_2}^{-1}(a_3)) \cdot \ldots \cdot \alpha_{g_1}^{-1}\Big(\ldots(\alpha_{g_{r-1}}^{-1}(a_r))\ldots\Big) \;=\; a.$$

We remark that for all geodesics $a \in A$ and $g \in G$, $ag = g\alpha_g(a)$ implies that $l_Y(a) = l_Y(\alpha_g(a))$. Indeed, if $l_Y(a) > l_Y(\alpha_g(a))$, then there exists another geodesic $a_2 \in G$ such that $ag \;=_{A\rtimes_\alpha G}\; a_2g$, ag and a_2g are geodesics of $A \rtimes_\alpha G$ and $l_{X\cup Y}(ag) > l_{X\cup Y}(a_2g)$. Then by Proposition 2.2,

$$\begin{aligned}\gamma_{A\rtimes_\alpha G,\, X\cup Y}(n) \;&=\; \sum_{k=0}^{n} \binom{n}{k} \gamma_Y(k) \cdot \gamma_X(n-k) \\ &=\; \gamma_{A\times G,\, X\cup Y}(n).\end{aligned}$$

□

For example, as *id* and *inv* are extensions of a bijection, the geodesic growth rates of S_1 and S_2 are 2.

We look now at a particularly important type of semi-direct product. Let $G = \langle X|R\rangle$ and $A = \langle Y|S\rangle$ be two finitely generated groups such that G acts on A. The wreath product of A and G, denoted by $A \wr G$, is defined by

$$A \wr G \;=\; \left(\bigoplus_{h\in G} A\right) \rtimes G. \tag{2.8}$$

For any element $w \in A \wr G$ there are a finite subset S of G, $g \in G$, and a map $h : S \to A$ such that

$$w \;=\; \left(\prod_{s\in S} s\, h(s)\, s^{-1}\right) \cdot g. \tag{2.9}$$

The form (2.9) is unique up to the ordering of S chosen in forming the product $\prod$. To ensure uniqueness, we give an ordering $(s_1, s_2, ..., s_n)$ to $S = \{s_1, ..., s_n\}$ in such a way that the integer m given by

$$m \;=\; l_X(s_1) \;+\; \sum_{i=1}^{n-1} l_X(s_i^{-1}s_{i+1}) \;+\; l_X(s_n^{-1}g) \tag{2.10}$$

is minimal. Notice that the minimality of m depends only on S and g, not on h.

In [37], D.L. Johnson proved that the form given in (2.9), subject to the minimality of (2.10), is a minimal length normal form of words representing elements in $A \wr G$.

Interesting examples of wreath products are the Lamplighter groups. For all $m \geq 2$, the Lamplighter group L_m is defined by

$$C_m \wr \mathbb{Z} \;=\; \left(\bigoplus_{h\in\mathbb{Z}} C_m\right) \rtimes \mathbb{Z},$$

where each element

$$S = (..., s_{-2}, s_{-1}, s_0, s_1, s_2, s_3, ...) \in \bigoplus_{h\in\mathbb{Z}} C_m,$$

$s_i \in C_m$ satisfies that $s_i = 0$ for all $i \in \mathbb{Z}$ except for finitely many.

The internal operation in L_m is given by

$$(S, n) \cdot (T, m) \;=\; (S \oplus (T + n)\,,\, n + m)\,,$$

where $S \oplus T$ is the operation defined elementwise in $\bigoplus_{h\in\mathbb{Z}} C_m$, and $T + n$ is the translation $T + n \;=\; \{t_{i+n}\}_{i\in\mathbb{Z}}$, where $T \;=\; \{t_i\}_{i\in\mathbb{Z}}$.

To simplify the notation, we denote by S the subset of $\mathbb{Z}$ such that $i \in S$ if and only if $s_i \neq 0$.

Let $t = (\emptyset, 1)$ and $a = (\{0\}, 0)$. Then we have that $ta = (\{1\}, 1)$ and $t^n a t^{-n} = (\{n\}, 0)$, and so a and t generate L_m, and the standard presentation of L_m is defined by

$$L_m \;=\; \left\langle\, a, t \mid a^m = 1, [t^i a t^{-i}\,,\, t^j a t^{-j}] = 1 \;\; \forall i, j \in \mathbb{Z} \,\right\rangle.$$

Observe that L_m is a finitely generated but not finitely presented group.

For all $m \geq 2$, we denote by a_k^i all elements of the form $a_k^i = t^k a^i t^{-k}$ for all $k \in \mathbb{Z}$ and $|i| \leq \lfloor \frac{m}{2} \rfloor$. Each element $w = (S, r) \in L_m$, where $|S| = n$, is represented by reduced words of the form

$$w \quad = \quad a_{k_1}^{i_1}\, a_{k_2}^{i_2}\, \dots\, a_{k_n}^{i_n}\, t^r,$$

where $k_j \in S$ for all $j = 1, 2, ..., n$, $k_p \neq k_q$ for all $p \neq q$, and $\frac{m}{2} < i_j \leq \frac{m}{2}$.

The "Right first" normal form and "Left first" normal form are given by

$$RF(w) \quad = \quad a_{k_1}^{i_1}\, a_{k_2}^{i_2}\, \dots\, a_{k_l}^{i_l}\, a_{-j_1}^{i_{l+1}}\, a_{-j_2}^{i_{l+2}}\, \dots\, a_{-j_n}^{i_{l+n}}\, t^r$$

and

$$LF(w) = a_{-j_1}^{i_{l+1}}\, a_{-j_2}^{i_{l+2}}\, \dots\, a_{-j_n}^{i_{l+n}} a_{k_1}^{i_1}\, a_{k_2}^{i_2}\, \dots\, a_{k_l}^{i_l}\, t^r,$$

where $r \in Z$, $0 \leq k_1 < k_2 < ... < k_l$, $0 \leq j_1 < j_2 ... < j_m$ and $|k_p| \leq \lfloor \frac{m}{2} \rfloor$, and we may omit $a^{-\frac{m}{2}}$ to unsure uniqueness if m is even, for all p.

Note that the sign of r in these normal forms implies the existence (or not) of a geodesic:

- If $r > 0$, then every word w' in Left first normal form is a geodesic. Moreover, a word w in Right first normal form is a geodesic if and only if $m = 0$.
- If $r < 0$, then every word w in Right first normal form is a geodesic. Moreover, a word w' in Left first normal form is a geodesic if and only if $l = 0$.
- If $r = 0$, then every word w in Right first normal form and every word w' in Left first normal form are geodesics.

However, geodesics are not unique in L_2 and not every geodesic is necessarily in normal form. For example, the words $tatatat^{-3} = a_1 a_2 a_3$ and $t^3 at^{-1}at^{-1}at^{-1} = a_3 a_2 a_1$ are two different geodesics which represent the element $(\{1, 2, 3\}; 0)$.

Since every geodesic is of the form $a_{k_1}^{i_1}\, a_{k_2}^{i_2}\, \dots\, a_{k_n}^{i_n}\, t^r$, where $k_j \in \mathbb{Z}$, and since every geodesic of this form is the image of a geodesic by the morphism

$$\begin{array}{rcl} \phi : L_m & \to & L_m \\ a & \mapsto & a \\ t & \mapsto & t^{-1}, \end{array}$$

then it suffices to understand the subset of geodesics which begin with $a_{k_1}^{i_1}$ where $k_1 \geq 0$.

The next proposition provides upper bounds of the geodesic growth rates of L_m.

Proposition 2.35. *With respect to the generating set $X = \{a, t\}$, the geodesic growth rate of L_m satisfies $\gamma(L_2, X) \leq 2$ and $\gamma(L_m, X) \leq 3$ for all $m > 2$. In particular,*

$$\gamma(L_3, X) \quad \leq \quad \frac{1 + \sqrt{17}}{2} \quad \cong \quad 2.56155.$$

Proof. Let $m \geq 2$ be an integer. Since a is of order m in L_m, we have

$$\gamma(L_m, X) \quad \leq \quad \gamma(C_m * \mathbb{Z}, X).$$

From Proposition 2.10, we have that

$$\Gamma(\mathbb{Z} * C_2, X) \quad = \quad \frac{(1 + z)^2}{1 - z - z^2}$$

and

$$\Gamma(\mathbb{Z} * C_3, X) \quad = \quad \frac{(1 + z)(1 + 2z)}{1 - z - 4z^2},$$

which have $\frac{1}{2}$ and $\frac{\sqrt{17}-1}{8}$ as radius of convergence, respectively. □

Now, we study the cases $m = 2$ or 3 in more detail and count the number of geodesics w in L_m of length n such that

- w represents an element $(S, r) \in L_m$, where $S \subset \mathbb{N}$ is of size k, and $r \geq 0$;
- $|w|_t = p \geq 0$;
- $|w|_{t^{-1}} \geq 0$.

We can represent such a word visually as in Figure 2.1, where yellow circles are the k lamps turned on and the red path is the path followed by w.

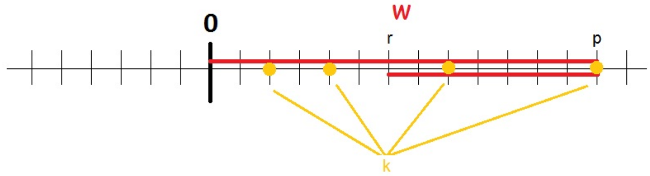

Figure 2.1: Representation of w, with k lamps.

For $m = 2$ or 3, we define the following sets

$$\begin{aligned} A_m(n,k,p) &= \{w \in Geo(L_m, \{a,t\}) \mid w \text{ is as before with } n,k,p \text{ fixed}\} \\ A_m(n,k) &= \{w \in Geo(L_m, \{a,t\}) \mid w \in A_m(n,k,p) \text{ for } p \geq 0\} \\ A_m(n) &= \{w \in Geo(L_m, \{a,t\}) \mid w \in A_m(n,k) \text{ for } k \geq 0\} \end{aligned}$$

We remark that if n, k and p are fixed, then the position of the lamplighter (r) is unique and completely determined by n, k and p.

Proposition 2.36. *For all $n, k, p \geq 1$ and $k - 1 \leq p \leq \lceil \frac{n}{2} \rceil$, we have*

$$|A_2(n,k,p)| = \sum_{q=0}^{k-1} \binom{p}{k-1} 2^{k-q-1} = \binom{p}{k-1}(2^k - 1),$$

and

$$|A_3(n,k,p)| = 2^k \sum_{q=0}^{k-1} \binom{p}{k-1} 2^{k-q-1} = \binom{p}{k-1}(4^k - 2^k),$$

where $\lceil x \rceil := \inf\{n \in \mathbb{Z} \mid n \geq x\}$.

Proof. The first equality comes from the fact that we chose $k-1$ lamps in the set $\{0, 1, ..., p-1\}$ and then we decide for the $k-q-1$ rightmost ones which we visit on the way out and which we visit on the way back. The last lamp must be turned on on the way out.

For $|A_3(n,k,p)|$, we have the choice for each lamp to "turn on" a or a^{-1}. Thus we multiply the result for $\#A_2(n,k,p)$ by 2^k. ☐

Then, we have that for all $n, k \geq 1$ and $k - 1 \leq \lceil \frac{n}{2} \rceil$,

$$|A_2(n,k)| = \sum_{p=k-1}^{n-k} \binom{p}{k-1}(2^k - 1)$$

and

$$|A_3(n,k)| = \sum_{p=k-1}^{n-k} \binom{p}{k-1}(4^k - 2^k).$$

We have the following classical result.

Proposition 2.37. *For all $0 \leq n \leq m$, we have that*

$$\sum_{i=n}^{m} \binom{i}{n} = \binom{m+1}{m-n}.$$

Proposition 2.37 implies that

$$|A_2(n,k)| = \binom{n-k+1}{k}(2^k - 1)$$

and

$$|A_3(n,k)| = \binom{n-k+1}{k}(4^k - 2^k)$$

Then, by definition,

$$|A_2(n)| = \sum_{k=1}^{\lceil \frac{n}{2} \rceil} \binom{n-k+1}{k}(2^k - 1) \tag{2.11}$$

and

$$|A_3(n)| = \sum_{k=1}^{\lceil \frac{n}{2} \rceil} \binom{n-k+1}{k}(4^k - 2^k). \tag{2.12}$$

Now we study three different the above sums in Propositions 2.38, 2.39 and 2.40.

Proposition 2.38. *For all $n \geq 1$, we have*

$$\sum_{k=1}^{\lceil \frac{n}{2} \rceil} \binom{n-k+1}{k} = \mathcal{F}(n+1) - 1,$$

where $\mathcal{F}(n)$ is the Fibonacci sequence with $\mathcal{F}(0) = \mathcal{F}(1) = 1$.

Proof. We use induction: If $n = 1, 2, 3$ the equality is immediate.

We suppose that the equality is correct for some $n \geq 1$. We have two cases:

1. if n is odd, then

$$\left\lceil \frac{n+1}{2} \right\rceil = \left\lceil \frac{n}{2} \right\rceil = \left\lceil \frac{n-1}{2} \right\rceil + 1.$$

   Then

$$\begin{aligned}
\sum_{k=1}^{\lceil \frac{n+1}{2} \rceil} \binom{n-k+2}{k} &= \sum_{k=1}^{\lceil \frac{n}{2} \rceil} \binom{n-k+2}{k} = \sum_{k=1}^{\lceil \frac{n}{2} \rceil} \binom{n-k+1}{k} + \sum_{k=1}^{\lceil \frac{n}{2} \rceil} \binom{n-k+1}{k-1} \\
&= \mathcal{F}(n+1) - 1 + \sum_{k=0}^{\lceil \frac{n}{2} \rceil - 1} \binom{n-k}{k} \\
&= \mathcal{F}(n+1) + \sum_{k=1}^{\lceil \frac{n}{2} \rceil - 1} \binom{n-k}{k} \\
&= \mathcal{F}(n+1) + \mathcal{F}(n) - 1 = \mathcal{F}(n+2) - 1
\end{aligned}$$

2. if n is even, then

$$\left\lceil \frac{n+1}{2} \right\rceil \;=\; \left\lceil \frac{n}{2} \right\rceil + 1 \;=\; \left\lceil \frac{n-1}{2} \right\rceil + 1.$$

This implies that

$$\begin{aligned}
\sum_{k=1}^{\lceil \frac{n+1}{2} \rceil} \binom{n-k+2}{k} &= \sum_{k=1}^{\lceil \frac{n}{2} \rceil} \binom{n-k+2}{k} + 1 \\
&= \sum_{k=1}^{\lceil \frac{n}{2} \rceil} \binom{n-k+1}{k} + \sum_{k=1}^{\lceil \frac{n}{2} \rceil} \binom{n-k+1}{k-1} + 1 \\
&= \mathcal{F}(n+1) - 1 + \sum_{k=0}^{\lceil \frac{n}{2} \rceil - 1} \binom{n-k}{k} + \binom{n/2}{n/2} \\
&= \mathcal{F}(n+1) + \sum_{k=1}^{\lceil \frac{n}{2} \rceil - 1} \binom{n-k}{k} \\
&= \mathcal{F}(n+1) + \mathcal{F}(n) - 1 = \mathcal{F}(n+2) - 1
\end{aligned}$$

□

Proposition 2.39. *For all $n \geq 1$, we have*

$$\sum_{k=1}^{\lceil \frac{n}{2} \rceil} 2^k \binom{n-k+1}{k} \;=\; \mathcal{G}(n),$$

where $\mathcal{G}(n)$ is the integer sequence defined recursively by $\mathcal{G}(0) = 1$, $\mathcal{G}(1) = 2$, $\mathcal{G}(2) = 4$ and $\mathcal{G}(n) = \mathcal{G}(n-1) + 2\mathcal{G}(n-2) + 2$ for all $n \geq 3$.

Proof. It is similar to the proof of Proposition 2.38. □

Proposition 2.40. *For all $n \geq 1$, we have*

$$\sum_{k=1}^{\lceil \frac{n}{2} \rceil} 4^k \binom{n-k+1}{k} \;=\; \mathcal{H}(n),$$

where $\mathcal{H}(n)$ is the integer sequence defined recursively by $\mathcal{H}(0) = 1$, $\mathcal{H}(1) = 4$, $\mathcal{H}(2) = 8$ and $\mathcal{H}(n) = \mathcal{H}(n-1) + 4\mathcal{H}(n-2) + 4$ for all $n \geq 3$.

Proof. It is similar to the proof of Proposition 2.38. □

Propositions 2.38, 2.39 and 2.40 and equations (2.11) and (2.12) imply that

$$|A_2(n)| \;=\; \mathcal{G}(n) - \mathcal{F}(n+1) + 1$$

and

$$|A_3(n)| \;=\; \mathcal{H}(n) - \mathcal{G}(n)$$

for all $n \geq 1$, and $|A_2(0)| = |A_3(0)| = 1$.

If we defined the integer sequence $f : \mathbb{N} \to \mathbb{N}$ by $f(n) := \mathcal{F}(n+1) - 1$ for all $n \geq 0$, we have then that the generating functions of $f(n)$, $\mathcal{G}$ and $\mathcal{H}$ have the following formulas.

Proposition 2.41. *The generating functions of the integer sequences $f(n)$, $\mathcal{G}(n)$ and $\mathcal{H}(n)$ are given by*

$$\mathbf{F}(z) = \frac{z^2}{(1-z)(1-z-z^2)},$$

$$\mathbf{G}(z) = \frac{1-z^2+2z^3}{(1-z)(1-z-2z^2)},$$

and

$$\mathbf{H}(z) = \frac{1+2z-3z^2+4z^3}{(1-z)(1-z-4z^2)},$$

respectively.

Proof. By recursion, we have that

$$\begin{aligned}
\mathbf{F}(z) &= f(0)+f(1)z+\sum_{k\geq 2} f(k)z^k = \sum_{k\geq 2}\left[f(k-1)+f(k-2)+1\right]z^k \\
&= z\cdot\sum_{k\geq 1} f(k)z^k + z^2\sum_{k\geq 0} f(k)z^k + \sum_{k\geq 2} z^k \\
&= z\cdot(\mathbf{F}(z)-f(0))+z^2\cdot(\mathbf{F}(z)) + \frac{1}{1-z}-1-z = \frac{z^2}{1-z} + \mathbf{F}(z)\cdot(z+z^2),
\end{aligned}$$

which gives the generating function of $f(n)$.

The generating functions of $\mathcal{G}(n)$ and $\mathcal{H}(n)$ follow similarly.

□

Proposition 2.41 implies the following.

