Topics

in

Combinatorial Group Theory

Preface

I gave a course on Combinatorial Group Theory at ETH, Zürich, in the Winter term of 1987/88. The notes of that course have been reproduced here, essentially without change. I have made no attempt to improve on those notes, nor have I made any real attempt to provide a complete list of references. I have, however, included some general references which should make it possible for the interested reader to obtain easy access to any one of the topics treated here. In notes of this kind, it may happen that an idea or a theorem that is due to someone other than the author, has inadvertently been improperly acknowledged, if at all. If indeed that is the case here, I trust that I will be forgiven in view of the informal nature of these notes.

Acknowledgements

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I would also like to acknowledge the help and insightful comments of Urs Stammbach. In addition, I would like to take this opportunity to express my thanks and appreciation to him and his wife Irene, for their friendship and their hospitality over many years, and to him, in particular, for all of the work that he has done on my behalf, making it possible for me to spend so many pleasurable months in Zurich.

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CHAPTER I History

1. Introduction

This course will be devoted to a number of topics in combinatorial group theory. I want to begin with a short historical account of the subject itself. This account, besides being of interest in its own right, will help to explain what the subject is all about.

2. The Beginnings

Combinatorial group theory is a loosely defined subject, with close connections to topology and logic. Its origins can be traced back to the middle of the 19th century. With surprising frequency problems in a wide variety of disciplines, including differential equations, automorphic functions and geometry, were distilled into explicit questions about groups. The groups involved took many forms – matrix groups, groups preserving e.g. quadratic forms, isometry groups and

numerous others. The introduction of the fundamental group by Poincaré in 1895, the discovery of knot groups by Wirtinger in 1905 and the proof by Tietze in 1908 that the fundamental group of a compact finite dimensional arcwise connected manifold is finitely presented served to underline the importance of finitely presented groups.

Just a short time earlier, in 1902, Burnside posed his now celebrated problem.

Problem 1 *Suppose that the group G is finitely generated and that for a fixed positive integer n*

$$x^n = 1 \ forall \ x \in G.$$

Is G finite?

Thus Burnside raised for the first time the idea of a finiteness condition on a group.

Then, in a series of extraordinarily influential papers between 1910 and 1914, Max Dehn proposed and partly solved a number of problems about finitely presented groups, thereby heralding in the birth of a new subject, combinatorial group theory. Thus the subject came endowed and encumbered by many of the problems that had stimulated its birth. The problems were generally concerned with various classes of groups and were of the following kind: Are all the groups in a given class finite (e.g., the Burnside Problem)? Finitely generated? Finitely presented? What are the conjugates of a given element in a given group? What are the subgroups of that group? Is there an algorithm for deciding for every pair of groups in a given class whether or not they are isomorphic? And so on. The objective of combinatorial group theory is the systematic development of algebraic techniques to settle such questions. In view of the scope of the subject and the extraordinary variety of groups involved, it is not surprising that no really general theory exists. However much has been accomplished and a wide variety of techniques and methods has been developed with wide application and potential. Some of these techniques have even found a wider use e.g. in the study of so-called free rings and their relations, in generalisations of commutative ring theory, in logic, in topology and in the theory of computing. The reader might wish to consult the book by Chandler, Bruce and Wilhelm Magnus, The History of Combinatorial Group Theory: *A Case Study in the History of Ideas*, **Studies in the History of Mathematics and the Physical Sciences 9** (1982), Springer-Verlag, New York, Heidelberg, Berlin.

I do not want to stop my historical account at this point. However, in order to make it intelligible also to those who are not altogether familiar with some of the terms and notation I will invoke, as well as some of the theorems and definitions I will later take for granted, I want to continue my discussion interspersing it with ingredients that I will call to mind as I need them.

3. Finitely presented groups

Let G be a group. We express the fact that H is a subgroup of G by writing $H \leq G$; if H is a normal subgroup of G we write $H \trianglelefteq G$.