Proposition 2.42. *The following hold:*

$$\limsup_{n\to\infty} \sqrt[n]{|A_2(n)|} = 2,$$

and

$$\limsup_{n\to\infty} \sqrt[n]{|A_3(n)|} = \frac{1+\sqrt{17}}{2}.$$

Proof. The generating functions of $|A_2(n)|$ and $|A_3(n)|$ are given by

$$\begin{aligned}
\mathbf{A}_2(z) &:= \sum_{n\geq 0}|A_2(n)|\,z^n = |A_2(0)| + \sum_{n\geq 1}|A_2(n)|\,z^n = 1+\sum_{n\geq 1}(\mathcal{G}(n)-f(n))\,z^n \\
&= \sum_{n\geq 0}(\mathcal{G}(n)-f(n))\,z^n = \mathbf{G}(z) - \mathbf{F}(z) \\
&= \frac{1-z^2+2z^3}{(1-z)(1-z-2z^2)} - \frac{z^2}{(1-z)(1-z-z^2)} \\
&= \frac{1-z-3z^2+4z^3+z^4-2z^5}{(1-z)(1-z-2z^2)(1-z-z^2)}
\end{aligned}$$

and

$$\begin{aligned}
\mathbf{A}_3(z) &:= \sum_{n\geq 0}|A_3(n)|\,z^n = \sum_{n\geq 0}(\mathcal{H}(n)-f(n))\,z^n \\
&= \mathbf{H}(z) - \mathbf{G}(z) = \frac{1+2z-3z^2+4z^3}{(1-z)(1-z-4z^2)} - \frac{1-z^2+2z^3}{(1-z)(1-z-2z^2)} \\
&= \frac{2z}{(1-z-2z^2)(1-z-4z^2)},
\end{aligned}$$

respectively.

Then the radii of convergence of $\mathbf{A}_2(z)$ and $\mathbf{A}_3(z)$ are $\frac{1}{2}$ and $\frac{\sqrt{17}-1}{8}$, respectively. □

We have then the following theorem.

Theorem 2.43. *With respect to the generating set $X = \{a, t\}$, the geodesic growth rates of L_2 and L_3 satisfy $\gamma(L_2, X) = 2$ and $\gamma(L_3, X) = \frac{1+\sqrt{17}}{2}$, respectively.*

Proof. Since $\gamma_{L_2}(n) \geq |A_2(n)|$ and $\gamma_{L_3}(n) \geq |A_3(n)|$ for all $n \geq 0$, the result follows from Proposition 2.42. □

A direct consequence of this corollary is that $\gamma(L_m, X) \geq \frac{1+\sqrt{17}}{2}$ for all $m \geq 3$. For all $m \geq n \geq 2$, we have that $\gamma(L_m, \{a,t\}) \geq \gamma(L_n, \{a,t\})$. In particular, we have the following Proposition.

Proposition 2.44. *For all $n \geq 2$, we have that $\gamma(L_{2n}, \{a,t\}) = \gamma(L_{2n+1}, \{a,t\})$.*

Proof. We have that $\gamma(L_{2n}, \{a,t\}) \leq \gamma(L_{2n+1}, \{a,t\})$.
To prove the equality, we need to show that every geodesic of L_{2n+1} is a geodesic of L_{2n}. But every geodesic of L_{2n+1} is of the form $a_{i_1}^{k_1} a_{i_2}^{k_2} \ldots a_{i_p}^{k_p} t^r$, where $a_i^j = t^i a^j t^{-i}$, $i_l \in \mathbb{Z}^*$ and $k_l \in \{-n, -n+1, ..., 0, 1, ..., n\}$ for all $l = 1, ..., p \geq 0$. This is exactly the same definition of geodesics of L_{2n}, then every geodesic of L_{2n+1} is a geodesic of L_{2n}. □

Now, we try to find the geodesic growth rate of L_m. To have a lower bound, we count the number of geodesics w in L_m such that

- w represents an element $(S, r) \in L_m$, where $S \subset \mathbb{N}$ is of size k, and $r \geq 0$;
- $|w|_t = p$;
- $\sum_{i=1}^{k} |s_i| = L$;

by generalizing the approach for L_2 and L_3. We can represent such a word visually as in Figure 2.2.

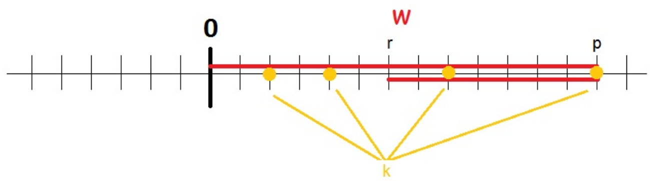

Figure 2.2: Representation of $(S, r) \in L_m$, $S \subset \mathbb{N}$ is of size k, and $r \geq 0$.

Denote now

$$\begin{aligned}
A_m(n,k,p,L) &= \{w \in Geo(L_m, \{a,t\}) \mid w \text{ is as before with } n,k,p \text{ fixed}\} \\
A_m(n,k,L) &= \{w \in Geo(L_m, \{a,t\}) \mid w \in A_m(n,k,p,L) \text{ for one } p \geq 0\} \\
A_m(n,k) &= \{w \in Geo(L_m, \{a,t\}) \mid w \in A_m(n,k,L) \text{ for one } L \geq k\} \\
A_m(n) &= \{w \in Geo(L_m, \{a,t\}) \mid w \in A_m(n,k) \text{ for one } k \geq 0\}
\end{aligned}$$

We remark that if n, k, p and L are fixed, then the position of the lamplighter (r) is unique and completely defined with n, k, p and L.

Let $\{t_i\}_{i=1}^{k}$ be a list of natural numbers, which sum up to a natural number n. It is called an integer composition of n. The set of all such lists, where the ordering of the summands matters, is the set of all integer compositions of n. Let $\mathcal{C}(n, k, b)$ denote the number of compositions of n such that summands t_i are natural numbers, with $1 \leq t_i \leq b$ for all i.

Proposition 2.45. *For all $n, k, p, L \geq 1$ and $k-1 \leq p \leq n-L$, we have*

$$|A_{2m}(n,k,p,L)| \;=\; \sum_{q=0}^{k-1} \binom{p}{k-1} 2^{k-q-1}\, \mathcal{C}(L,k,m)\, 2^k \;=\; \binom{p}{k-1} (4^k - 2^k)\, \mathcal{C}(L,k,m).$$

Proof. The first equality comes from the fact that we choose $k-1$ lamps in the set $\{0, 1, ..., p-1\}$, each positive or negative, for which the $k-q-1$ rightmost ones could be turned on on the way out or on the way back. Finally, we multiply the result by the number of compositions of L into the k lamps such that each lamp has length at most m. □

To finish this chapter, since from Theorem 2.43 we have $\gamma(L_2, X) = \gamma(\mathbb{Z} * C_2, \{a, b\})$ and $\gamma(L_2, X) = \gamma(\mathbb{Z} * C_2, \{a, b\})$ and since we know that $\gamma(L_{2m}, \{a, t\}) = \gamma(L_{2m+1}, \{a, t\})$ for all $m \geq 2$ from Proposition 2.44, we would like to prove that the geodesic growth rate of L_{2m} with respect to this generating set is the same as the geodesic growth rate of $\mathbb{Z} * C_m$ with respect to the standard generating set.

Conjecture 2.46. *For all $m \geq 2$, $\gamma(L_{2m}, \{a, t\}) \;=\; \gamma(C_m * \mathbb{Z}, \{a, t\})$.*

Chapter 3

Groups acting on regular rooted trees

In this chapter, we study a family of groups acting on regular rooted trees. In particular, we prove that the following groups have exponential geodesic growth.

1. The Grigorchuk groups $\mathcal{G}_\omega$;
2. The Gupta-Sidki p-groups;
3. The Square group $\mathbb{S}$;
4. The Spinal groups.

Grigorchuk proved in [28] that under certain conditions, $\mathcal{G}_\omega$ are infinite torsion groups of intermediate growth. It is an open question whether the Gupta-Sidki p-groups, which belong to the "splitter-mixer" class of groups defined by Bartholdi in [3], have intermediate spherical growth or not. Furthermore, we study the Square group $\mathbb{S}$, an example proposed by R. Grigorchuk, who stated its spherical growth as an interesting open question. Finally, in [6], Bartholdi and Šunik proved that all Spinal groups have intermediate spherical growth.

For all these groups, the proof showing their exponential geodesic growth uses the Schreier graphs $\mathcal{S}_n$ corresponding to the level-transitive action of each group on a k-regular rooted tree. We generalise the idea of Elder, Gutierrez and Šunić in [20] that each geodesic path's label in some $\mathcal{S}_n$ is a geodesic word in the group.

Finally, we point out that our method above does not apply to the Gupta-Fabrykowski group, whose geodesic growth type is still unknown. Bartholdi and Pochon proved in [5] that the Gupta-Fabrykowski group has intermediate spherical growth.

3.1 Preliminaries

First, we recall the basic notions and facts about rooted trees, their automorphism groups and the Schreier graphs, using the notation in the book Self-Similar groups of Nekrashevych [46].

Let X be a finite set. The set X^* corresponds naturally to the vertex set of a rooted tree $\mathcal{T}$, in which two words are connected by an edge if and only if they are of the form v and

vx, where $v \in X^*$ and $x \in X$. We often use X^* to refer to the tree $\mathcal{T}$. The empty word ϵ is the root of the tree $\mathcal{T}$. See Figure 3.1 for the case $X = \{0, 1\}$.

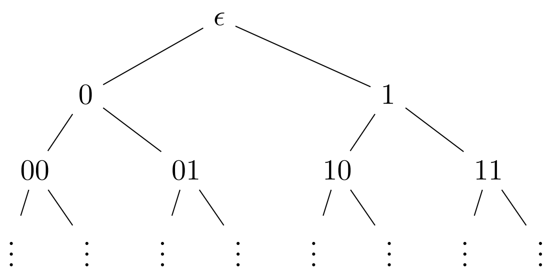

Figure 3.1: Binary rooted tree

The set $X^n \subset X^*$ of words of length n is called nth level of the tree X^*. A map $f : X^* \to X^*$ is an endomorphism of the tree X^* if it preserves the root and adjacency of the vertices. If f is an endomorphism of the tree X^*, then $f(X^n) \subseteq X^n$. An automorphism is a bijective endomorphism.

The boundary of the tree is the set of all its ends, i.e. infinite simple paths starting at some fixed vertex (e.g. at the root). The boundary of the tree $\mathcal{T}$ is naturally identified with the set ∂X (or $X^{\mathbb{N}}$) of all infinite words

$$\partial X \;=\; \{\, x_1 x_2 \ldots \mid x_i \in X\}\,.$$

The sets vX^* and $v\,\partial X$ are the sets of (respectively finite and infinite) words starting with the word $v \in X^*$.

Let G be a group acting on X^*. The image of a point $x \in X^*$ under the action of an element g is denoted $g(x)$ and in the product $g_1 g_2$ the element g_2 acts first, i.e $g_1 g_2(x) = g_1(g_2(x))$. Let us denote by $Aut(X^*)$ the group of all automorphisms of the rooted tree X^*. An action of a group G by automorphisms of the tree X^* is said to be level-transitive if it is transitive on every level X^n of the tree X^*.

If G is finitely generated by a set S, the sequence of Schreier graphs $\mathcal{S}_n = \mathcal{S}(S, X^n)$ of the level-transitive action is a sequence of graphs with $V(\mathcal{S}_n) = X^n$ and $E(\mathcal{S}_n) \subseteq S \times X^n$, where the edge (s, v) starts at v and ends at $s(v)$.

Definition 3.1 (c.f. [29])**.** Let us consider a family $\{(X_n; v_n)\}$ of marked graphs, i.e. graphs with chosen vertices $v_n \in X_n$. We define the metric $Dist$ by

$$Dist((X_1; v_1);\ (X_2; v_2)) \;=\; \inf\left\{\frac{1}{n+1} \mid \mathcal{B}_{X_1}(v_1, n) \text{ is isometric to } \mathcal{B}_{X_2}(v_2, n)\right\}.$$

Then a marked graph $(X; v)$ is the limit graph of the sequence $\{(X_n; v_n)\}$ if

$$\lim_{n\to\infty} Dist((X; v); (X_n; v_n)) \;=\; 0.$$

The limit graph is unique up to isometry. The limit space of G is then the limit graph of its sequence of Schreier graphs $\mathcal{S}_n$.

Remark 3.2. Notice that if G acts level-transitively on X^*, a geodesic path in one of these graphs $\mathcal{S}_n$ is a geodesic in the Cayley graph of the group G, since if there were a shorter path in the group there would be a shorter connection in the graph $\mathcal{S}_n$.

3.2 The Grigorchuk groups G_ω

Let $A = \langle a | a^2 = 1\rangle \simeq Sym(2)$ and $B = \langle b, c, d | bcd = 1\rangle \simeq A \times A$ be two finite groups. There are only 3 non-trivial epimorphisms $B \twoheadrightarrow A$:

$$\alpha : \begin{array}{rcl} b & \mapsto & 1 \\ c & \mapsto & a \\ d & \mapsto & a \end{array} \qquad \beta : \begin{array}{rcl} b & \mapsto & a \\ c & \mapsto & 1 \\ d & \mapsto & a \end{array} \qquad \delta : \begin{array}{rcl} b & \mapsto & a \\ c & \mapsto & a \\ d & \mapsto & 1 \end{array}$$

Let $X = \{0, 1\}$ and $\omega := (\omega_n)_{n\in\mathbb{N}^*}$ be an infinite sequence where $\omega_n \in \{\alpha, \beta, \delta\}$ for all $n \geq 1$. A and B act faithfully on ∂X as follows

$$\begin{array}{rcl} 1_A(x_1\, x_2\, x_3\, x_4\, \ldots) & = & x_1\, x_2\, x_3\, x_4\, \ldots \\ a(0\, x_2\, x_3\, x_4\, \ldots) & = & 1\, x_2\, x_3\, x_4\, \ldots \\ a(1\, x_2\, x_3\, x_4\, \ldots) & = & 0\, x_2\, x_3\, x_4\, \ldots\ , \end{array}$$

and

$$\begin{array}{rcl} y\left(1^{n-1}\, 0\, x_{n+1}\, x_{n+2}\, \ldots\right) & = & 1^{n-1}\, 0\, \omega_n(y)(x_{n+1}\, x_{n+2}\, x_{n+3}\, \ldots) \\ y(x) & = & x \text{ for all words } x \text{ not starting with } 1^{n-1}0 \end{array}$$

for all $n \geq 2$ and for all $y \in B$.

Let $\mathcal{G}_\omega < Aut(X^*)$ be the group generated by A and B with respect to the sequence $\omega \in \{\alpha, \beta, \delta\}^{\mathbb{N}^*}$.

Grigorchuk proved in [28] that $\mathcal{G}_\omega$ is an infinite torsion group of intermediate spherical growth if each of the epimorphisms α, β and δ appears infinitely often in ω. These particular examples are called the Grigorchuk groups.

Example 3.3. The first Grigorchuk group, denoted by $\mathcal{G}$, is defined by the periodic infinite sequence

$$\omega \ = \ (\delta\, \beta\, \alpha)^\infty.$$

It was proved by Elder, Gutierrez and Šunić in [20] that the geodesics in the Schreier graphs $\mathcal{S}_n$ form a language called the Grigorchuk-Schreier-Zimin words and this language has exponential growth of rate $\sqrt{2}$.

For any $\omega \in \{\alpha, \beta, \delta\}^{\mathbb{N}^*}$, notice the following.

Remark 3.4. The Schreier graphs corresponding to the action of the group $\mathcal{G}_\omega$ on levels 0 and 1 of the tree are shown in Figure 3.2 and are independant of ω.

Figure 3.2: $\mathcal{S}_0$ and $\mathcal{S}_1$ for $\mathcal{G}_\omega$

Theorem 3.5. *For any $\omega \in \{\alpha, \beta, \delta\}^{\mathbb{N}^*}$, the group $\mathcal{G}_\omega$ has exponential geodesic growth with respect to the generating set $A \cup B$.*

Proof. By Remark 3.2, since the action is level transitive, it suffices to prove that the growth of the geodesics in the group which are geodesic paths in some Scheier graphs $\mathcal{S}_n$ is exponential.

By Remark 3.4, the Schreier graphs corresponding to the action on level 0 and 1 of the tree are shown in Figure 3.2.

We can obtain the Schreier graph $\mathcal{S}_{n+1}$ from two copies of the graph $\mathcal{S}_n$ joined together by a bi-edge labeled either b/c, b/d or c/d according to the following rule.

Let $\mathcal{S}_n^i$, $i \in \{0, 1\}$, be two graphs obtained from $\mathcal{S}_n$ by adding i at the end of each vertex label in $\mathcal{S}_n$. Then let $\mathcal{S}_n^1$ be the right half of $\mathcal{S}_{n+1}$, flip $\mathcal{S}_n^0$ by 180° and let it be the left half of $\mathcal{S}_{n+1}$ and finally, connect the two halves by "unwrapping" the loops from the ends of the two halves at the vertices $1^{n-2}00$ in the left half and $1^{n-2}01$ in the right half with the rule:

$$\begin{aligned} \text{If } \omega_n = \alpha &\Rightarrow \text{unwrap the loops } c \;\&\; d \\ \text{If } \omega_n = \beta &\Rightarrow \text{unwrap the loops } b \;\&\; d \\ \text{If } \omega_n = \delta &\Rightarrow \text{unwrap the loops } b \;\&\; c \end{aligned}$$

Then for all $n \geq 1$, the number of geodesic paths of length n from the vertex $1^{n-1}0$ in $\mathcal{S}_n$ is at least $2^{\lfloor \frac{n}{2} \rfloor}$. Then $\mathcal{G}_\omega$ has exponential geodesic growth of rate at least $\sqrt{2}$. □

Example 3.6. The Schreier graphs corresponding to the action of the first Grigorchuk group $\mathcal{G}$ on levels 0, 1, 2 and 3 of the tree $\{0, 1\}^*$ are given by Figures 3.2, 3.3 and 3.4, respectively.

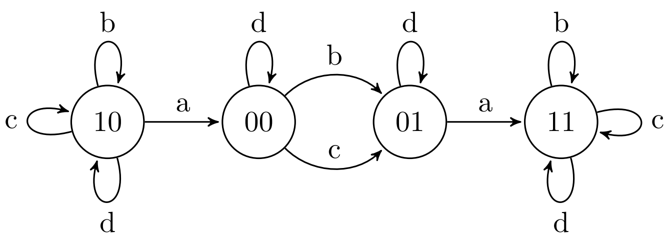

Figure 3.3: $\mathcal{S}_2$ for $\mathcal{G}$

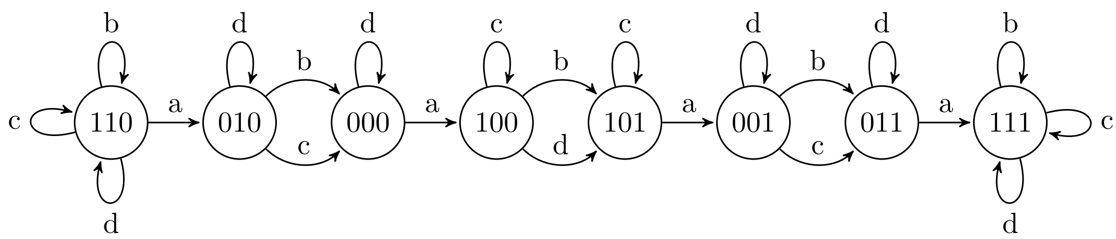

Figure 3.4: $\mathcal{S}_3$ for $\mathcal{G}$

3.3 The Gupta-Sidki p-groups

Let $p \geq 3$ be a prime, and $X = \{0, 1, 2..., p-1\}$. We define $t, z \in Sym(X^*)$ as

$$\begin{aligned} t(\,x_1\,x_2\,x_3\,x_4\,\cdots) &= ((x_1+1) \mod p)\,x_2\,x_3\,x_4\,\ldots \\ z(0^r\,i\,j\,w) &= 0^r\,i\,(i+j)\,w, \end{aligned}$$

where $x_i \in X$ for all i, $r \geq 0$, $i \neq 0$, $i, j, i + j \in X$ and $w \in X^*$.

We call $GS_p := \langle t, z \rangle$ the p-th Gupta-Sidki group and $GS = GS_3$ the Gupta-Sidki group [32, 33]. For all p, these groups are infinite, infinitely presented, and all elements have order p. It is still an open question whether the GS_p groups have intermediate spherical growth or not.

We study below the group GS in more detail. We note that the action of GS on X^* is level-transitive. The Schreier graphs $\{\mathcal{S}_n\}_{n \geq 0}$ of GS are recursively defined as follows, where we use the notation (x, y, l) for the edge from the vertex x to the vertex y, with label l:

- $\mathcal{S}_0$ is the graph with a single vertex labeled ϵ and two loops (ϵ, ϵ, t) and (ϵ, ϵ, z).
- For all $n \geq 0$, $\mathcal{S}_{n+1}$ is constructed by creating 3 copies of $\mathcal{S}_n$, called $\mathcal{S}_n^0$, $\mathcal{S}_n^1$ and $\mathcal{S}_n^2$. For each vertex $x \in \mathcal{S}_n$ and $i = 0, 1, 2$, let $x_i \in \mathcal{S}_n^i$ be defined as $x_i := ix$. For all $x, y, \in V(\mathcal{S}_n)$, the edges of $\mathcal{S}_{n+1}$ are constructed by letting
    * (x_0, y_0, z) be an edge in $\mathcal{S}_n^0 \subset \mathcal{S}_{n+1}$ if and only if (x, y, z) is an edge of $\mathcal{S}_n$.
    * (x_1, y_1, z) be an edge in $\mathcal{S}_n^1 \subset \mathcal{S}_{n+1}$ if and only if (x, y, t) is an edge of $\mathcal{S}_n$.
    * (x_2, y_2, z) be an edge in $\mathcal{S}_n^2 \subset \mathcal{S}_{n+1}$ if and only if (x, y, t^{-1}) is an edge of $\mathcal{S}_n$.
    * (x_i, x_j, t) be an edge in $\mathcal{S}_{n+1}$ if and only if $j = i + 1 (\mod 3)$.

Example 3.7. The Schreier graphs corresponding to the action of the Gupta-Sidki group GS on levels 0, 1 and 2 of the tree $\{0, 1, 2\}^*$ are given by Figures 3.5, 3.6 and 3.7.

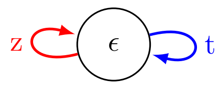

Figure 3.5: $\mathcal{S}_0$ for GS.

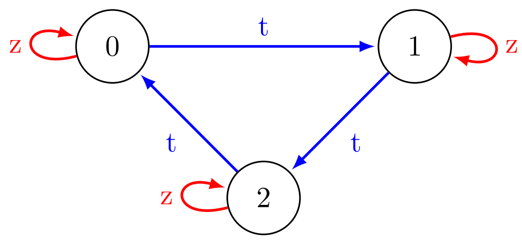

Figure 3.6: $\mathcal{S}_1$ for GS.

Notice that every geodesic of GS is of the form

$$(z^{e_1})\, t^{f_1}\, z^{e_2}\, t^{f_2} \ldots z^{e_n}\, t^{f_n}\, (z^{e_{n+1}}),$$

where $e_i, f_j \in \{\pm 1\}$ for all i, j, and the first and last "z" are optional. However, not every word of this form is necessarily a geodesic, e.g $(tz)^3 = 1$. It implies that $1 \leq \gamma(GS, \{t, z\}) \leq 2$.

Theorem 3.8. *GS_p has exponential geodesic growth with respect to the generating set $\{t, z\}$ for all $p \geq 3$ prime.*

Proof. It suffices to prove the result for $p = 3$, since every geodesic in GS is a geodesic in GS_p for $p > 3$ prime.

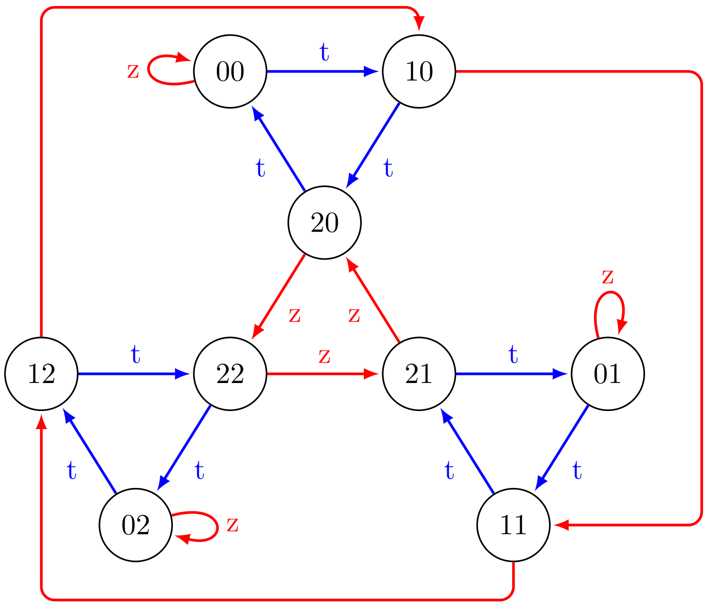

Figure 3.7: $\mathcal{S}_2$ for GS.

Let $n \geq 1$ be big enough. In GS, every edge labelled by t in the Schreier graph $\mathcal{S}_n$ induces a switch between the subgraphs S^0_{n-1} and $S^1_{n-1} \cup S^2_{n-1}$.

Let

$$w = t^{f_1}\, z^{e_2}\, t^{f_2}\, z^{e_3}\, t^{f_3} \ldots z^{e_{2k-1}}\, t^{f_{2k-1}}\, z^{e_{2k}}\, t^{f_{2k}} \tag{3.1}$$

be the label of a geodesic path in $\mathcal{S}_n$ starting at the vertex 0^n. This path corresponds to a sequence of switches of the form

$$S^0_{n-1} \to S^{i_1}_{n-1} \to S^0_{n-1} \to S^{i_2}_{n-1} \to S^0_{n-1} \to S^{i_3}_{n-1} \to S^0_{n-1} \to \ldots,$$

where for each j, $i_j = 1$ or 2 if and only if $f_{2j-1} = 1$ or -1, respectively.

Moreover, the end vertex of the path labelled by tzt^{-1} beginning in S^0_{n-1} is the same as the end vertex of the path labelled by $t^{-1}z^{-1}t$. This implies that, for all $j \geq 1$, each subword of the form $t^{f_{2j-1}}\, z^{e_{2j}}\, t^{f_{2j}}$ can be replaced by the subword $t^{-f_{2j-1}}\, z^{-e_{2j}}\, t^{-f_{2j}}$ and the new word is then a geodesic path in $\mathcal{S}_n$ with the same end vertex as before. Then

$$\begin{aligned}
\gamma(GS_p, \{t, z\}) &\geq \gamma(GS, \{t, z\}) \\
&\geq \lim_{k\to\infty} \sqrt[4k-1]{|\{\, w \text{ geodesic path in } \mathcal{S}_n \text{ from } 0^n \text{ of the form (3.1)} \,\}|} \\
&\geq \sqrt[4]{2} \;>\; 1
\end{aligned}$$

□

3.4 The Square group

Let $X = \{0, 1, 2, 3\}$. We define $t, z \in Sym(X^*)$ as

$$\begin{aligned}
t(\, x_1\, x_2\, x_3\, x_4\, \cdots) &= ((x_1 + 1) \mod 4)\, x_2\, x_3\, x_4 \ldots \\
z(3^r \cdot 0 \cdot w) &= 3^r \cdot 0 \cdot t(w) \\
z(3^r \cdot 2 \cdot w) &= 3^r \cdot 2 \cdot t^{-1}(w),
\end{aligned}$$

where $x_i \in X$ for all i, $r \geq 0$ and $w \in X^*$.

We call the group $\mathbb{S}$ generated by t and z the Square group.

This group is an example proposed by Grigorchuk, who stated its spherical growth as an interesting open question. We note that the action of $\mathbb{S}$ is level-transitive. The Schreier graphs $\{\mathcal{S}_n\}_{n\geq 0}$ of $\mathbb{S}$ are recursively defined as follows, where we use the notation (x, y, l) for the edge from the vertex x to the vertex y, with label l:

- $\mathcal{S}_0$ is the graph with a single vertex labeled ϵ and two loops (ϵ, ϵ, t) and (ϵ, ϵ, z).
- For all $n \geq 0$, $\mathcal{S}_{n+1}$ is constructed by creating 4 copies of $\mathcal{S}_n$, called $\mathcal{S}_n^0$, $\mathcal{S}_n^1$, $\mathcal{S}_n^2$ and $\mathcal{S}_n^3$. For each vertex $x \in \mathcal{S}_n$ and $i = 0, 1, 2, 3$, let $x_i \in \mathcal{S}_n^i$ be defined as $x_i := ix$. For all $x, y, \in V(\mathcal{S}_n)$, the edges of $\mathcal{S}_{n+1}$ are constructed by letting
  - (x_0, y_0, z) be an edge in $\mathcal{S}_n^0 \subset \mathcal{S}_{n+1}$ if and only if (x, y, t) is an edge of $\mathcal{S}_n$.
  - (x_1, x_1, z) be an loop in $\mathcal{S}_n^1 \subset \mathcal{S}_{n+1}$ for all $x \in \mathcal{S}_n$.
  - (x_2, y_2, z) be an edge in $\mathcal{S}_n^2 \subset \mathcal{S}_{n+1}$ if and only if (x, y, t^{-1}) is an edge of $\mathcal{S}_n$.
  - (x_3, y_3, z) be an edge in $\mathcal{S}_n^3 \subset \mathcal{S}_{n+1}$ if and only if (x, y, z) is an edge of $\mathcal{S}_n$.
  - (x_i, x_j, t) be an edge in $\mathcal{S}_{n+1}$ if and only if $j = i + 1 (\mod 3)$.

Example 3.9. The Schreier graphs corresponding to the action of the Square group $\mathbb{S}$ on levels 0,1 and 2 of the tree $\{0, 1, 2, 3\}^*$ are given by Figures 3.8, 3.9 and 3.10.

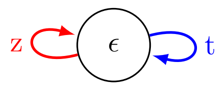

Figure 3.8: $\mathcal{S}_0$ for $\mathbb{S}$.

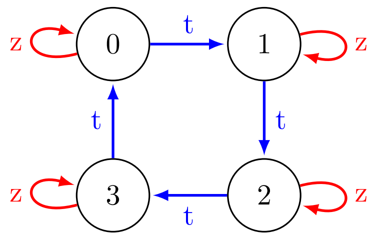

Figure 3.9: $\mathcal{S}_1$ for $\mathbb{S}$.

Theorem 3.10. $\mathbb{S}$ *has exponential geodesic growth with respect to the generating set* $\{t, z\}$.

Proof. Every geodesic path in $\mathcal{S}_n$ from the vertex 3^n begin with t and every edge labelled by t (except maybe the last) induces a switch between the subgraphs S_n^3 and $S_n^0 \cup S_n^2$.

The end of the proof is exactly the same as the proof of Theorem 3.8 for Gupta-Sidki p-groups, and $\mathbb{S}$ has exponential geodesic growth of rate at least $\sqrt[4]{2}$. □

3.5 Spinal groups

Let $p \geq 2$ and $X = \{0, 1, 2, ..., p-1\}$. Let A be a group (called the root group) acting faithfully and transitively on X and B be a finite group (called the level group) such

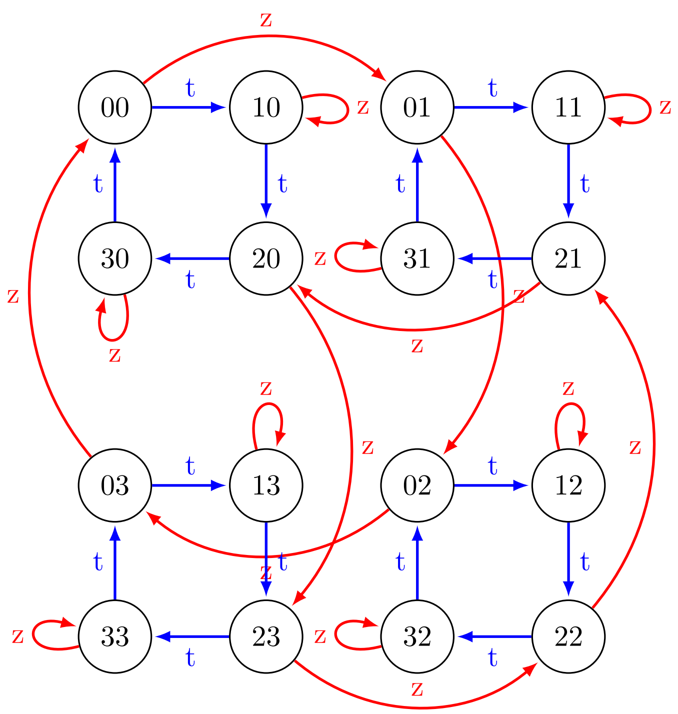

Figure 3.10: $\mathcal{S}_2$ for $\mathbb{S}$.

that $|B| > |A|$ and such that the set $Epi(B, A)$ of epimorphisms $B \twoheadrightarrow A$ is non empty. For all $\omega_i \in Epi(B, A)$, we denote $K_i := Ker(\omega_i)$.

Remark 3.11. Since $|B| > |A|$, from the First isomorphism Theorem, we have $|B| = k|A|$, where $k = |K_i| > 1$ for all $\omega_i \in Epi(B, A)$.

Moreover, if $b \in B$ verifies $b \in K_i$ for some i, then there are $c \neq d \in B$ such that $d = bc$ and $\omega_i(c) = \omega_i(d) = a \in A \setminus \{1_A\}$.

We define now Ω as the set of infinite sequences of epimorphisms $B \twoheadrightarrow A$, i.e

$$\Omega \quad = \quad \{\omega = (\omega_1, \omega_2, \omega_3, \dots) \ \mid \omega_n \in Epi(B, A) \quad \forall n \geq 1 \ \}$$

and

$$\widehat{\Omega} \quad := \quad \left\{ \omega \in \Omega \ \middle| \ \bigcup_{j \geq i} K_j = B \quad \text{and} \quad \bigcap_{j \geq i} K_j = \{1\} \quad \forall i \geq 1 \right\}.$$

By definition, A acts faithfully on ∂X as

$$a(x_1 x_2 x_3 x_4 \dots) \quad = \quad a(x_1) x_2 x_3 x_4 \dots \qquad \forall a \in A$$

and for each $\omega \in \Omega$ fixed, the faithful action of B on ∂X is defined by

$$\begin{aligned} b\Big((p-1)^{n-1}\, 0\, x_{n+1}\, x_{n+2} \dots\Big) \quad &= \quad (p-1)^{n-1}\, 0\, \omega_n(b)(x_{n+1})\, x_{n+2}\, x_{n+3} \dots \\ b(x) \quad &= \quad x \ \text{ for all words } x \text{ not starting with } (p-1)^{n-1} 0 \end{aligned}$$

for all $n \geq 1$ and for all $b \in B$.

More precisely:

(i) $a \in A$ permutes the subtrees $0X^*$, $1X^*$,...,$(p-1)X^*$, and

(ii) $b \in B$ is determined by ω: b acts on the subtree $0X^*$ as $\omega_1(b)$ would act on X^*, on the subtree $(p-1)0X^*$ as $\omega_2(b)$ would act on X^*, ..., and acts trivially on subtrees not of the form $(p-1)^k 0X^*$. (C.f. Figure 3.11)

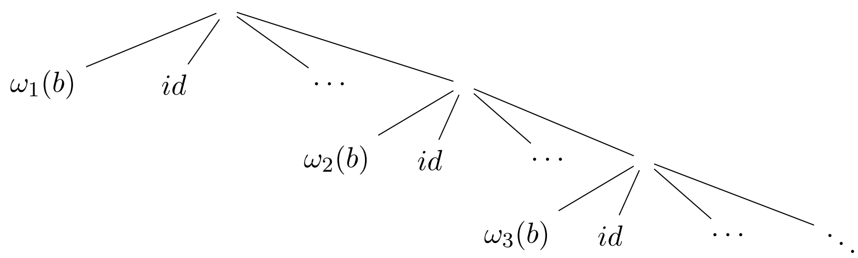

Figure 3.11: The action of b on $\mathcal{T}$

For any sequence $\omega \in \widehat{\Omega}$, the subgroup of $Aut(\mathcal{T})$ generated by A and B is denoted by $\mathcal{G}_\omega$ and is called the spinal group defined by the sequence ω.

Remark 3.12. As the Grigorchuk groups are particular examples of Spinal groups, we use the same notation to define both families of groups.

Remark 3.13. For any sequence $\omega \in \Omega$, the action of $\mathcal{G}_\omega$ is level-transitive on X^*.