Let $X \subseteq G$. Then the subgroup of G generated by X is denoted by $\mathrm{gp}(X)$. Thus, by definition, $\mathrm{gp}(X)$ is the least subgroup of G containing X. It follows that

$$\mathrm{gp}(X) \;=\; \{{x_1}^{\varepsilon_1} \dots {x_n}^{\varepsilon_n} \mid \; x_i \in X,\; \varepsilon_i = \pm 1\}.$$

We call the product

$${x_1}^{\varepsilon_1} \dots {x_n}^{\varepsilon_n} \qquad (x_i \in X,\; \varepsilon_i = \pm 1)$$

an X-product. An X-product is termed *reduced* if

$$x_i = x_{i+1} \qquad \textit{implies} \qquad \varepsilon_i + \varepsilon_{i+1} \neq 0 \qquad (i = 1, \dots, n-1).$$

If $G = \mathrm{gp}(X)$ and every non-empty reduced X-product is $\neq 1$ then we term X a free set of generators of G and G itself is termed free; we also say that X freely generates G or that G is free on X. Notice that if G is free on X, then two reduced X-products are equal if and only if they are identical.

Theorem 1 *(i) If G is free on X and also on Y, then $|X| = |Y|$; this common cardinal number is termed the rank of the free group G.*
(ii) Let X be a set. Then there exists a free group G freely generated by X, the so-called free group on X.

(iii) Let G be free on X. Then for every group H and every map $\theta : X \longrightarrow H$ there exists a homomorphism $\varphi : G \longrightarrow H$ such that $\varphi \mid X = \theta$

Corollary 1 *Every group is isomorphic to a factor group of a free group.*

A group is termed an α-generator group if it can be generated by a set of cardinality α, it is termed finitely generated if it can be generated by a set of finite cardinality.

Let G again be a group, $X \subseteq G$. Then the least normal subgroup of G containing X, the so-called normal closure of X in G, is denoted by $\mathrm{gp}_{_G}(X)$. So

$$\mathrm{gp}_{_G}(X) \;=\; \mathrm{gp}(g^{-1}xg \;\mid\; g \in G,\; x \in X).$$

Now suppose G is a group, F a free group on X, θ a map from X into G such that

$$G \;=\; \mathrm{gp}(X\theta).$$

Then the extension φ of θ to F maps F onto G with kernel K. Suppose

$$K \;=\; \mathrm{gp}_{_F}(R).$$

Then we write

$$G \;=\; \langle X; R\rangle \tag{1}$$

and term $\langle X; R\rangle$ a presentation of G. Notice that such a presentation (1) comes with an implicit map $\theta : X \longrightarrow G$ such that the extension of θ to the free group F on X yields a homomorphism φ with kernel $\mathrm{gp}_{_F}(R)$.

If we identify X with its image in G then (1) simply means that X generates G and everything about G can be deduced from the fact that $r = 1$ in G for every $r \in R$.

Example 1 *Let*

$$G \;=\; \langle\; a, b \;;\; a^{-1}bab^{-2}, b^{-1}aba^{-2} \;\rangle.$$

Notice that in G

$$a^{-1}ba \;=\; b^2 \;,\; b^{-1}ab \;=\; a^2.$$

So

$$a \;=\; a^{-1}a^2 \;=\; a^{-1}b^{-1}ab \;=\; (a^{-1}ba)^{-1}b \;=\; b^{-2}b \;=\; b^{-1}.$$

But then this implies

$$b \;=\; b^2 \qquad or \qquad b \;=\; 1.$$

So

$$a \;=\; 1.$$

In other words

$$G \;=\; \{1\}$$

is the so-called trivial group.

The lesson here is that groups given by presentations can be very tricky.

Definition 1 *A group is finitely presented if it has a finite presentation i.e.*

$$G \;=\; \langle X; R\rangle$$

where X and R are both finite.

It is time now to return to more history and to Dehn.

4. More history

In his paper in 1912 Dehn explicitly raised three problems about finitely presented groups.

The word problem

Let G be a group given by a finite presentation

$$G \;=\; \langle X; R\rangle.$$

Is there an algorithm which decides whether or not any given unworked out X-product – often referred to as a word – is the identity in G?

The conjugacy problem

Let G be a group given by a finite presentation

$$G \;=\; \langle X; R\rangle.$$

Is there an algorithm which decides whether or not any pair of words v, w are conjugate in G i.e. if there exists an X-word z such that

$$w \;=\; z^{-1}vz \;\; \textit{in } G$$

Finally:

The isomorphism problem

Is there an algorithm which determines whether or not any pair of groups (in some well-defined class of groups) given by finite presentations are isomorphic?