In [6, pp. 17-19], Bartholdi and Šunić proved that every spinal group has intermediate spherical growth, using a generalisation of the idea of the proof for the first Grigorchuk group. With the following theorem, we prove that all of these groups have exponential geodesic growth.

Theorem 3.14. *Let $\omega \in \Omega$. With respect to the generating set $A \cup B$, $\mathcal{G}_\omega$ has exponential geodesic growth of rate at least $\sqrt{2}$.*

Proof. The proof is divided into two cases: (i) $p = 2$ and (ii) $p > 2$.

(i) Suppose $p = 2$. Then $A = \{1, a\} \simeq C_2$ and $|B| = k|A|$, where $k \geq 2$. For all epimorphisms $\omega_i : B \twoheadrightarrow A$, there exist at least one non-trivial element $b_i \in B$ such that $\omega_i(b_i) = 1$ and at least two different non-trivial elements $\tilde{b}_i, \bar{b}_i \in B$ such that $\omega_i(\tilde{b}_i) = \omega_i(\bar{b}_i)$ (it suffices to take $\bar{b} = b \cdot \tilde{b}$). The Schreier graphs $\mathcal{S}_0$ and $\mathcal{S}_1$ corresponding to the action on levels 0 and 1 of the tree are shown in Figure 3.12 and are independant of ω.

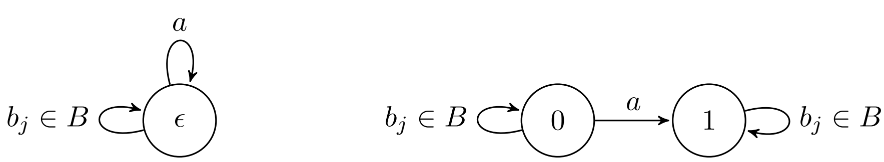

Figure 3.12: $\mathcal{S}_0$ and $\mathcal{S}_1$ for G_ω

We can obtain the Schreier graph $\mathcal{S}_{n+1}$ from two copies of the graph $\mathcal{S}_n$ joined together by multi-edges labeled according to the following rule:

Let $\mathcal{S}_n^i$, $i \in \{0, 1\}$ and $n \geq 1$, be two graphs obtained from $\mathcal{S}_n$ by adding i at the end of each vertex label in $\mathcal{S}_n$. Then let $\mathcal{S}_n^1$ be the right half of $\mathcal{S}_{n+1}$, flip $\mathcal{S}_n^0$ by 180° and let it be the left half of $\mathcal{S}_{n+1}$ and finally, connect the two halves by "unwrapping" the loops labeled by b verifying $b \notin K_n$ from the ends of the two halves at the vertices $1^{n-2}00$ in the left half and $1^{n-2}01$ in the right half.

Finally, since there exist exactly $N := |B| - k \geq 2$ such b's, the number of geodesic paths of length n in $\mathcal{S}_n$ is at least $N^{\lfloor \frac{n}{2} \rfloor}$. Then if $p = 2$, $\mathcal{G}_\omega$ has exponential geodesic growth of rate at least $\sqrt{N}$.

(ii) Suppose now that $p > 2$. Since $\mathcal{G}_\omega$ is generated by $A \cup B$, the following types of relations are verified:

$$\text{For all } a \in A \subset \mathcal{G}_\omega, \text{ there exist } \tilde{a} \in A \subset \mathcal{G}_\omega \text{ such that } a\tilde{a} \;=\; 1_{G_\omega}, \tag{3.2}$$

$$\text{For all } a_1, a_2 \in A \subset \mathcal{G}_\omega, \text{ there exist } a_3 \in A \subset \mathcal{G}_\omega \text{ such that } a_1 a_2 \;=\; a_3, \tag{3.3}$$

and similarly for $B \subset \mathcal{G}_\omega$. It implies that each element in $\mathcal{G}_\omega$ could be written as

$$(a_1)\, b_1\, a_2\, b_2\, a_3\, b_3 \ldots a_k\, b_k\, (a_{k+1}),$$

where $a_i \in A$, $b_i \in B$ for all i and the first and the last a_i are optional. We remark that a word of this form is not necessarily a geodesic. E.g. the word *adadad* in the first Grigorchuk group $\mathcal{G}$ studied in section 3.2 is not a geodesic.

Moreover, by Remark 3.13, we know that the action of $\mathcal{G}_\omega$ is level-transitive. By Remark 3.2, it suffices to show that the Schreier graphs have exponential geodesic growth. The Schreier graphs $\{\mathcal{S}_n\}_{n \geq 0}$ of $\mathcal{G}_\omega$ are recursively defined as follows, where we use the notation (x, y, l) for the edge from the vertex x to the vertex y, with label l:

- $\mathcal{S}_0$ is the graph with a single vertex labeled ϵ and loops (ϵ, ϵ, x) for all $x \in A \cup B$.
- $\mathcal{S}_1$ is the graph with a p vertices labeled $0, 1, \ldots, (p-1)$ with the edges constructed by letting for all $i, j = 0, 1, \ldots, p-1$, $b \in B$ and $a \in A$ be loops (i, i, b) and edges (i, j, a) if $a(i) = j$.
- For all $n \geq 1$, $\mathcal{S}_{n+1}$ is constructed by creating p copies of $\mathcal{S}_n$, called $\mathcal{S}_n^0$, $\mathcal{S}_n^1$, ... $\mathcal{S}_n^{p-1}$. For each vertex $x \in \mathcal{S}_n$ and $i = 0, 1, \ldots, p-1$, let $x_i \in \mathcal{S}_n^i$ be defined as $x_i := ix$. For all $x, y, \in V(\mathcal{S}_n)$, the edges of $\mathcal{S}_{n+1}$ are constructed by letting
  - $*$ (x_0, y_0, b) be an edge in $\mathcal{S}_n^0 \subset \mathcal{S}_{n+1}$ if and only if $\omega_1(b)(x) = y$.
  - $*$ (x_i, x_i, b) be an loop in $\mathcal{S}_n^i \subset \mathcal{S}_{n+1}$ for all $x \in \mathcal{S}_n$, $b \in B$ and $i = 1, \ldots, p-2$.
  - $*$ For all $b \in B$, (x_{p-1}, y_{p-1}, b) be an edge if and only if there exists $1 \leq k \leq n-2$ such that $x = (p-1)^k 0\tilde{x}$, $y = (p-1)^k 0\tilde{y}$ and $\omega_{k+2}(b)(\tilde{x}) = \tilde{y}$. If not, (x_{p-1}, x_{p-1}, b) be a loop.
  - $*$ For all $a \in A$, (x_i, x_j, a) be an edge in $\mathcal{S}_{n+1}$ if and only if $a(i) = j$.

For example, the Schreier graphs corresponding to the action of $\mathcal{G}_\omega$ on levels 0,1 and 2 of the tree X^* can be represented in Figures 3.13, 3.14 and 3.15, where the double, blue edges represent edges of the form (x_i, x_j, a) for all $i, j \leq p-1$ and $a \in A$ such that $a(i) = j$.

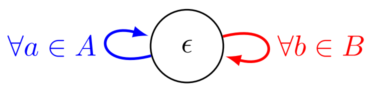

Figure 3.13: $\mathcal{S}_0$ for $\mathcal{G}_\omega$.

By induction, we can see the subgraph $\mathcal{S}_{n-k}^v$ of $\mathcal{S}_{n+1}$ as the intersection $V(\mathcal{S}_{n+1}) \cap vX^*$, where $v \in X^{k+1}$ and $1 \leq k < n$.

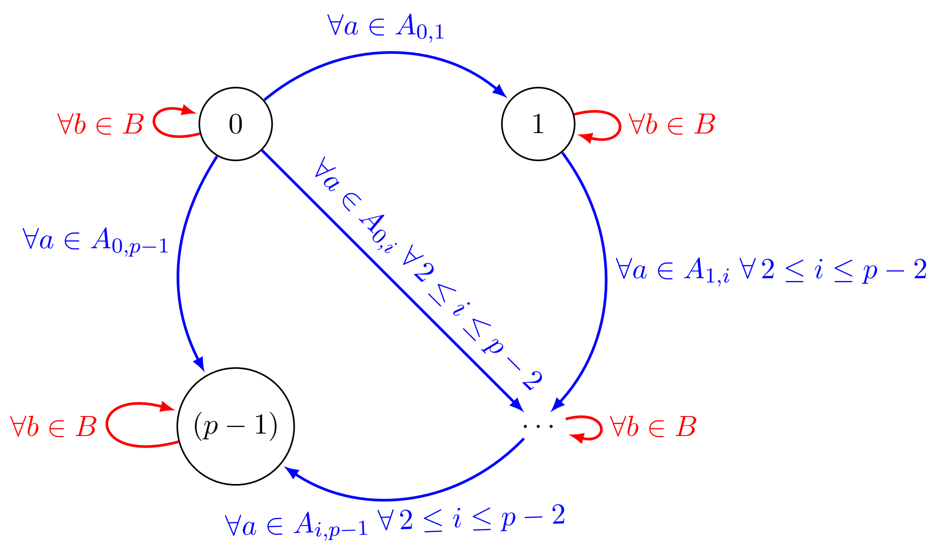

Figure 3.14: $\mathcal{S}_1$ for $\mathcal{G}_\omega$, where $A_{i,j} := \{a \in A | a(i) = j\}$.

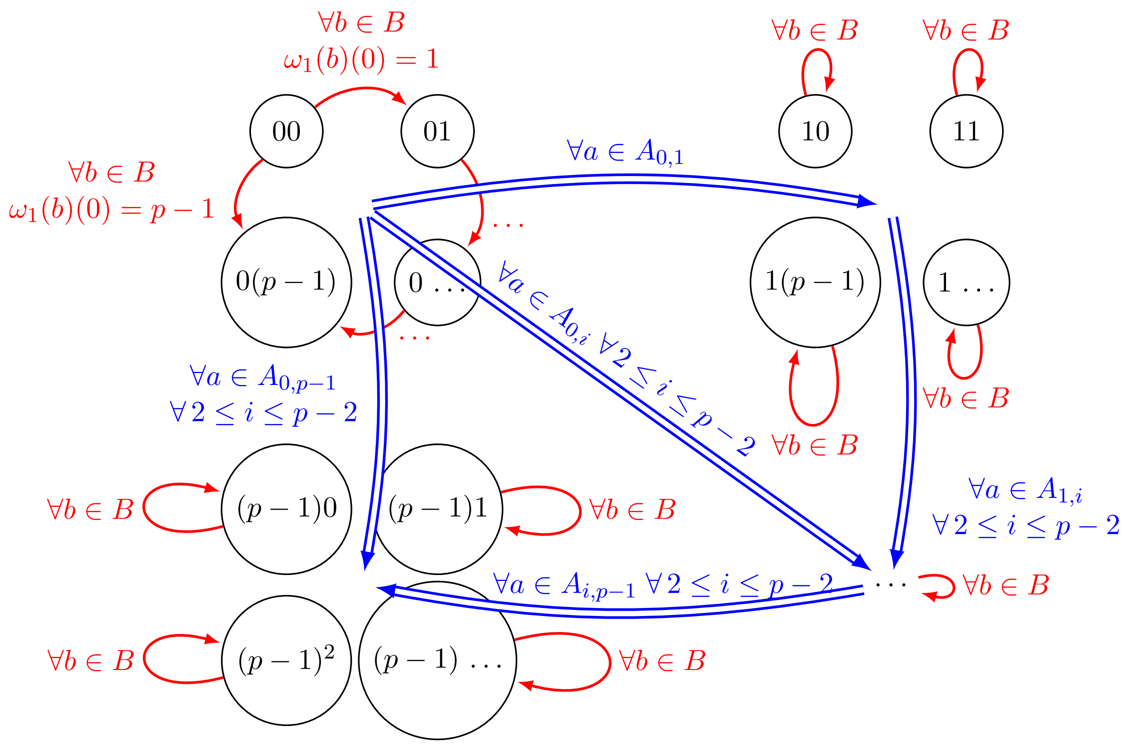

Figure 3.15: $\mathcal{S}_2$ for $\mathcal{G}_\omega$, where $A_{i,j} := \{a \in A | a(i) = j\}$ and the double, blue edges represent edges of the form (x_i, x_j, a) for all $i, j \leq p-1$ and $a \in A$ such that $a(i) = j$.

By construction, notice that all geodesic paths in $\mathcal{S}_n$, $n \geq 2$, from the vertex 0^n are of the form

$$a_0\, b_1\, a_1^{-1}\, b_2\, a_2\, b_3\, a_3^{-1}\, b_4\, a_4 \ldots a_{k-1}\, b_k\, (a_k),$$

where $a_j \in A$ verifies $a_j(0) = p-1$ for all j, $b_i \in B$ for all i and the last a_j is optional. Observe that all of these a_j are not necessarily unique.

By Remark 3.11, for all $j \geq 1$ there are $b, c \in B$ non-trivial such that $\omega_i(b) = 1_A$ and $\omega_i(c) \neq 1_A$, and for the edge labelled by c in $E(\mathcal{S}_{n-i+1}^{(p-1)^{i-1}0}) \subset E(\mathcal{S}_{n+1})$ there is another edge labelled by $d = bc \neq c$ with the same end vertex c.

Finally, each geodesic path of length $2k$ from the vertex 0^n in $\mathcal{S}_n$ with n large enough, which begins by a letter in A (respectively in B) if k is even (resp. odd), could be replaced by another geodesic path in $\mathcal{S}_n$ which has the same end vertex as the initial path. Thus for all $k \geq 1$ the number of geodesic paths of length k is at least $2^{\lfloor \frac{n}{2} \rfloor}$, so $\mathcal{G}_\omega$ has exponential geodesic growth of rate at least $\sqrt{2}$.

□

Theorem 3.14 implies the following result.

Corollary 3.15. *Let $\mathcal{G}_\omega$ be a spinal group. With respect to the generating set $A \cup B$, $\mathcal{G}_\omega$ has exponential geodesic growth of rate at least $\sqrt{2}$.*

We remark that if there are k distinct elements $a_1, a_2, \ldots, a_k \in A$ such that

$$a_1(0) = a_2(0) = \ldots = a_k(0) = p-1,$$

then by Remark 3.11 each letter of a geodesic path in a Schreier graph could be replaced by $k-1$ another letters such that the new geodesic has the same end vertex as the initial path. Thus $\gamma(\mathcal{G}_\omega) > k-1$.

3.6 Gupta-Fabrykowski

Let $X = \{0,1,2\}$. We define $t, z \in Sym(X^*)$ as follows

$$\begin{aligned} t(\,x_1\, x_2\, x_3\, x_4\, \cdots) &= ((x_1+1) \mod 3)\, x_2\, x_3\, x_4 \ldots \\ z(2^r\, 0\, w) &= 2^r\, 0\, t(w), \end{aligned}$$

where $x_i \in X$ for all i, $r \geq 0$, and $w \in X^*$.

We call $GF := \langle t, z \rangle$ the Gupta-Fabrykowski group. In [22], Fabrykowski and Gupta stated that GF has intermediate growth and proved it in [23].

Let $x = tz$ and $y = zt$ be elements in GF, and let K be the subgroup of GF generated by x and y. Then K is normal in GF, because $y^z = x^{-1}y^{-1}$, $y^{t^{-1}} = y^{-1}x^{-1}$ and $y^{z^{-1}} = y^t = t$, and similar relations hold for x. Moreover K is of index 3 in GF, with transversal $\langle t \rangle$.

The Schreier graphs $\{\mathcal{S}_n\}_{n \geq 0}$ of GF are recursively defined as follows, where we use the notation (x, y, l) for the edge from the vertex x to the vertex y, with label l:

- $\mathcal{S}_0$ is the graph with a single vertex labeled ϵ and four loops $(\epsilon, \epsilon, t^{\pm 1})$ and $(\epsilon, \epsilon, z^{\pm 1})$.
- $\mathcal{S}_1$ is the graph with three vertices labeled $0, 1$ and 3, and vertices $(i, i+1\,(mod\,3), t)$ and loops (i, i, z) for all $i = 0, 1, 2$.

- For all $n \geq 1$, $\mathcal{S}_{n+1}$ is constructed by creating three copies of $\mathcal{S}_n$, called $\mathcal{S}_n^0$, $\mathcal{S}_n^1$ and $\mathcal{S}_n^2$. For each vertex $x \in \mathcal{S}_n$ and $i = 0, 1, 2$, let $x_i \in \mathcal{S}_n^i$ be defined as $x_i := ix$. For all $x, y, \in V(\mathcal{S}_n)$, the edges of $\mathcal{S}_{n+1}$ are constructed by letting
  - (x_0, y_0, z) be an edge in $\mathcal{S}_n^0 \subset \mathcal{S}_{n+1}$ if and only if (x, y, t) is an edge of $\mathcal{S}_n$.
  - (x_1, x_1, z) be an loop in $\mathcal{S}_n^1 \subset \mathcal{S}_{n+1}$ for all $x \in \mathcal{S}_n$.
  - (x_2, y_2, z) be an edge in $\mathcal{S}_n^2 \subset \mathcal{S}_{n+1}$ if and only if (x, y, z) is an edge of $\mathcal{S}_n$.
  - (x_i, x_j, t) be an edge in $\mathcal{S}_{n+1}$ if and only if $j = i + 1(\mod 3)$.

Example 3.16. In [4, pp. 40 - 42], there is a second recursive definition of the Schreier graphs $\{\mathcal{S}_n\}_{n\geq 0}$ of GF, using substitutional rules. Figure 3.16 is the 6-th Schreier graph $\mathcal{S}_6$ of GF. The red and blue edges represent the generators z and t respectively (c.f. [4]).