Notice that the point of the example I worked out is made more clear in the light of these three problems of Dehn. Dehn came to them while looking at the fundamental groups of two-dimensional surfaces. The question as to whether a given loop is homotopic to the identity is the word problem, whether two loops are freely homotopic is the conjugacy problem and whether the fundamental groups of two surfaces are isomorphic reflects the problem as to whether the spaces are homeomorphic.

Some 20 years after Dehn proposed these problems Magnus proved his famous Freiheitssatz:

Theorem 2 (W. Magnus 1930) *Let G be a group with a single definition relator i.e.*

$$G \;=\; \langle x_1, \ldots, x_q; r\rangle.$$

Suppose r is cyclically reduced i.e., the first and last letters in r are not inverses of each other. If each of $x_1, \ldots, x_q$ actually appears in r, then any proper subset of $\{x_1, \ldots, x_q\}$ freely generates a free group.

This led to the first major break-through on solving the word problem.

Theorem 3 (W. Magnus 1932) *The word problem for a* 1*-relator group has a positive solution.*

It took almost 50 years before all of Dehn's questions were finally answered.

First, Novikov proved, in 1954, the remarkable

Theorem 4 *There exists a finitely presented group with an insoluble word problem.*

Notice that such a group with an insoluble word problem also has an insoluble conjugacy problem. Novikov's proof was a combinatorial tour-de-force. New and simpler proofs were obtained by Boone in 1959 and Britton in 1961.

The very existence of a finitely presented group with an insoluble word problem led Adyan in 1957 to prove a most striking negative theorem about finitely presented groups. In order to explain we need the notion of a Markov property.

Definition 2 *An algebraic property (i.e., one preserved under isomorphism) of finitely presented groups is termed a Markov property if*
(i) there exists a finitely presented group with the property,
(ii) there exists a finitely presented group which cannot be embedded in, i.e. is not isomorphic to a subgroup of, a group with the property.

Here is Rabin's formulation, in 1958, of Adyan's theorem:

Theorem 5 (Adyan 1957, Rabin 1958) *Let $\mathcal{M}$ be a Markov property. Then there is no algorithm which decides whether or not any finitely presented group has this property $\mathcal{M}$.*

To illustrate, notice that the following are Markov properties:

(i) *triviality;*
(ii) *finiteness;*
(iii) *commutativity;*
(iv) *having solvable word problem;*
(v) *simplicity;*
(vi) *freeness.*

It is obvious that being trivial i.e., being of order 1, is a Markov property, as are finiteness, commutativity, having solvable word problem as well as freeness. Now a finitely presented simple group has a solvable word problem. Hence a finitely presented group with an insoluble word problem cannot be embedded in a finitely presented simple group. This means that being simple is also a Markov property.

Notice that the seemingly haphazard proof that the group in Example 1 is trivial was no accident or lack of skill – the insolubility of the triviality problem makes such proofs ad hoc by necessity!

Adyan's theorem was followed in 1959 by similar, much easier, theorems about elements and subgroups of a group in work of Baumslag, Boone, B.H. Neumann.

Theorem 6 *There is a finitely presented group G_0 such that no effective procedure exists to determine whether or not a word in the generators of G_0 represents*
(i) *an element in the center of G_0;*
(ii) *an element permutable with a given element of G_0;*
(iii) *an n-th power with $n > 1$ an integer;*
(iv) *an element whose conjugacy class is finite;*
(v) *an element of a given subgroup of G_0;*
(vi) *a commutator i.e. of the form $x^{-1}y^{-1}xy$;*
(vii) *an element of finite order.*

Theorem 7 *Let $\mathcal{P}$ be an algebraic property of groups. Suppose*
(i) *there is a finitely presented group that has $\mathcal{P}$;*
(ii) *there exists an integer n such that no free group of rank r has $\mathcal{P}$ if $r \geq n$. Then*

there is a finitely presented group G such that there is no algorithm which determines whether or not any finite set of elements of G generates a subgroup with $\mathcal{P}$.

So e.g., there is a finitely presented group G such that there is no algorithm which decides whether or not any finitely generated subgroup of G is finite.