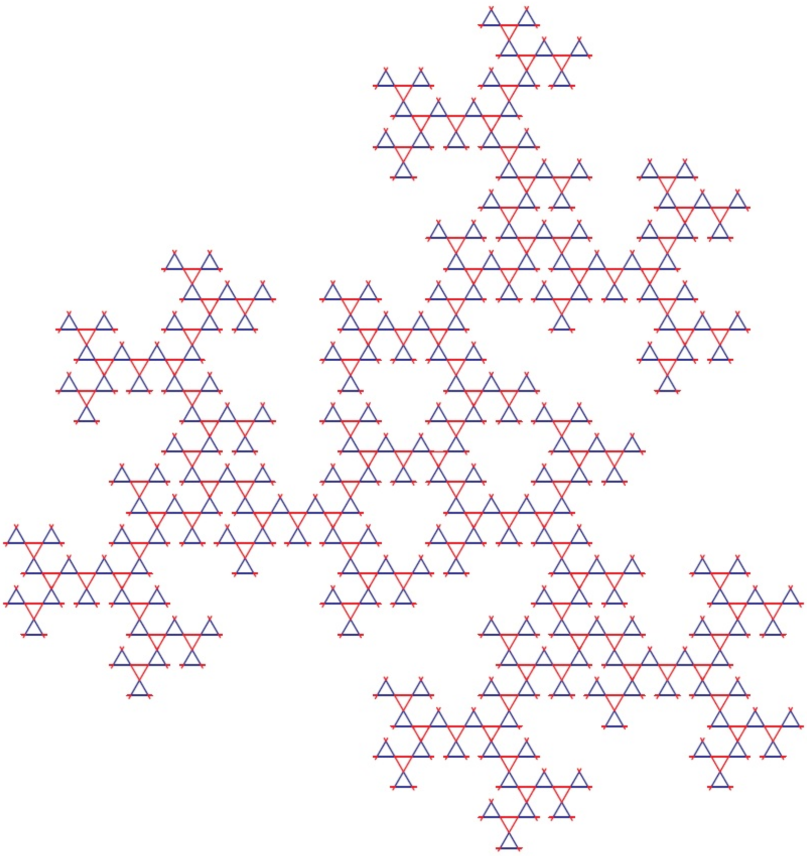

Figure 3.16: $\mathcal{S}_6$ of GF.

Theorem 3.17 (Bartholdi and Grigorchuk, [4])**.** *For all $n \geq 0$, the Schreier graph $\mathcal{S}_n$ of GF is a subgraph of the Cayley graph*

$$Cay\left(C_3 * C_3 = \langle a, b | a^3 = b^3 = 1\rangle,\ \{a, b\}\right)$$

of diameter $2^n - 1$ and $|V(\mathcal{S}_n)| = 3^n$.

It was proved in [4, p. 42] that the limit space $\mathcal{S}$ of GF has polynomial spherical growth of degree $\log_2(3)$. Then, since geodesics in $Cay\,(C_3 * C_3, \{a, b\})$ are unique, the geodesic growth function of the language of words in the limit space is polynomial as well.

The Gupta-Fabrykowski is then the first group of intermediate spherical growth in our study for which it is not feasible to prove that the geodesic growth is exponential using Schreier graphs in the same manner as for the other examples in this thesis.

Chapter 4

Formal geodesic growth

Within the area of formal languages, the Chomsky hierarchy is a classification of formal grammars and the languages that they define. This hierarchy of grammars was defined by N. Chomsky in 1956 [12].

In this chapter we give a basic introduction to formal grammars and formal languages (based on the article [36]) and study the set of geodesics in a group from a formal language point of view. We also determine the algebraic complexity of the geodesic growth series for certain classes of groups.

Finally, we consider the falsification by fellow traveller property (FFTP) and one of its generalisations, the $h(n)$-FFTP property. We study the groups $\mathbb{F}_k \times \mathbb{F}_k$, giving specific generating sets of $\mathbb{F}_k \times \mathbb{F}_k$ for which these groups have the $h(n)$-FFTP property for $h(n) \neq n$ linear but do not have the standard FFTP property.

4.1 Preliminaries

We define a grammar G as a 4-tuple $(N, \Sigma, P, \$)$, where:

- N is a finite set of nonterminal symbols;
- Σ is a finite set of terminal symbols, disjoint from N;
- P is a finite set of production rules, where each rule p is of the form

$$(N \cup \Sigma)^* \; N \; (N \cup \Sigma)^* \quad \to \quad (N \cup \Sigma)^*;$$

- $\$ \in N$ is a distinguished symbol, called the start symbol.

Grammars can be divided into four classes by gradually increasing the restrictions on the productions. Let $G = (N, \Sigma, P, \$)$ be a grammar.

1. G is a Type-0 grammar if it is unrestricted.
2. G is a Type-1 grammar, or context-sensitive grammar if each production $\alpha \to \beta$ in P satisfies $|\alpha| \leq |\beta|$. The production $\$ \to \epsilon$ is allowed if $\$$ does not appear on the right-hand side of any other production.
3. G is a Type-2 grammar, or context-free grammar if each production $\alpha \to \beta$ in P satisfies $|\alpha| = 1$.
4. G is a Type-3 grammar, or regular grammar if each production in P is of one of the following two forms:

$$A \to aB, \qquad A \to a,$$

   where $A, B \in N$ and $a \in \Sigma \cup \{\epsilon\}$.

We have that for each $i = 0, 1, 2$, the class of Type-i languages contains the class of Type-$(i+1)$ languages.

Let $G = (N, \Sigma, P, \$)$ be a grammar, and let w_1, w_2 be two words on $N \cup \Sigma$. We say that w_1 directly derives w_2, written $w_1 \Rightarrow w_2$, if $w_1 = \eta\alpha\nu$, $w_2 = \eta\beta\nu$, where $\alpha, \beta, \eta, \nu \in (N \cup \Sigma)^*$ and $\alpha \to \beta$ is a production in P. We say that w_1 derives w_2, written $w_1 \Rightarrow^* w_2$ if there exists a sequence of words $v_1, v_2, v_3, ..., v_n$ on $N \cup \Sigma$ such that

$$w_1 \;\Rightarrow\; v_1 \;\Rightarrow\; v_2 \;\Rightarrow\; \ldots \;\Rightarrow\; v_n \;\Rightarrow\; w_2.$$

The formal language generated by G, denoted by $L(G)$, is then defined as

$$L(G) \;=\; \{\, w \,|\, w \in \Sigma^*,\ \$ \Rightarrow^* w \,\}\,.$$

We say that $L(G)$ is regular, context-free or context-sensitive if G is regular, context-free or context-sensitive, respectively.

Let L, L_1, L_2 be three formal languages over Σ. The concatenation of L_1 and L_2, denoted by $L_1\,L_2$, is the language

$$L_1 L_2 \;=\; \{\, xy \,|\, x \in L_1,\ y \in L_2 \,\}\,.$$

The Kleene closure of L, denoted by L^*, is the language defined by

$$L^* \;=\; \bigcup_{i \geq 0} L^i,$$

where $L^0 = \{\epsilon\}$ and $L^i = LL^{i-1}$ for $i \geq 1$, respectively. In other words, the Kleene closure of a formal language L consists of all strings that can be formed by concatenating zero or more words from L.

Another important concept is that of regular expressions. The regular expressions over Σ and the languages they represent are defined recursively as follows:

1. The symbol $\emptyset$ is a regular expression and represents the empty language;
2. The symbol ϵ is a regular expression and represents the language whose only member is the empty word;
3. For each $x \in \Sigma$, x is a regular expression and represents the language $\{x\}$;
4. If r and s are regular expressions representing the languages R and S over Σ, then $(r+s)$, (rs) and r^* are regular expressions representing the languages $R \cup S$, RS and R^*, respectively.

We have the following relation between formal languages generated by a grammar and regular expressions.

Theorem 4.1 (Hopcroft and Ullman, [35]). *Let L be a formal language over Σ. Then there exists a regular grammar $G = (N, \Sigma, P, \$)$ which generates L if and only if L can be represented by a regular expression.*

The classes of languages defined above satisfy various closure properties, as seen below.

Theorem 4.2 (Hopcroft and Ullman, [35]).

1. *The union, concatenation and Kleene star operations of context-free languages are context-free;*

2. *The union, concatenation, intersection and Kleene star operations of regular languages are regular.*

3. *The intersection of a context-free language with a regular language is context-free.*

There are several classical results, the most important being the Pumping Lemma and Ogden's Iteration Lemma, which give a way to disprove that a language is regular and context-free, respectively.

Lemma 4.3 (Pumping Lemma, [35]). *Let $L \subset X^*$ be a regular language. Then there exists an integer $N \geq 1$ depending only on L such that every word $w \in L$ of length at least N can be written as $w = xyz$, satisfying the following conditions:*

1. $|y| \geq 1$;

2. $|xy| \leq N$;

3. *for all* $i \geq 0$, $xy^iz \in L$.

Lemma 4.4 (Ogden's Iteration Lemma for context-free languages, [7]). *Let $L \subset X^*$ be a context-free language. Then there exists an integer $N \geq 1$ such that any word $w \in L$ and for any choice of at least N marked positions in w, w admits a factorization $w = \alpha u \beta v \gamma$, where $\alpha, \beta, \gamma, u, v \in X^*$, satisfying*

1. $\alpha u^n \beta v^n \gamma \in L$ *for all* $n \geq 0$;

2. uv *contains at most* N *marked positions;*

3. *(each of α, u, β) or (each of β, v, γ) contains at least one marked position.*

We now turn to the concept of the growth of a formal language: let L be a formal language over Σ. We define the growth function $s_L : \mathbb{N} \to \mathbb{N}$ of L by

$$s_L(n) \;\; := \;\; | \; \Sigma^n \; \cap L \; | \, .$$

Since $(s_L(n))_{n\geq 0}$ is an integer sequence, we can define the ordinary growth series $S_L : \mathbb{C} \to \mathbb{C}$ of $(s_L(n))_{n\geq 0}$ as

$$S_L(z) \;\; = \;\; \sum_{n\geq 0} s_L(n) z^n .$$

An integer series $A(z)$ is said to be rational if there exist two polynomials P, Q with integer coefficients such that $A(z) = \frac{P(z)}{Q(z)}$ for all $z \in \mathbb{C}$. If S_L is rational, we say that L has rational growth. A classical result linking languages and growth is the following.

Theorem 4.5. *Let L be a formal language. If L is regular, then it has rational growth.*

4.2 Rationality of growth series for products of groups

Let G be a group and X a finite generating set. The set $Geo(G, X)$, called the language of geodesics, or the geodesic language, is the set of words over X which are geodesics in G with respect to the generating set X. We say that G, with respect to X, has rational geodesic growth if the language $Geo(G, X)$ has rational growth.

In this section, we study more precisely the type, and rationality, of the growth of geodesic languages $Geo(G, X)$ for products of groups.

4.2.1 Graph products

In [39], Loeffler, Meier and Worthington show that the regularity of the language of geodesics is preserved by graph products.

Let G be a group and X a finite generating set. The pair (G, X) is called a Cannon pair if the language of geodesics $Geo(G, X)$ is regular. Let $\mathcal{G} = \{G_1, G_2, ..., G_n\}$ be a finite set of finitely generated groups, $\mathcal{T}$ a graph with n vertices and a bijection $\phi : \mathcal{G} \to \mathcal{T}$. The graph product of $G_1, G_2, \ldots, G_n$ with respect to $\mathcal{T}$, denoted by $\prod_{\mathcal{T}} \mathcal{G}$, is the group generated by the elements G_i, $i = 1, 2, \ldots, n$, modulo relations implying that elements of G_i and G_j commute if there is an edge between $\phi(G_i)$ and $\phi(G_j)$.

A first result about graph products is the following.

Theorem 4.6 (Loeffler, Meier and Worthington, [39])**.** *Let G_i, $i = 1, ..., n$, be groups generated by finite sets A_i, and $\mathcal{T}$ a graph with n vertices. Let $A = \bigcup_{i=1}^n A_i$. Then $\prod_{\mathcal{T}} \mathcal{G}$ is a Cannon pair if and only if each (G_i, A_i) is a Cannon pair.*

Since direct and free products, with respect to standard generating sets, are examples of graph products, Theorem 4.6 implies that the geodesic languages of these two products are regular if the geodesic language of each factor is regular.

Let H and K be two groups generated by finite sets X and Y, respectively. Let $Geo(H, X)$ and $Geo(K, Y)$ be the two languages of geodesics of these groups. From Proposition 2.1 given in Chapter 2, we have that the geodesic growth series for the direct product $H \times K$, with respect to the generating set $X \cup Y$, is given by

$$\Gamma_{H \times K}(z) \;=\; \sum_{n=0}^{\infty} \gamma_{X \cup Y}(n) z^n,$$

where

$$\gamma_{X \cup Y}(n) \;=\; \sum_{j=0}^{n} \binom{n}{j} \gamma_X(j)\, \gamma_Y(n-j) \tag{4.1}$$

for $n \geq 0$, which is the shuffle product, denoted by $A \sqcup\!\sqcup B$, of the two integer sequences $A := (\gamma_X(n))_{n \geq 0}$ and $B := (\gamma_X(n))_{n \geq 0}$.

For the free product, we have from Proposition 2.10 that the geodesic growth series for the free product $H * K$, with respect to the generating set $X \cup Y$, is given by

$$\frac{1}{\Gamma_{H*K}(z)} \;=\; \frac{1}{\Gamma_H(z)} + \frac{1}{\Gamma_K(z)} - 1.$$

These formulas imply the following, based on a proposition of Bacher [2].

Proposition 4.7. *Let H and K be two groups generated by finite sets X and Y, respectively. Suppose that $\Gamma_X(z)$ and $\Gamma_Y(z)$ are rational. Then the free product $H * K$ and direct product $H \times K$, with respect to the generating set $X \cup Y$, have rational geodesic growth.*

Proof. The result is obvious for the free product and for the direct product of two groups whose geodesic growth series are polynomials.

Moreover, by decomposing into simple fractions over $\mathbb{C}$, each rational series in $\mathbb{R}[z]$ can be seen as a finite sum of the form $\frac{\lambda}{(z-\alpha)^k}$ where $\lambda, \alpha \in \mathbb{C}$ and $k \in \mathbb{N}^*$. Since the shuffle product is bilinear, it is then enough to consider shuffle products of the form

$$z^h \ \sqcup\!\sqcup \ \sum_{n=0}^{\infty} n^k \alpha^n z^n \ = \ \sum_{n=0}^{\infty} \binom{n+h}{k} n^k \alpha^n z^{n+h}, \tag{4.2}$$

where $h, k \in \mathbb{N}^*$ and $\alpha \in \mathbb{C}$, and of the form

$$\sum_{n=0}^{\infty} n^h \alpha^n z^n \ \sqcup\!\sqcup \ \sum_{n=0}^{\infty} n^k \beta^n z^n \ = \ \sum_{0 \le m \le n}^{\infty} \binom{n}{m} m^h \, (n-m)^k \, \alpha^n \, \beta^{n-m} z^n, \tag{4.3}$$

where $h, k \in \mathbb{N}^*$ and $\alpha, \beta \in \mathbb{C}$.

Since $\sum_{n=0}^{\infty} n^k \alpha^n z^n$ is a finite sum of derivatives of $\frac{1}{1-\alpha z}$ and (4.3) are evaluations at $x = \alpha$, $y = \beta$ of

$$\left(x \frac{\partial}{\partial x} \right)^h \left(y \frac{\partial}{\partial y} \right)^k \left(\frac{1}{1-(x+y)z} \right),$$

(4.2) and (4.3) are rational and the shuffle product of two rational series is rational. □

4.2.2 Wreath product

Let G and A be two groups generated by finite sets X and Y, respectively, such that G acts on A. As in Section 2.3, equation (2.8), the wreath product of A and G is defined by

$$A \wr G \ = \ \left(\bigoplus_{h \in G} A \right) \ \rtimes \ G.$$

As in Johnson's article [37], we see $A \wr G$ as the split extension

$$1 \ \to \ \bigotimes_{g \in G} gAg^{-1} \ \to \ A \wr G \ \to \ G \ \to \ 1.$$

It follows that $A \wr G$ is generated by $X \cup Y$ and each element $w \in A \wr G$ can be written in the form

$$w \ = \ \left(\prod_{s \in S} s \, h(s) \, s^{-1} \right) \ \cdot \ g, \tag{4.4}$$

where S is a finite subset of G, $g \in G$ and $h : S \to A$ is a map.

Furthermore, Johnson proved that all geodesics are of this form (4.4) and are unique up to the ordering $(s_1, s_2, \ldots, s_n)$ of S, where $n = |S|$, chosen in forming the product in such a way that the integer m given by

$$m \ = \ l_X(s_1) \ + \ \sum_{i=1}^{n-1} l_X(s_i^{-1} s_{i+1}) \ + \ l_X(s_n^{-1} g) \tag{4.5}$$

is minimal [37].

Suppose now that G is finite. Johnson proved that the spherical growth of $A \wr G$ is rational if the spherical growth of A is rational and G is finite [37]. For geodesic growth, we need to account for the number of orderings of each S which preserve the minimality of m. Since G is finite, there are exactly $2^{|G|}$ subsets of G, and each subset S is of size $\le |G|$. Thus there are at most $|G|!$ orderings of the $s_i \in S$ which preserve the minimality of m and we get the following result.

Theorem 4.8. *Let G be a finite group which acts on a finitely generated group A of rational geodesic growth. Then $A \wr G$ has rational geodesic growth with respect to the standard generating set.*

Proof. Since G is finite, there are a finite number of geodesic representatives for each $s_i \in S$. So for all $S \subseteq G$ of size n and $g \in G$, there is a finite number, denoted by $A_{S,g}$, of arrangements and geodesic representatives such that m, defined in (4.5), is preserved.

Since the length of an element w defined in (4.4) is given by $l(w) = m + \sum_{s \in S} l_Y(h(s))$, the number of words of this form of length $m + k$ is exactly

$$A_{S,g} \cdot \sum_{\substack{\sum_{i=1}^{n} l_Y(h(s_i)) = k, \\ h(s_i) \neq 0}} \gamma_Y(l_Y(h(s_i))).$$

As the sum of $l_Y(h(s_i)) = k$ is the k-th coefficient of the integer series $(\Gamma_Y(z) - 1)^n$, we have then that

$$\Gamma_{S,g}(z) = A_{S,g} \cdot z^m \, (\Gamma_Y(z) - 1)^n.$$

Finally, $\Gamma_{A \wr G}(z)$ is a finite sum over S and g. It is thus rational. □

The most studied example of wreath products are the Lamplighter groups $L_m = C_m \wr \mathbb{Z}$ with respect to the standard generating set X. Since $\mathbb{Z}$ is infinite, we can't use Theorem 4.8. However, we have the following result.

Theorem 4.9. *The groups L_2 and L_3 have rational geodesic growth with respect to their standard generating sets.*

Proof. In Section 2.3, we proved that all geodesics in L_m, for all $m \geq 2$, can be seen as particular cases of reduced words

$$a_{k_1}^{i_1} \, a_{k_2}^{i_2} \, \ldots \, a_{k_n}^{i_n} \, t^r,$$

where $k_j \in \mathbb{Z}$ and $a_k^i := t^k a^i t^{-k}$ for all $k \in \mathbb{Z}$ and $|i| \leq \lfloor \frac{m}{2} \rfloor$. Moreover, since the image of a geodesic by the map ϕ defined by $\phi(a) = a$ and $\phi(t) = t^{-1}$ is another geodesic, we have that the number of geodesics in L_m of length n is the double of the number of geodesics in L_m such that $k_1 \geq 1$ or $k_1 = 0$ and $k_2 > 0$.