It follows that, from an algorithmic standpoint, finitely presented groups represent a completely intractable class. For a general reference to this subject se the paper by C.F. Miller III: *Decision problems for groups–survey and reflections* in **Algorithms and Classification in Combinatorial Group Theory** MSRI Publications No. 23, edited by G. Baumslag and C.F. Miller III, Springer-Verlag (1991).

5. Higman's marvellous theorem

In a sense one aspect of the theory of finitely presented groups was brought to a close in 1961 when Graham Higman proved the following extraordinary

Theorem 8 *Let G be a finitely generated group. Then G is a subgroup of a finitely presented group if and only if G can be presented in the following form*

$$G \;=\; \langle x_1, \ldots, x_q \;;\; r_1, r_2, \ldots \rangle \qquad (q < \infty)$$

where $r_1, r_2, \ldots$ is a recursively enumerable set of defining relations.

This theorem (G. Higman: *Subgroups of finitely presented groups*, **Proc. Royal Soc. London Ser. A 262**, 455-475 (1961)) establishes a bond between recursive function theory and the subgroup structure of finitely presented groups.

I want to briefly illustrate just how remarkable Higman's theorem is by concocting a finitely presented group with an insoluble word problem.

To this end let f be a function with domain and range the positive integers and suppose that

(i) given any positive integer n we can effectively compute $f(n)$;
(ii) given any positive integer m there is no effective method which decides whether or not there is a positive integer n such that $f(n) = m$.

So f is a recursive (or computable) function whose range is not a recursive subset of the positive integers.

Now form

$$G = \langle a, b, c, d \; ; \; b^{-f(n)}ab^{f(n)} = c^{-f(n)}dc^{f(n)} \quad (n \geq 1)\rangle.$$

G then is a finitely generated, recursively presented group. Moreover it can be shown that

$$b^{-m}ab^m = c^{-m}dc^m \text{ if and only if } m = f(n).$$

Thus

$$b^{-m}ab^mc^{-m}d^{-1}c^m = 1 \text{ if and only if } m = f(n).$$

This means that in order to solve the word problem in G we need to know the range of f. But this, by assumption, is not a recursive set. So G has an insoluble word problem. By Higman's theorem

$$G \leq H$$

with H finitely presented. So H has an insoluble word problem.

The one major problem still unsolved is: What can one say about the general subgroup structure of finitely presented groups?

This brings me to the next part of the history.

6. Varieties of groups

I want to turn my attention to an extremely interesting topic in group theory that received much attention in the 1960's. Although the subject lapsed into disfavour for a while, a lot of old and seemingly impossible open problems have been solved recently. The results are intriguing enough to merit some discussion. (See the book by Hanna Neumann: *Varieties of Groups*, **Ergebnisse der Mathematik und ihrer Grenzgebiete 37**, Springer-Verlag Berlin Heidelberg New York (1967) for an introduction to varieties.)

Let me start out then with a definition.

Definition 3 *A non-empty class $\mathcal{V}$ of groups is termed a variety (of groups) if it is closed under homomorphic images, subgroups and cartesian products.*

In order to give some examples, I need to introduce some notation. Suppose that G is a group, $x, y \in G$. Then the commutator $x^{-1}y^{-1}xy$ of x and y is denoted by $[x, y]$ and the conjugate of x by y is denoted by x^y. Thus

$$[x, y] = x^{-1}y^{-1}xy\ , \quad x^y = y^{-1}xy\ .$$

It is easy to check that the following identities, which we will refer to as the basic commutator identities, hold:

$$\begin{aligned} x^y &= x[x, y] \\ (xy)^z &= x^z y^z \\ [x, y]^{-1} &= [y, x] \\ [xy, z] &= [x, z]^y [y, z] \\ [x, yz] &= [x, z][x, y]^z\ . \end{aligned}$$

These basic identities can be verified by direct calculation.

Suppose H and K are subgroups of G. Then we define

$$[H, K] = gp([h, k] \mid h \in H,\ k \in K).$$

The commutator subgroup or derived group of G is denoted by G' and is defined by

$$G' \ = \ [G, G].$$

Note that if $H \trianglelefteq G,\ K \trianglelefteq G$, then $[H, K] \trianglelefteq G$. So $G' \trianglelefteq G$ and indeed is the smallest normal subgroup of G with abelian factor group. Inductively we define

$$G^{(n)} \ = \ (G^{(n-1)})' \qquad (n \geq 2)$$

where $G^{(1)} = G'$, and the series

$$G \ = \ G^{(0)} \ \geq \ G' \ \geq \ G^{(2)} \ \geq \ \ldots \ \geq \ G^{(n)} \ \geq \ \ldots$$

is termed the *derived series* of G. G is termed *solvable* if $G^{(n)} = 1$ for some n, the least such n being termed the derived length of G. Notice that subgroups, homomorphic images and cartesian products of solvable groups of derived length at most d are again solvable of derived length at most d. Thus the class $\mathcal{S}_d$ of all solvable groups of derived length at most d is a variety.

Now let K be a field. Then $\mathrm{GL}(n, K)$ denotes the group of all $n \times n$ invertible matrices over K and $\mathrm{Tr}(n, K)$ the subgroup of $\mathrm{GL}(n, K)$ of all lower triangular matrices (i.e. zeroes above the main diagonal). We denote, in the case where K is commutative, the subgroup of $\mathrm{GL}(n, K)$ of matrices of determinant 1 by $\mathrm{SL}(n, K)$. Note that in the commutative case $\mathrm{Tr}(n, K)$ is solvable. In particular, if $\mathrm{K} = \mathbf{Q}(x)$, the field of fractions of the polynomial algebra $\mathbf{Q}[x]$ in a single variable, we have the two triangular groups

$$N \ = \ gp(\ a \ = \ \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix},\ d \ = \ \begin{pmatrix} 3/2 & 0 \\ 0 & 1 \end{pmatrix}\ )$$

and

$$W \ = \ gp(\ a,\ t \ = \ \begin{pmatrix} x & 0 \\ 1 & 1 \end{pmatrix}\ ).$$

We leave it to the reader to prove

Exercise 1

(a) $N' \cong \mathbf{Z}[3/2][2/3] = \mathbf{Z}\ [1/6]$ *the additive group of the ring of integers with* $1/6$ *adjoined;*

(b) W' *is a free abelian group of infinite rank;*

(c) $N'' = W'' = 1$ *i.e.* N *and* W *are metabelian.*

Thus it might be noted that subgroups of even finitely generated metabelian groups need not be finitely generated.

The *centre* of G is denoted by ζG; so

$$\zeta G \ = \ \{x \in G \ \big| \ [x, y] \ = \ 1 \qquad for\ all \qquad y \in G\}.$$

The *upper central series* of G is defined to be the series

$$1 \ = \ \zeta_0 G \ \leq \ \zeta_1 G \ \leq \ \ldots \ \zeta_n G \ \leq \ \ldots$$

where inductively

$$\zeta_{i+1} G/\zeta_i G \ = \ \zeta(G/\zeta_i G) \qquad (i \geq 0).$$

So $\zeta_1 G = \zeta G$. The *lower central series* of G is defined to be the series

$$G \ = \ \gamma_1 G \ \geq \ \gamma_2 G \ \geq \ \ldots \ \geq \ \gamma_n G \ \geq \ \ldots$$

where inductively

$$\gamma_{n+1} G \ = \ [\gamma_n G, G] \qquad (n \geq 1).$$

G is *nilpotent* if $\gamma_{c+1} G = 1$ for some c with the least such c the class of G.

Note

$$\zeta_c G \ = \ G \qquad \text{if and only if} \qquad \gamma_{c+1} G \ = \ 1 \ .$$

Notice subgroups, homomorphic images and cartesian products of nilpotent groups of class at most c are again nilpotent of class at most c. Thus the class $\mathcal{N}_c$ of all nilpotent groups of class at most c is another variety. Notice that a group that is nilpotent of class at most c is solvable of derived length at most c.