If we study the cases $m = 2$ or 3 in more detail, each geodesic w in this subset, representing an element $(S, r) \in L_m$, where $|S| = k$, is exactly of one of these forms:

$$|w|_{t^{-1}} = 0 \quad \text{that is} \quad |w|_t = n - k = r \geq 0 \tag{4.6}$$
$$|w|_{t^{-1}} > |w|_t \quad \text{that is} \quad r = min\{k_i\} < 0 \tag{4.7}$$
$$0 < |w|_{t^{-1}} \leq |w|_t \ \text{ and } \ r \geq 0 \quad \text{that is} \quad k_i \geq 0 \; \forall \, i \tag{4.8}$$
$$0 < |w|_{t^{-1}} \leq |w|_t \quad \text{and} \quad 0 > r > min\{k_i\}. \tag{4.9}$$

We can represent all of these words as in Figures 4.1, 4.2, 4.3, and 4.4.

For $m = 2$ or 3, we define the following sets

$$\begin{aligned} A_m^\alpha(n,k,p) &= \#\{\, w \in Geo(L_m, \{a,t\}) \mid w \text{ is of the form } \alpha \text{ with } n,k,p \text{ fixed} \,\}, \\ A_m^\alpha(n,k) &= \#\{\, w \in Geo(L_m, \{a,t\}) \mid w \in A_m^\alpha(n,k,p) \text{ for } p \geq 0 \,\}, \\ A_m^\alpha(n) &= \#\{\, w \in Geo(L_m, \{a,t\}) \mid w \in A_m^\alpha(n,k) \text{ for } k \geq 0 \,\}, \end{aligned}$$

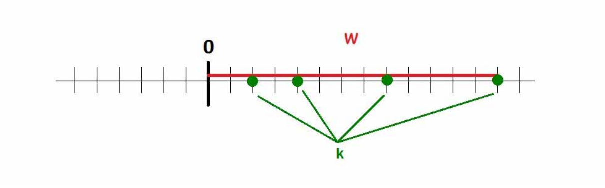

Figure 4.1: Representation of the case (4.6).

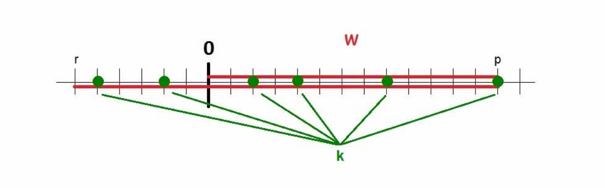

Figure 4.2: Representation of the case (4.7).

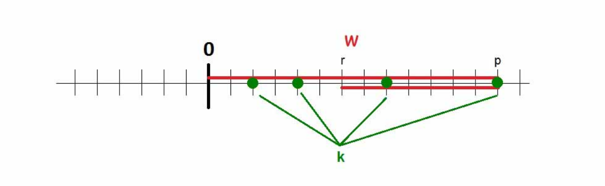

Figure 4.3: Representation of the case (4.8).

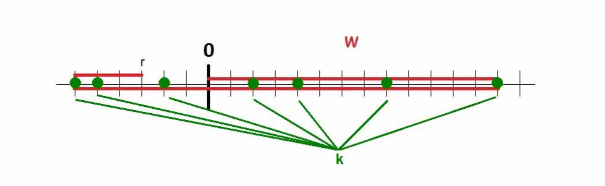

Figure 4.4: Representation of the case (4.9).

where α is one of the four cases defined in (4.6),(4.7),(4.8) or (4.9) , and

$$p \;=\; \begin{cases} max\{k_i\} - min\{k_i\} & \text{if } \alpha \text{ is the case (4.9)} \\ p = max\{k_i\} & \text{otherwise} \end{cases}$$

It is easy to prove that

$$A_m^{(4.7)}(n,k,p) \;=\; A_m^{(4.8)}(n,k,p) \;=\; A_m^{(4.9)}(n,k-1,p)$$

for all $n, k \geq 2$ and $p \geq 0$. This implies that

$$A_m^{(4.7)}(n) \;=\; A_m^{(4.8)}(n) \;=\; A_m^{(4.9)}(n) \;+\; K,$$

where $K = 2^{\frac{n}{2}} - 1$ if n is even and $K = 0$ if n is odd.

In the same way, it is easy to prove that

$$A_2^{(4.6)}(n,k,p) \;=\; \binom{p}{k-1},$$

and

$$A_3^{(4.6)}(n,k,p) \;=\; \binom{p}{k-1} 2^k,$$

which implies that

$$A_2^{(4.6)}(n,k) \;=\; \binom{n-k+1}{k},$$

and

$$A_3^{(4.6)}(n,k) \;=\; \binom{n-k+1}{k} 2^k,$$

thus

$$A_2^{(4.6)}(n) \;=\; \mathcal{F}(n+1) - 1,$$

and

$$A_3^{(4.6)}(n) \;=\; \mathcal{G}(n),$$

where $\mathcal{F}(n)$ is the Fibonacci sequence with $\mathcal{F}(0) = \mathcal{F}(1) = 1$ and $\mathcal{G}(n)$ is the integer sequence defined recursively by $\mathcal{G}(0) = 1$, $\mathcal{G}(1) = 2$, $\mathcal{G}(2) = 4$ and $\mathcal{G}(n) = \mathcal{G}(n-1) + 2\mathcal{G}(n-2) + 2$ for all $n \geq 3$..

By Propositions 2.36, 2.38, 2.39, 2.40 and 2.41 and equations (2.11) and (2.12) we have that

$$\gamma_{L_m,\{a,t\}}(n) \;=\; 2 \cdot \left(A_m^{(4.6)}(n) \;+\; A_m^{(4.7)}(n) \;+\; A_m^{(4.8)}(n) \;+\; A_m^{(4.9)}(n) \right)$$

is rational for $m = 2$ or 3. □

Notice that the rationality of the geodesic growth series does not follow from the regularity of the language of geodesics, since the language is not regular, as the results below show.

Theorem 4.10 (Cleary, Elder and Taback, [14]). *The geodesic language of L_2, with respect to the generating set $\{a,t\}$ is not regular but context-free.*

Theorem 4.11 (Cleary, Elder and Taback, [14]). *The language of all geodesics for L_m with the generating set $\{a,t\}$ is not regular.*

To finish this section, since from Theorem 4.9 we have that L_2 and L_3 have rational geodesic growth and since we know that $\gamma(L_{2m}, \{a,t\}) = \gamma(L_{2m+1}, \{a,t\})$ for all $m \geq 2$ from Proposition 2.44, the next logical question would be to know if the geodesic growth of L_m is rational for all $m \geq 2$. This question is still open.

Conjecture 4.12. *The geodesic growth series of L_m with respect to the generating set $\{a,t\}$ is rational for all $m \geq 2$.*

4.3 FFTP-property

Let G be a group with a finite generating set X. We denote by d_X the word metric on G with respect to X. In [47], Neumann and Shapiro defined the synchronous and asynchronous falsification by fellow traveller properties as the following.

Definition 4.13. Let $k \geq 0$ be a real number and $u = x_1 x_2 \ldots x_n \in X^*$ be the label of a directed path in $\mathcal{C}ay(G,X)$. Then u can be seen as the image of the function $\tilde{u} : \mathbb{N} \to X^*$, where

$$\tilde{u}(i) = \begin{cases} x_1\, x_2\, \ldots\, x_i & \text{if } i \leq n \\ u & \text{otherwise.} \end{cases}$$

Two words $u_1, u_2 \in X^*$ are said to synchronously k-fellow travel if $d_X(\tilde{u}_1(i), \tilde{u}_2(i)) \leq k$ for all $i \in \mathbb{N}$, and to asynchronously k-fellow travel if there is a non-decreasing continuous function $\phi : \mathbb{N} \to \mathbb{N}$ such that $d_X(\tilde{u}_1(i), \tilde{u}_2(\phi(i))) \leq k$ for all $i \in \mathbb{N}$.

A group G with finite generating set X has the (asynchronous) falsification by fellow traveller property ((asynchronous) FFTP-property) if there is a constant k such that every non-geodesic word in G with respect to X is (asynchronously) k-fellow travelled by a shorter word (not necessarily a geodesic).

In [47], Neumann and Shapiro proved that if (G,X) has the FFTP-property then the full language $Geo(G,X)$ is regular, and used this relation to prove the rationality of the geodesic growth in hyperbolic groups. In [19], Elder proved that the converse is false, that is there is a group G and finite generating set X such that $Geo(G,X)$ is regular but X fails to have the FFTP-property.

Definition 4.14. Let $h : \mathbb{N} \to \mathbb{R}$ be a monotone increasing function such that there is a constant $N \in \mathbb{N}$ so that $h(n) \geq n$ for all $n \geq N$. We say that (G,X) has the $h(n)$-FFTP property if there is a constant $k \in \mathbb{N}$ so that each word $w \in X^*$ with $\ell(w) > h(|w|)$ is k-fellow travelled by a shorter word (not necessarily a geodesic).

Elder proved in [18] that the asynchronous version of the original FFTP-property is equivalent to its synchronous version. Moreover, although an asynchronous version of the $h(n)$-FFTP property 4.14 also exists, we focus on the synchronous version, since the asynchronous version is equivalent to the synchronous version of the $h(n)$-FFTP property. It is an open question whether the asynchronous version of the $h(n)$-FFTP-property is equivalent to its synchronous version.

For example, if we define $h(n) := n$ for all $n \geq 0$, we recover the original FFTP-property defined in Definition 4.13.

Notice that if (G,X) has the $h(n)$-FFTP property, then it has the $\tilde{h}(n)$-FFTP for all monotone increasing maps $\tilde{h} : \mathbb{N} \to \mathbb{R}$ such that $\tilde{h}(n) \geq h(n)$ for all n.

Remark 4.15. If (G,X) has the FFTP property, then it has the $h(n)$-FFTP property for all h superlinear.

In this section, we study in more detail the groups $G_k = \mathbb{F}_k \times \mathbb{F}_k$ with presentation

$$\mathcal{P}_k \quad = \quad \langle a_1, b_1, c_1, ..., a_k, b_k, c_k \mid [a_i, b_j] = 1, c_i = a_i b_i \ \forall i, j = 1, 2, ..., k \ \rangle ,$$

where, for all $k \geq 2$, we denote by $X_k \ := \ \{a_i^{\pm 1}, b_i^{\pm 1}, c_i^{\pm 1}, i = 1, 2, ..., n\}$ the generating set of G_k. For easier notation, we define $A_k := \{a_1^{\pm 1}, a_2^{\pm 1}, ..., a_k^{\pm 1}\}$, $B_k := \{b_1^{\pm 1}, b_2^{\pm 1}, ..., b_k^{\pm 1}\}$ and $C_k = \{c_1^{\pm 1}, c_2^{\pm 1}, ..., c_k^{\pm 1}\}$.

If a word $w \in X_k^*$ contains positive and negative powers, we need to look at positive and negative powers separately. Then the normal form of $w \in X_k^*$ is

$$NF(w) \quad := \quad w_{1,1}\, w_{2,1}\, w_{1,2}\, w_{2,2} \ \ldots \ w_{1,k}\, w_{2,k}, \tag{4.10}$$

where $w_{1,i} \in A_k^*$, $w_{2,i} \in B_k^*$ for all $i = 1, 2, ..., k$ are non empty except maybe the first and last, and if i is odd (respectively even), then $w_{1,i}$ and $w_{2,i}$ contain only positive powers (respectively negative powers). It is unique for each $w \in X_k^*$ by construction.

In the particular case if $g \in X_k^*$ is a positive word (i.e with only positive powers), the normal form of g is then

$$NF(g) = w_1 w_2, \tag{4.11}$$

where $w_1 \in A_k^*$ and $w_2 \in B_k^*$ are freely reduced and positive.

Remark 4.16. Notice that the only way to find a shorter word from a non-geodesic w on X_k is to move letters (respecting the relations of the presentation $\mathcal{P}_k$) to create pairs $(a_i b_i)^{\pm 1}$ for $i \in \{1, 2, ..., n\}$ and replace them by $c_i^{\pm 1}$. Since the normal form given in (4.10) is unique for each geodesic, if we have a normal form given in (4.10), then all the geodesics found by creating pairs are counted exactly once in $Geo(G_k, X_k)$. Thus listing the complete geodesic language $Geo(G_k, X_k)$ is equivalent to listing recursively all normal forms given in (4.10) and, for each of these forms, find the geodesics.

We make remarks that are useful later on.

1. Every geodesic which begins with a negative letter is the image of a geodesic which begins with a positive letter by the morphism $\phi : X_k \to X_k$, $x \mapsto x^{-1}$ for all $x \in X_k$.
2. Every positive geodesic which begins with a letter $b_i \in B_k$ is the image of a positive geodesic which begins with a letter $a_i \in A_k$ by the morphism $\lambda : X_k \to X_k$, $b_i \leftrightarrow a_i$ and $c_i \mapsto c_i$ for all $i \in \{1, 2, ..., k\}$.
3. In G_{2k}, every positive geodesic which begins by a letter $a_{2i-1} \in A_{2k}$ is the image of a positive geodesic which begins by a letter $a_{2i} \in A_{2k}$ by the morphism $\nu : X_k \to X_k$, $a_{2i} \leftrightarrow a_{2i-1}$, $b_{2i} \leftrightarrow b_{2i-1}$ and $c_{2i} \leftrightarrow c_{2i-1}$ for all $i = 1, 2, ..., k$.

From the two first remarks, it suffices then to understand the geodesics which begin with a positive $a_i \in A_k$.

Theorem 4.17. *The group G_k, with respect to the generating set X_k, has the $(2n)$-FFTP property for all $k \geq 2$.*

Proof. Let $w \in X^*$ have $\ell(w) > 2|w|$. The proof is separated into two parts. First, suppose that w does not contain letters of C_k:

i) If w is not freely reduced, then removing a pair xx^{-1} where $x \in A_k \cup B_k$ gives a shorter word which 2-fellow travels w.

ii) If w is freely reduced and if $NF(w) = w_1w_2$ is not freely reduced, then (without loss of generality) $w_1 = u_1a_1a_1^{-1}u_2$. So the word $u_1u_2w_2$ is shorter and 2-fellow travels w.

iii) Finally, if w and $NF(w) = w_1w_2$ are freely reduced, to get a geodesic equal to w, the only thing we can do is move letters next to each other to create many pairs a_ib_i to be replaced by c_i. But the best we could do is join all the letters together, which would give a word of length $\frac{\ell(w)}{2}$, but since $|w| < \frac{\ell(w)}{2}$, the new word couldn't be a geodesic. This is a contradiction because we can not reduce this new word since all letter in this new word is in C_k, which is the definition of a geodesic in G_k. So this case cannot occur.

Now, suppose that w contains letters of C_k:

i) If w is not freely reduced, then removing one pair xx^{-1} where $x \in X_k$ gives a shorter word which 2-fellow travels w.

ii) If w is freely reduced, then define the <u>second normal form</u> of w by

$$SNF(w) = u_1v_1z_1u_2v_2z_2 \dots u_kv_kz_k$$

where $z_i \in C_k^*$, $u_i \in A_k^*$ and $v_i \in B_k^*$ by moving letters in A_k and B_k and fixing the positions of letters in C_k. If this word is not freely reduced, then (without loss of generality) $u_1 = t_1a_1a_1^{-1}t_2$. So the word $t_1t_2v_1z_1u_2v_2z_2 \dots u_kv_kz_k$ is shorter and 2-fellow travels w .

Now replace each z_j by $c_i \mapsto a_ib_i$ for all $i = 1, 2, ..., k$. After possible free cancellations, we have a new word w'. If $l(w') < l(w)$, it implies that (after a possible move of letters in $A_k \cup B_k$ with respect to relations in $\mathcal{P}_k$ and without loss of generality) $w = w_1a_1b_1c_1^{-1}w_2$ where w_1, w_2 are words on X_k. If $l(w') \geq l(w)$, by the first step of this proof w' is either not freely reduced or its normal form isn't freely reduced.

Then (after a possible move of letters in $A_k \cup B_k$ with respect to relations in $\mathcal{P}_k$ and without loss of generality) w contains $a_1^{-1}c_1$ as subword. So the word $\tilde{w}$ where we replace this subword by b_1 is shorter and 2-fellow travels w .

Then (G_k, X_k) has the (2n)-FFTP property with constant $k = 2$. □

If we take the group $G_2 = \mathbb{F}_2 \times \mathbb{F}_2$ generated by the set X_2, then the language L_2 defined by

$$L_2 \ := \ Geo(G_2, X_2) \ \cap \ a_1^*c_2^*b_1^*$$

verifies $L_2 = \{a_1^lc_2^mb_1^n | m \geq l \text{ or } n\}$ and is therefore context-free but not regular, so by Theorem 4.2, the language of geodesics $Geo(G_2, X_2)$ is not regular. This is the first example of a group which satisfies the $h(n)$-FFTP property and does not have a regular language.

In particular, it shows that $h(n)$-FFTP property for h different from the identity doesn't imply regularity of the language of geodesics.

If

$$L_3 \ := \ Geo(G_3, X_3) \ \cap \ a_1^*a_2^*c_3^*b_2^*b_1^* \, ,$$

then we have that

$$L_3 \ = \ \left\{a_1^l \, a_2^m \, c_3^n \, b_2^p \, b_1^q \mid n \geq [(m \text{ or } p) \text{ and } (l \text{ or } q)]\right\}.$$

Now, suppose L_3 is context-free, and let $N \geq 1$ be the integer of Lemma 4.4. Let

$$w \;=\; a_1^N a_2^N c_3^N b_2^{N+1} b_1^{N+1},$$

where all letters a_1 are marked.

If w is factored as $\alpha u \beta v \gamma$ such that $\alpha u^n \beta v^n \gamma \in L$ for all $n \geq 0$, Lemma 4.4 implies that u and v are powers of a letter.