Exercise 2 *If G is a finitely generated nilpotent group of class c, prove $\gamma_c G$ is finitely generated and hence that every subgroup of G is also finitely generated.*

Now suppose that $\mathcal{U}$ and $\mathcal{V}$ are varieties of groups. Then we define their product $\mathcal{U}.\mathcal{V}$ to be the class of all groups G which contain a normal subgroup N such that $N \in \mathcal{U}$ and $G/N \in \mathcal{V}$. We refer to the groups in $\mathcal{U}.\mathcal{V}$ as $\mathcal{U}$ by $\mathcal{V}$ groups. It is not hard to see that the product $\mathcal{U}.\mathcal{V}$ is again a variety. It is also not hard to see that this product is associative and that it turns the set of all varieties of groups into a semigroup with an identity. In 1962 B. H., Hanna and Peter M. Neumann proved that this semigroup of varieties of groups is free. In more detail, let $\mathcal{E}$ denote the variety of trivial groups and let $\mathcal{O}$ denote the variety of all groups. Moreover let us term a variety $\mathcal{V}$ of groups, $\mathcal{V} \neq \mathcal{O}, \mathcal{V} \neq \mathcal{E}$ indecomposable if it cannot be expressed as a product of varieties different from $\mathcal{O}, \mathcal{E}$. Then we have the following

Theorem 9 *The set of all varieties of groups with binary operation product is a semigroup with the property that every variety different from $\mathcal{E}$ and $\mathcal{O}$ can be written as a product of indecomposable varieties in exactly one way.*

Olshansky, and independently at about the same time Vaughan-Lee (see the article by Kovacs and Newman at the end of this chapter) proved in 1970 the

Theorem 10 *There exist continuously many different varieties of groups.*

More recently Kleiman has obtained a number of remarkable negative results about varieties of groups (see the paper by C.F. Miller cited earlier for further details).

There is in each variety of groups a counterpart to the notion of a free group, defined in terms of a universal mapping property. More precisely

Definition 4 *Let $\mathcal{V}$ be a variety of groups. Then a group F in $\mathcal{V}$ is said to be free in $\mathcal{V}$ or a free $\mathcal{V}$-group, if it is generated by a set X such that for every group G in $\mathcal{V}$ and every mapping θ from X into G, there exists a homomorphism ϕ of F into G which agrees with θ on X.*

We shall need the

Definition *A group G is termed hopfian if $G \cong G/N$ implies that $N=1$.*

Finitely generated free groups have this property and it was also the case for the free groups in an arbitrary variety. However in 1989 S.V. Ivanov proved the surprising

Theorem 11 *There exists a variety $\mathcal{V}$ such that all of the non-cyclic free groups in $\mathcal{V}$ are not hopfian.*

This remarkable theorem is proved by using a variation of *small cancellation theory*. I will have a little more to say about this later. Many of the ideas involved can be traced back to Dehn, Tartakovskii, Adian and Novikov and Olshansky (see the book by Roger C. Lyndon and Paul E. Schupp: *Combinatorial Group Theory*, **Ergebnisse der Mathematik und ihrer Grengebiete 89**, Springer-Verlag Berlin Heidelberg New York (1977)). Let me mention one more extraordinary result, which is also due to Olshansky.

Theorem 12 *For every sufficiently large prime p (e.g. $p > 10^{75}$) there exists an infinite group all of whose proper subgroups are of order p.*

Theorem 12 provides another negative answer to one of Burnside's problems. It also establish the existence of a so-called Tarski monster.

We concentrate next on the class of solvable groups. Observe to begin with that a finitely generated abelian group is a direct product of a finite number of cyclic groups. So all the algorithmic problems mentioned at the outset can be solved for such groups. Solvable groups can be viewed as generalisations of these finitely generated abelian groups. For a long time the one remaining outstanding word problem concerned finitely presented solvable groups. Then in 1981 O. Kharlampovich proved the

Theorem 13 *There exists a finitely presented solvable group with an insoluble word*

problem.

This allied with more recent work of Baumslag, Strebel and Gildenhuys in 1985 puts the class of finitely presented solvable groups in much the same place as that of finitely presented groups.

Theorem 14 *The isomorphism problem for finitely presented solvable groups is recursively undecidable.*

Positive algorithmic results about finitely presented are few and far between. Perhaps the most outstanding of these is due to Grunewald and Segal who proved in 1980 the

Theorem 15 *The isomorphism problem for finitely generated nilpotent groups has a positive solution.*

In fact they proved somewhat more. In particular their techniques, which make use of the theory of arithmetic and algebraic groups, give rise to a positive solution to the isomorphism problem for finite dimensional Lie algebras over $\mathbf{Q}$, a problem that had been open for almost a century.