By the third point of the Lemma 4.4, there are only two possibilities:

1. if each of β, v, γ contains at least one marked position, α, u, β, v are powers of a_1 (α and u can be empty). Then the word $\alpha u^2 \beta v^2 \gamma \notin L$ because the power of the letter c_3 in this new word is stricly smaller than the powers of a_1 and b_1.
2. if each of α, u, β contains at least one marked position, α and u are powers of a_1. Then we have 6 choices for v:
   (i) if v is a power of the letter a_1, we have exactly the same case as before.
   (ii) if v is a power of the letter a_2, then the word $\alpha u^2 \beta v^2 \gamma \notin L$ for the same reason as in the previous case.
   (iii) if v is a power of the letter c_3, then the word $\alpha u^2 \beta v^2 \gamma \notin L$ because the power of the letter c_3 in this new word is stricly smaller than the powers of a_2 and b_2.
   (iv) if v is a power of the letter b_2, then the word $\alpha u^2 \beta v^2 \gamma \notin L$ for the same reason as in case (i).
   (v) if v is a power of the letter b_1, then the word $\alpha u^2 \beta v^2 \gamma \notin L$ for the same reason as in case (i).
   (vi) if v is empty, then the word $\alpha u^2 \beta v^2 \gamma \notin L$ for the same reason as in case (i).

Since L is not context-free, then $Geo(G_3, X_3)$ is not context-free by Theorem 4.2.

In the same way, we can prove the following proposition.

Proposition 4.18. *Let L be the language defined by*

$$L \;:=\; Geo(G_2, X_2) \;\cap\; a_1^* \, c_2^* \, c_1^* \, b_1^* \, b_2^*.$$

Then

$$L \;=\; \left\{ a_1^l \, c_2^m \, c_1^n \, b_1^p \, b_2^q \mid \neg I_1 \text{ and } \neg I_2 \text{ and } \neg I_3 \text{ and } \neg I_4 \text{ and } \neg I_5 \text{ and } \neg I_6 \right\}, \qquad (4.12)$$

where

$$\begin{aligned}
I_1 &= (n+p \geq l > n \;\; and \;\; m \leq q) \\
I_2 &= (n+p \geq l \;\; and \;\; m \geq q \;\; AND \; m+n < l+q\;) \\
I_3 &= (n+p \leq l \;\; and \;\; m \leq q) \\
I_4 &= (n+p \leq l \;\; and \;\; q \leq m < p+q) \\
I_5 &= (p \geq l \;\; and \;\; m < l) \\
I_6 &= (p \leq l < p+n \;\; and \;\; m < p)
\end{aligned}$$

Proof. Let $w = a_1^l \, c_2^m \, c_1^n \, b_1^p \, b_2^q$. By definition, if we replace each $c_i \in C_k$ by $a_i b_i \in A_k B_k$, its normal form is

$$a_1^l \, a_2^m \, a_1^n \, b_2^m \, b_1^{n+p} \, b_2^q.$$

Let us define a block as a maximal power of a letter. There are, then, three possibilities to create pairs from this normal form which contains 6 blocks:

1. Merge the block of a_2's with the first block of b_2's and the second block of a_1's with the block of b_1's. We obtain w, which is of length $l+m+n+p+q$.
2. Merge the first block of a_1's with the block of b_1's and the block of a_2's with the second block of b_2's. Here, the result depends on powers:
   - If $m \leq q$ and $n+p \geq l$, then we obtain the new word $b_2^m\, c_1^l\, b_1^{n+p-l}\, c_2^m\, b_2^{q-m}\, a_1^n$ which has length $m+2n+p+q$.
   - If $m \geq q$ and $n+p \geq l$, then we obtain the new word $b_2^m\, c_1^l\, b_1^{n+p-l}\, a_2^{m-q}\, c_2^q\, a_1^n$, which has length $2m+2n+p$.
   - If $m \leq q$ and $n+p \leq l$, then we obtain the new word $b_2^m\, a_1^{l-n-p}\, c_1^{n+p}\, c_2^m\, b_2^{q-m}\, a_1^n$, which has length $l+m+n+q$.
   - If $m \geq q$ and $n+p \leq l$, then we obtain the new word $b_2^m\, a_1^{l-n-p}\, c_1^{n+p}\, a_2^{m-q}\, c_2^q\, a_1^n$, which has length $l+2m+n$.
3. Merge the block of a_1's with the block of b_1's. Here, again, the result depends on powers:
   - If $p \geq l$, then we obtain the new word $b_2^m\, c_1^l\, a_2^m\, c_1^n\, b_1^{p-l}\, b_2^q$ which has length $2m+n+p+q$.
   - If $p \leq l < p+n$, then the new word $b_2^m\, c_1^l\, a_2^m\, c_1^{n+p-l}\, a_1^{l-p}\, b_2^q$ has length $l+2m+n+q$.
   - If $l \geq p+n$, we find exactly the same possibilities as previously to create pairs.

Thus w is not a geodesic if and only if the powers l, m, n, p, q verify one of the inequalities I_j, $j = 1, 2, ..., 6$. □

Proposition 4.18 implies the following theorem.

Theorem 4.19. *For all $k \geq 2$, Geo(G_k, X_k) is not context-free.*

Proof. Let L be the language defined in Proposition 4.18, that is

$$L \ := \ Geo(G_2, X_2) \ \cap \ a_1^*\, c_2^*\, c_1^*\, b_1^*\, b_2^*.$$

Suppose L is context-free, and let $N \geq 1$ be the integer defined in the Lemma 4.4. Let

$$w \ = \ a_1^N c_2^N c_1^N b_1^{3N} b_2^{3N},$$

where all letters a_1 are marked. By definition of L, $w \notin I_j$ for all $j = 1, 2, .., 6$. Then $w \in L$.
If w is factored as $\alpha u \beta v \gamma$ such that $\alpha u^n \beta v^n \gamma \in L$ for all $n \geq 0$, Lemma 4.4 implies that u and v are powers of a letter.
By the third point of Lemma 4.4, there are only two possibilities:

1. if each of β, v, γ contains at least one marked position, then α, u, β, v are powers of a_1 (α and u can be empty). Thus the word $\alpha u^2 \beta v^2 \gamma \notin L$ because $\alpha u^2 \beta v^2 \gamma$ verifies I_5.
2. if each of α, u, β contains at least one marked position, then α and u are powers of a_1. So we have 6 choices for v:

- if v is a power of the letter a_1, we have exactly the same case as before.
- if $u = a^{k_1}$ and $v = c_2^{k_2}$, then there are 3 possibilities:
  - if $k_1 > k_2$, then the word $\alpha u^2 \beta v^2 \gamma$ verifies I_5 and is not in L.
  - if $k_1 < k_2$, then the word $\alpha\beta\gamma$ verifies I_5 and is not in L.
  - if $k_1 = k_2$, then the word $\alpha u^2 \beta v^2 \gamma$ verifies I_1 and is not in L.
- if v is a power of the letter c_1, then the word $\alpha u^2 \beta v^2 \gamma$ verifies I_5 and is not in L.
- if v is a power of the letter b_1, then the word $\alpha u^2 \beta v^2 \gamma$ verifies I_5 and is not in L.
- if v is a power of the letter b_2, then the word $\alpha u^2 \beta v^2 \gamma$ verifies I_5 and is not in L.
- if v is empty, then the word $\alpha u^2 \beta v^2 \gamma$ verifies I_5 and is not in L.

Since L is not context-free, $Geo(G_2, X_2)$ is not context-free.
As

$$Geo(G_2, X_2) \;=\; Geo(G_k, X_k) \cap X_2^*$$

for all $k \geq 2$, the language $Geo(G_k, X_k)$ is not context-free for all $k \geq 2$. □

Since $Geo(G_k, X_k)$ is not context-free for all $k \geq 2$, the logical next question would be to know if it is context-sensitive.

Proposition 4.20 (Personal communication, [17])**.** *The language of geodesics of $\mathbb{F}_k \times \mathbb{F}_k$ is context-sensitive for all $k \geq 2$.*

Proof. We give here an informal proof.
Computationally, a context-sensitive language is equivalent to a linearly bounded tape nondeterministic Turing machine. Assume a word w in generators X_k is written on the tape. Write $\#w^{-1}$ at the end of this word. Now, enumerate systematically all words v in X_k of length less than $l(w)$ and write them at the end of the tape. We have now $w\#w^{-1}v$.

For each v, take the image of $w^{-1}v$ in $\mathbb{F}_k \times \mathbb{F}_k$ with standard generating set. Then call the algorithm to solve the Word problem, which can be done in Logspace. If you find a v, answer "Not geodesic", and if no, accept w. □

Note that the time of this algorithm seems very long, but that doesn't matter since we only used at most $3n+1$ squares of the tape, where $n = |w|$.

Since the Chomsky hierarchy does not adequately describe the language $Geo(G_k, X_k)$, we would like to know if $Geo(G_k, X_k)$ belongs to one of the subclasses of the context-sensitive languages, like indexed languages. Unfortunately, proving that a language is (or not) indexed is complex.

To have a better understanding of $Geo(G_2, X_2)$, we give an algorithm in Appendix B able to count all positive geodesics in G_2, i.e such that the normal form $w_1 w_2$ is positive and of the form (4.11). The idea behind the algorithm is to read the two words w_1 and w_2 from left to right and decide whether or not stopping and creating a pair gives us the maximum of the number of pairs we could create.

This idea could be generalised for all geodesics in G_2 in two steps:

1. From the normal form $w_{1,1}\, w_{2,1}\, w_{1,2}\, w_{2,2}\, \ldots\, w_{1,k}\, w_{2,k}$ given in (4.10), we read the two vectors $w_{1,1}\, w_{1,2}\, \ldots\, w_{1,k}$ and $w_{2,1}\, w_{2,2}\, \ldots\, w_{2,k}$ from left to right and decide whether or not stopping and studying in more detail the pairs given by the normal form $w_{1,i}w_{2,j}$ (where the signs of the powers in $w_{1,i}$ and $w_{2,j}$ are the same) gives us the maximality of the number of pairs we could create with the same idea as the algorithm in Appendix B.

2. If we stop and $w_{1,i}$,$w_{2,j}$ are positive, then use the algorithm in Appendix B. If they are negative, then use the algorithm in Appendix B on $\phi(w_{1,i})$ and $\phi(w_{2,j})$, where ϕ is defined above.

Since the definition of L in equation (4.12) is completely given by an intersection of languages verifying an inequality, we have the following conjectures.

Conjecture 4.21. *The language L defined in equation* (4.12) *is indexed.*

Conjecture 4.22. *The language $Geo(G_2, X_2)$ is indexed.*

By the article of Loeffler, Meier and Worthington [39], we know that $Geo(G_k, A_k \cup B_k)$ is regular for all $k \geq 2$ and by Neumann and Shapiro [47], the n-FFTP property implies the regularity of the language of geodesics. We then found groups G_k such that:

1. The groups, with respect to the generating set $A_k \cup B_k$, have the FFTP property, so their languages of geodesics $Geo(G_k, A_k \cup B_k)$ are regular;

2. The groups, with respect to the generating set X_k, have the $h(n)$-FFTP property where $h(n) \neq n$ is linear but have their languages of geodesics $Geo(G_k, X_k)$ which are not regular. Hence (G_k, X_k) do not have the FFTP property.

Finally, another interesting example for the $h(n)$-FFTP property is given by Cannon: Let $\mathcal{G}$ be the group defined by

$$\mathcal{G} \;:=\; \Big\langle\, a, t \mid t^2 = 1,\; atat = tata \,\Big\rangle.$$

$\mathcal{G}$ is the split extension of $\mathbb{Z}^2$, generated by its standard generating set $\{a, b\}$, by C_2 , generated by its standard generating set $\{t\}$, such that t conjugates a to b and b to a. Neumann and Shapiro studied $\mathcal{G}$ in [47] to prove that the FFTP property does depend on the generating sets. This group verifies then:

1. With respect to the generating set $\{a, b, t\}$, $\mathcal{G}$ has the FFTP property, so its language of geodesics $Geo(\mathcal{G}, \{a, b, t\})$ is regular;

2. The group, with respect to the generating set $\{a, t\}$, has the $h(n)$-FFTP property where $h(n) = \left(1 + \frac{1}{c}\right) n$ is linear (c.f. [1, Proposition 5]), has its language of geodesics $Geo(\mathcal{G}, \{a, t\})$ which is regular (c.f. [19]), but does not have the FFTP property (c.f. [19]).

Antolín, Ciobanu, Elder and Hermiller asked in [1] if $\mathcal{G}$ has $h(n)$-FFTP property where $h(n) = n + c$, $c \in \mathbb{N}$. This question is still open.

Conjecture 4.23 (Antolín, Ciobanu, Elder and Hermiller, [1])**.** *The group $\mathcal{G}$, with respect to the generating set $\{a, t\}$, has the $(n + c)$-FFTP property where $c \in \mathbb{N}$ is fixed.*

Another conjecture is the following.

Conjecture 4.24 (Antolín, Ciobanu, Elder and Hermiller, [1])**.** *The $(n + c)$-FFTP property does not imply the FFTP property for all $c \in \mathbb{N}^*$.*

Appendix A

List of open questions

Here is a list of open questions and conjectures, listed by topic, which are prompted by the results in this thesis.

Products of groups

The following four conjectures were stated in Chapters 2 and 4.

Conjecture 2.8. *Let H and K be two finitely generated groups. Then the minimal geodesic growth rate of the direct product $H \times K$ is given by*

$$\gamma(H \times K) \; = \; \gamma(H) \; + \; \gamma(K).$$

Conjecture 2.13. *Let H and K be two groups generated by the finite sets X and Y, respectively. Then*

$$\gamma(H * K, X \cup Y) \; \geq \; \gamma(H, X) \; + \; \gamma(K, Y) \; + \; 1.$$

Conjecture 2.46. *For all $m \geq 2$,*

$$\gamma(L_{2m}, \{a, t\}) \; = \; \gamma(C_m * \mathbb{Z}, \{a, t\}).$$

Conjecture 4.12. *The geodesic growth series of L_m with respect to the generating set $\{a, t\}$ is rational for all $m \geq 2$.*

Groups of intermediate spherical growth

We proved that many examples given by Bartholdi in [3], of intermediate spherical growth, have exponential geodesic growth (c.f. Chapter 3 of this thesis). It is interesting to try to find an example of group that has intermediate geodesic growth. The Gupta-Fabrykowski group could be a good candidate. It was showed in Chapter 3 of this thesis that it is not feasible to prove that the geodesic growth is exponential using Schreier graphs in the same manner used for the other examples.

Question 1. Does the Gupta-Fabrykowski group have exponential geodesic growth for some generating set?

Formal geodesic growth

Question 2. Let L be the regular set of all geodesics in a hyperbolic group with respect to a finite generating set, and $s : \mathbb{N} \to \mathbb{N}$ be a map defined by $s(n) := \#\{w \in L \mid |w| = n\}$ for all $n \geq 0$. Are there some constants A, B and α such that

$$A \cdot \alpha^n \ \leq \ s(n) \ \leq \ B \cdot \alpha^n \ ?$$

Ideas for the answer:

1. Coornaert gave a geometric proof to bound the spherical growth. Is it possible to find a similar proof for the geodesic growth?
2. Since L is regular, we could study the adjacency matrix via the Perron-Frobenius Theorem.

□

Question 3. Is there a group with solvable Word Problem and irrational geodesic growth?

Here are 4 ideas to find an example:

1. Work with $\mathbb{F}_k \times \mathbb{F}_k$ with a non standard generating set;
2. The Heisenberg group H_2. We know that it has a non regular normal form, non regular geodesic language and rational spherical growth for all generating sets [15].
3. The Heisenberg groups $H_n, n \geq 3$;
4. Virtually abelian groups. We know that the spherical growth is rational for all generating sets. Furthermore, Cannon has given an example of a generating set for a virtually abelian group such that the geodesic language is not regular (c.f. [39] and [47]). Since all examples studied of virtually abelian groups have rational geodesic growth, it is conjectured that all virtually abelian groups have rational geodesic growth, for all generating sets.

Question 4. Is the set of possible geodesic growth rates dense in $[1, \infty[$?

The $h(n)$-FFTP property

Conjecture 4.22. *The language of geodesics of* $\mathbb{F}_2 \times \mathbb{F}_2$*, generated by the set*

$$\{a_1, a_2, b_1, b_2, (a_1 b_1), (a_2 b_2)\},$$

is indexed.

We showed that the geodesic language of $\mathbb{F}_2 \times \mathbb{F}_2$ with respect to the generating set

$$\{a_1, a_2, b_1, b_2, (a_1 b_1), (a_2 b_2)\},$$

is not context free by focusing on the language L by

$$L \ := \ \left\{ a_1^l \, c_2^m \, c_1^n \, b_1^p \, b_2^q \mid \neg I_1 \cap \neg I_2 \cap \neg I_3 \cap \neg I_4 \cap \neg I_5 \cap \neg I_6 \right\},$$

where

$$\begin{aligned}
I_1 &= (n+p \geq l > n \ \text{ AND } \ m \leq q) \\
I_2 &= (n+p \geq l \ \text{ AND } \ m \geq q \ \text{ AND } m+n < l+q \ ) \\
I_3 &= (n+p \leq l \ \text{ AND } \ m \leq q) \\
I_4 &= (n+p \leq l \ \text{ AND } \ q \leq m < p+q) \\
I_5 &= (p \geq l \ \text{ AND } \ m < l) \\
I_6 &= (p \leq l < p+n \ \text{ AND } \ m < p)\,.
\end{aligned}$$

Conjecture 4.21. *The language L is indexed.*

Conjecture 4.23. *The group $\mathcal{G} := \langle\, a,t \mid t^2 = 1,\ atat = tata \,\rangle$, with respect to the generating set $\{a,t\}$, has the $(n+c)$-FFTP property where $c \in \mathbb{N}$ is fixed.*

Conjecture 4.24. *For all $c \in \mathbb{N}_+$, the $(n+c)$-FFTP property does not imply the FFTP property.*

Appendix B

Algorithm

In this Appendix, we give the code to the Algorithm implemented in C able to count (without listing) all geodesics representatives of a positive word $w \in \mathbb{F}_2 \times \mathbb{F}_2$ if the first normal form is given by $w_1 \cdot w_2$, $w_1 \in \{a,b\}^*$ and $w_2 \in \{x,y\}^*$, with comments and remarks.

We begin by translating the two words w_1 and w_2 to vectors v_1, v_2 with integer coefficients by

$$\begin{array}{rcl} w_1 &=& a^{i_1}\, b^{i_2}\, a^{i_3}\, \ldots\, b^{i_{2k}}\, a^{i_{2k+1}} \;\rightarrow\; v_1 := (\, i_1,\, i_2,\, i_3,\, \ldots,\, i_{2k},\, i_{2k+1},\, -1\,) \\ w_2 &=& x^{j_1}\, y^{j_2}\, x^{j_3}\, \ldots\, y^{i_{2l}}\, x^{j_{2l+1}} \;\rightarrow\; v_2 := (\, j_1,\, j_2,\, j_3,\, \ldots,\, i_{2l},\, i_{2l+1},\, -1\,), \end{array}$$

where $i_1 = 0$ or $i_{2k+1} = 0$ if w_1 begins or ends with the letter b, respectively, and in the same way for w_2. Notice that v_1 and v_2 are vectors with positive integer components, except for the last one, which is -1 in order to end the loop in the algorithm.

Firstly, we have to test if a vector is empty, and if not, if it begins with a in the case of v_1 or x in the case of v_2,

```
int Empty( int vector[])
{
    if( vector[0]==-1 )
    {
        return 1;
    }
    else if(vector[0]==0 )
    {
        if(vector[1] == -1 || vector[1] == 0)
        {
            return 1;
        }
        else
        {
            return 0;
        }
    }
    else
    {
        return 0;
    }
}


int FirstLetter( int vector[])
```

```
{
    if( Empty(vector)==1)
    {
        return -1;
    }
    else if( vector[0] == 0)
    {
        return 0;
    }
    else
    {
        return 1;
    }
}
```

In the same way, we need to be able to find the length of a vector.

```
int Length(int vector[])
{
    if(Empty(vector) != 1)
    {
        int i;

        for(i=0 ; vector[i] != -1 ; i = i+1)
        {
            // Do nothing
        }
        return i;
    }
}
```

Finally, we have the Algorithm which give the number of geodesics such that the first normal form is w_1w_2.

```
int Algorithm(int vector1[], int vector2[], int number)
{
    int Temp, min=0;

    if( Empty(vector1)==1 || Empty(vector2)==1)
    {
        return number;
    }
    else
    {
        if(FirstLetter(vector1)==1 && FirstLetter(vector2) ==1)
        {
            if(vector1[0] <= vector2[0]){min=vector1[0];}
            else {min=vector2[0];}
            vector1[0]   = vector1[0]- min;
            vector2[0]   = vector2[0]-min;
            return Algorithm(vector1, vector2, number);
        }
        else if(FirstLetter(vector1)==0 && FirstLetter(vector2) ==0)
        {
            if(vector1[1] <= vector2[1]){min=vector1[1];}
            else {min=vector2[1];}
            vector1[1]   = vector1[1]-min;
            vector2[1]   = vector2[1]-min;

            if (vector1[1]==0)
```

```
    {
        for(Temp=2; vector1[Temp] != -1 ; Temp++)
        {
            vector1[Temp-2] = vector1[Temp];
        }
        vector1[Temp-2]=-1;
        vector1[Temp-1]=-1;
        vector1[Temp]=-1;
    }
    if (vector2[1]==0)
    {
        for(Temp=2; vector2[Temp] != -1 ; Temp++)
        {
            vector2[Temp-2] = vector2[Temp];
        }
        vector2[Temp-2]=-1;
        vector2[Temp-1]=-1;
        vector2[Temp]=-1;
    }

    return Algorithm(vector1, vector2, number);

}

else if((FirstLetter(vector1)==1 && FirstLetter(vector2) ==0))
{
    int V1[Length(vector1)], V2[Length(vector2)];
    int i=0,j=2, choice1=0, choice2=0, choice3=0, newnumber=0;

    for(i=0; vector1[i]!= -1 ; i++)
    {
        V1[i]=vector1[i];
    }
     for(j=2; vector2[j]!= -1 ; j++)
    {
        V2[j-2]=vector2[j];
    }
    V1[i]=-1;
    V1[0]=0;

    V2[j-2]=-1;
    V2[j-1]=-1;
    V2[j]=-1;

    newnumber=number * 2;
    choice1 = Algorithm(V1,vector2, number);
    choice2 = Algorithm(vector1, V2, number);
    choice3 = Algorithm(vector1, V2, newnumber);
    if( choice1 > choice2 )
    {
        return choice1;
    }
    else if ( choice1 < choice2 )
    {
        return choice2;
    }
    else
    {
        return choice3;
    }
}
else
{
```

```
            int V1[Length(vector1)], V2[Length(vector2)];
            int i,j, choice1=0, choice2=0, choice3=0, newnumber2=0;

            for(i=2; vector1[i]!= -1 ; i++)
            {
                V1[i-2]=vector1[i];
            }
             for(j=0; vector2[j]!= -1 ; j++)
            {
                V2[j]=vector2[j];
            }
            V2[j]=vector2[j];
            V2[0]=0;
            V1[i-2]=-1;
            V1[i-1]=-1;
            V1[i]=-1;

            newnumber2=number*2;
            choice1 = Algorithm(V1,vector2, number);
            choice2 = Algorithm(vector1, V2, number);
            choice3 = Algorithm(vector1, V2, newnumber2);

            if( choice1 > choice2 )
            {
                return choice1;
            }
            else if ( choice1 < choice2 )
            {
                return choice2;
            }
            else
            {
                return choice3;
            }
        }
    }
}
```

In the same way, we have the Algorithm which gives the number of pairs we create from two vectors v_1 and v_2.

```
int NumberOfPairs(int vector1[], int vector2[], int pairs)
{
    int Temp, min=0;

    if( Empty(vector1)==1 || Empty(vector2)==1)
    {
        return pairs;
    }
    else
    {
        if(FirstLetter(vector1)==1 && FirstLetter(vector2) ==1)
        {
            if(vector1[0] <= vector2[0]){min=vector1[0];}
            else {min=vector2[0];}
            pairs       = pairs + min;
            vector1[0]  = vector1[0]- min;
            vector2[0]  = vector2[0]-min;
            return NumberOfPairs(vector1, vector2, pairs);
        }
        else if(FirstLetter(vector1)==0 && FirstLetter(vector2) ==0)
        {
```

```
    if(vector1[1] <= vector2[1]){min=vector1[1];}
    else {min=vector2[1];}
    pairs        = pairs + min;
    vector1[1]   = vector1[1]-min;
    vector2[1]   = vector2[1]-min;

    if (vector1[1]==0)
    {
        for(Temp=2; vector1[Temp] != -1 ; Temp++)
        {
                vector1[Temp-2] = vector1[Temp];
        }
        vector1[Temp-2]=-1;
        vector1[Temp-1]=-1;
        vector1[Temp]=-1;
    }
    if (vector2[1]==0)
    {
        for(Temp=2; vector2[Temp] != -1 ; Temp++)
        {
                vector2[Temp-2] = vector2[Temp];
        }
        vector2[Temp-2]=-1;
        vector2[Temp-1]=-1;
        vector2[Temp]=-1;
    }

    return NumberOfPairs(vector1, vector2, pairs);

}

else if((FirstLetter(vector1)==1 && FirstLetter(vector2) ==0))
{

    int V1[Length(vector1)], V2[Length(vector2)];
    int i=0,j=2, choice1=0, choice2=0;

    for(i=0; vector1[i]!= -1 ; i++)
    {
        V1[i]=vector1[i];
    }
     for(j=2; vector2[j]!= -1 ; j++)
    {
        V2[j-2]=vector2[j];
    }
    V1[i]=-1;
    V1[0]=0;

    V2[j-2]=-1;
    V2[j-1]=-1;
    V2[j]=-1;

    choice1 = NumberOfPairs(V1,vector2, pairs);

    choice2 = NumberOfPairs(vector1, V2, pairs);

    if( choice1 >= choice2 )
    {
        return choice1;
    }
    else
    {
        return choice2;
```

```
            }
        }

        else
        {

            int V1[Length(vector1)], V2[Length(vector2)];
            int i,j, choice1=0, choice2=0;

            for(i=2; vector1[i]!= -1 ; i++)
            {
                V1[i-2]=vector1[i];
            }
             for(j=0; vector2[j]!= -1 ; j++)
            {
                V2[j]=vector2[j];
            }
            V2[j]=vector2[j];
            V2[0]=0;
            V1[i-2]=-1;
            V1[i-1]=-1;
            V1[i]=-1;


            choice1 = NumberOfPairs(V1,vector2, pairs);

            choice2 = NumberOfPairs(vector1, V2, pairs);

            if( choice1 >= choice2 )
            {
                return choice1;
            }
            else
            {
                return choice2;
            }
        }
    }
}
```

List of Figures

Bibliography

[1] Antolín, Y., Ciobanu, L., Elder, M., and Hermiller, S. Extending the falsification by fellow traveler property, in preparation.

[2] Bacher, R. Arithmetic properties related to the shuffle-product. ArXiv (2007). http://arxiv.org/pdf/0705.3632v2.pdf.

[3] Bartholdi, L. Groups of intermediate growth. ArXiv (2002). http://arxiv.org/pdf/math/0201293.pdf.

[4] Bartholdi, L., and Grigorchuk, R. I. On the spectrum of Hecke type operators related to some fractal groups. Tr. Mat. Inst. Steklova 231 (2000), 5–45.

[5] Bartholdi, L., and Pochon, F. On growth and torsion of groups. Groups Geom. Dyn. 3, 4 (2009), 525–539.

[6] Bartholdi, L., and Šunić, Z. On the word and period growth of some groups of tree automorphisms. Comm. Algebra 29, 11 (2001), 4923–4964.

[7] Berstel, J. Transductions and context-free languages, vol. 38 of Leitfäden der Angewandten Mathematik und Mechanik [Guides to Applied Mathematics and Mechanics]. B. G. Teubner, Stuttgart, 1979.

[8] Bridson, M. R., Burillo, J., Elder, M., and Šunić, Z. On groups whose geodesic growth is polynomial. Internat. J. Algebra Comput. 22, 5 (2012), 1250048, (13 pages).

[9] Brieussel, J. Croissance et moyennabilité de certains groupes d'automorphismes d'un arbre enraciné. PhD thesis, Paris 7, 2008.

[10] Bucher, M., and Talambutsa, A. Exponential growth rates of free and amalgamated products. Israel J. Math. 212, 2 (2016), 521–546.

[11] Carlitz, L., Wilansky, A., Milnor, J., Struble, R. A., Felsinger, N., Simoes, J. M. S., Power, E. A., Shafer, R. E., and Maas, R. E. Problems and Solutions: Advanced Problems: 5600-5609. Amer. Math. Monthly 75, 6 (1968), 685–687.

[12] Chomsky, N. Three models for the description of language. Information Theory, IRE Transactions on 2 (1956), 113–124.

[13] Claudon, B. Le théorème de Gromov d'après Kleiner, Shalom et Tao. http://www.iecn.u-nancy.fr/~claudon/gromov.pdf.

[14] Cleary, S., Elder, M., and Taback, J. Cone types and geodesic languages for lamplighter groups and Thompson's group F. J. Algebra 303, 2 (2006), 476–500.

[15] Duchin, M., and Shapiro, M. Rational growth in the heisenberg group. ArXiv:1411.4201 (2014).

[16] Eilenberg, S., and Mac Lane, S. On the groups of $H(\Pi, n)$. I. Ann. of Math. (2) 58 (1953), 55–106.

[17] Elder, M. Personal communication.

[18] Elder, M. Finiteness and the falsification by fellow traveler property. Geom. Dedicata 95 (2002), 103–113.

[19] Elder, M. Regular geodesic languages and the falsification by fellow traveler property. Algebr. Geom. Topol. 5 (2005), 129–134.

[20] Elder, M., Gutierrez, M., and Šuniḱ, Z. Geodesics in the first Grigorchuk group, in preparation.

[21] Epstein, D. B. A., Cannon, J. W., Holt, D. F., Levy, S. V. F., Paterson, M. S., and Thurston, W. P. Word processing in groups. Jones and Bartlett Publishers, Boston, MA, 1992.

[22] Fabrykowski, J., and Gupta, N. On groups with sub-exponential growth functions. J. Indian Math. Soc. (N.S.) 49, 3-4 (1985), 249–256 (1987).

[23] Fabrykowski, J., and Gupta, N. On groups with sub-exponential growth functions. II. J. Indian Math. Soc. (N.S.) 56, 1-4 (1991), 217–228.

[24] Flajolet, P., Gardy, D., and Thimonier, L. Birthday paradox, coupon collectors, caching algorithms and self-organizing search. Discrete Appl. Math. 39, 3 (1992), 207–229.

[25] Flajolet, P., and Sedgewick, R. Analytic combinatorics. Cambridge University Press, Cambridge, 2009.

[26] Greiner, L. T. Fekete's subadditive lemma revisited. ArXiv (2014). http://arxiv.org/abs/1403.8134.

[27] Grigorchuk, R. I. On the Milnor problem of group growth. 30–33.

[28] Grigorchuk, R. I. Degrees of growth of finitely generated groups and the theory of invariant means. Izv. Akad. Nauk SSSR Ser. Mat. 48, 5 (1984), 939–985.

[29] Grigorchuk, R. I., and Żuk, A. On the asymptotic spectrum of random walks on infinite families of graphs. In Random walks and discrete potential theory (Cortona, 1997), Sympos. Math., XXXIX. Cambridge Univ. Press, Cambridge, 1999, pp. 188–204.

[30] Gromov, M. Groups of polynomial growth and expanding maps. Inst. Hautes Études Sci. Publ. Math., 53 (1981), 53–73.

[31] Gromov, M. Hyperbolic groups. In Essays in group theory, vol. 8 of Math. Sci. Res. Inst. Publ. Springer, New York, 1987, pp. 75–263.

[32] Gupta, N., and Sidki, S. Some infinite p-groups. Algebra i Logika 22, 5 (1983), 584–589.

[33] Gupta, N., and Sidki, S. Extension of groups by tree automorphisms. In Contributions to group theory, vol. 33 of Contemp. Math. Amer. Math. Soc., Providence, RI, 1984, pp. 232–246.

[34] Hermiller, S., Holt, D. F., and Rees, S. Groups whose geodesics are locally testable. Internat. J. Algebra Comput. 18, 5 (2008), 911–923.

[35] Hopcroft, J. E., and Ullman, J. D. Introduction to automata theory, languages, and computation. Addison-Wesley Publishing Co., Reading, Mass., 1979. Addison-Wesley Series in Computer Science.

[36] Jiang, T., Li, M., Ravikumar, B., and Regan, K. W. Formal grammars and languages. In Algorithms and theory of computation handbook. CRC, Boca Raton, FL, 1999, pp. 25–1–25–26.

[37] Johnson, D. L. Rational growth of wreath products. 309–315.

[38] Koubi, M. Croissance uniforme dans les groupes hyperboliques. Ann. Inst. Fourier (Grenoble) 48, 5 (1998), 1441–1453.

[39] Loeffler, J., Meier, J., and Worthington, J. Graph products and Cannon pairs. Internat. J. Algebra Comput. 12, 6 (2002), 747–754.

[40] Lothaire, M. Combinatorics on words. Cambridge Mathematical Library. Cambridge University Press, Cambridge, 1997. With a foreword by Roger Lyndon and a preface by Dominique Perrin, Corrected reprint of the 1983 original, with a new preface by Perrin.

[41] Lyndon, R. C., and Schupp, P. E. Combinatorial group theory. Classics in Mathematics. Springer-Verlag, Berlin, 2001. Reprint of the 1977 edition.

[42] Lysenok, I. G. A system of defining relations for a Grigorchuk group. Mathematical Notes 38, 4 (1985), 784–792.

[43] Mann, A. How groups grow, vol. 395 of London Mathematical Society Lecture Note Series. Cambridge University Press, Cambridge, 2012.

[44] Milnor, J. Growth of finitely generated solvable groups. J. Differential Geometry 2 (1968), 447–449.

[45] Milnor, J. A note on curvature and fundamental group. J. Differential Geometry 2 (1968), 1–7.

[46] Nekrashevych, V. Self-similar groups, vol. 117 of Mathematical Surveys and Monographs. American Mathematical Society, Providence, RI, 2005.

[47] Neumann, W. D., and Shapiro, M. Automatic structures, rational growth, and geometrically finite hyperbolic groups. Invent. Math. 120, 2 (1995), 259–287.

[48] Nicaud, C. Personal communication.

[49] Rockafellar, R. T. Convex analysis. Princeton Mathematical Series, No. 28. Princeton University Press, Princeton, N.J., 1970.

[50] Sambusetti, A. Minimal growth of non-Hopfian free products. C. R. Acad. Sci. Paris Sér. I Math. 329, 11 (1999), 943–946.

[51] Sambusetti, A. Growth tightness of free and amalgamated products. Ann. Sci. École Norm. Sup. (4) 35, 4 (2002), 477–488.

[52] Shalom, Y., and Tao, T. A finitary version of Gromov's polynomial growth theorem. Geom. Funct. Anal. 20, 6 (2010), 1502–1547.

[53] Shapiro, M. Pascal's triangles in abelian and hyperbolic groups. J. Austral. Math. Soc. Ser. A 63, 2 (1997), 281–288.

[54] Švarc, A. S. A volume invariant of coverings. Dokl. Akad. Nauk SSSR (N.S.) 105 (1955), 32–34.

[55] Talambutsa, A. L. Attainability of the exponent of exponential growth in free products of cyclic groups. Mathematical Notes 78 (2005), 569–572.

[56] Talambutsa, A. L. On the attainability of the minimal exponential growth rate for free products of finite cyclic groups. Tr. Mat. Inst. Steklova 274 (2011), 314–328.

[57] Wolf, J. A. Growth of finitely generated solvable groups and curvature of Riemanniann manifolds. J. Differential Geometry 2 (1968), 421–446.

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Education

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July 2012	MSc in mathematics (summa cum laude), Université de Fribourg » Supervisor: Prof. Ruth Kellerhals Thesis: *Sur quelques contributions de Zagier autour de la fonction zêta de Dedekind.*
July 2010	BSc in mathematics (summa cum laude), Université de Neuchâtel

Professional Experiences

08.2012 - 08.2016	Assistant-PhD student at the University of Neuchâtel. Data sessions exercises : *topology, metric geometry, groups and algorithms, differential and integral calculus, vectorial analysis, graph theory.*
01.2010 - 07.2013	Teacher of Mathematics and Microsoft Office in « Centre de Formation du Sacré-Cœur », at Estavayer-le-lac, Switzerland. Level: 11th Harmos.
10.2011 - 06.2012	Tutor at University of Fribourg, 50%.

Languages

French (Mother tongue), German (Level A2) and English (Level B2)

Technical strengths

Computer languages	C, C++, Java, HTML and CSS.
Graphic design	Photoshop and Illustrator.
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