In 1978 Bieri and Strebel introduced an invariant of finitely generated metabelian groups which detects whether or not such groups are finitely presented. Subsequently they showed that there is a similar, less discriminating, invariant of an arbitrary finitely generated group, now termed the Bieri-Strebel invariant. This work of Bieri and Strebel is extremely important and the interested reader is referred to the survey article by Strebel (Ralph Strebel: *Finitely Presented Soluble Groups*, in **Group Theory, Essays for Philip Hall** (1984) 257-314) for a detailed discussion of this invariant and finitely presented solvable groups as a whole (see the references at the end of the chapter).

7. Small Cancellation Theory

In this work on the solution of the word and conjugacy problems in 1912 Dehn solved these problems by veryfying that, in a sense, not too much cancellation takes place on forming products of certain sets of defining relators. This point of view has led to what is now called *Small Cancellation Theory* and it is with this theory that I want now to turn. (See the book by Lyndon and Schupp cited above for more details.)

To this end then suppose G has a presentation

$$G \;=\; \langle X; R\rangle \;.$$

I have already termed the elements of R defining relators. Assume that R is *symmetrized* i.e. closed under inverses and cyclic permutations. It is called a *one-sixth presentation* if it is symmetrized and satisfies the following condition: if $r, s \in R$ and if either more than one sixth the length of s cancels on computing the reduced word representing rs or more than one sixth the length of r cancels on computing the reduced word representing rs, then $r = s^{-1}$.

The following theorem of Greendlinger (1960) holds:

Theorem 16 *Suppose that the group G has a one-sixth presentation as above and that w is a non-empty reduced X-word. If $w = 1$ in G, then there exists $r \;=\; a_1 \dots a_l \;\in R$ such that*

$$w \;=\; b_1 \dots b_m a_1 \dots a_n c_1 \dots c_s$$

where

$$n \;>\; \frac{l}{2} \quad ,$$

i.e. w contains more than half a relator (here $b_i, a_j, c_k \;\in\; X \cup X^{-1}$).

This theorem provides an algorithm for solving the word problem in G when the above presentation is finite. For if w is a reduced X-product, by inspecting the finitely many elements of R, we can determine whether or not more than half of one of them is a sub-X-product of w. If not $w \neq 1$. Otherwise, $w = tuv$ where u is more than one half of $r = us \in R$. So

$$w \;=\; ts^{-1}v \qquad in\ G$$

Now the reduced X-product w_1 representing w is of smaller length than that of w. So we can repeat the process with w replaced by w_1. Inductively we can therefore determine whether or not $w = 1$ in G. A similar argument applies also to the conjugacy problem.

This approach to the study of groups given by generators and relators was carried further by Lyndon (1965) who re-introduced diagrams into the study of such groups allowing for the use of geometric-combinatorial arguments in handling cancellation phenomena in group theory. These methods and ideas have now been systematized and generalized, yielding important and powerful theorems in group theory. I have already mentioned the work of Olshanski. I want also now to mention the work of Rips, who independently, has created his own version of smallcancellation theory and has used this theory to prove a number of remarkable results about groups.

More recently Gromov has created a beautiful theory of what he terms Hyperbolic Groups (M. Gromov: *Hyperbolic Groups*, in **Essays on group theory**, MSRI Publications No. 8, edited by S. Gersten, Springer-Verlag (1987)). These groups are, to a certain extent, modelled on 'small cancellation groups 'and discrete groups of isometries of hyperbolic spaces.

There is another related theory of so-called automatic groups, that is of also of recent origin, due to Cannon, Epstein, Holt, Paterson and Thurston (J.W. Cannon, D.B.A. Epstein, D.F. Holt, M.S. Paterson and W.P. Thurston: *Word processing and group theory*, preprint, University of Warwick (1991)). Both of the above two theories are likely to have a profound effect on Combinatorial Group Theory.

There are some other important developments in the study of Combinatorial Group Theory. These include the so-called Bass-Serre theory of groups acting on trees (see Chapter VII), the cohomology of groups, in particular the extraordinary proof by Stallings and Swan that groups of cohomological dimension one are free (J.R. Stallings: *Group Theory and 3-dimensional Manifold*, **Yale Monographs 4** (1971) and the graph-theoretic methods of Stallings with applications by Gersten to the automorphisms of free groups, yielding for example his fixed point theorem of 1984 (S.M. Gersten: *On Fixed Points of Certain Automorphisms of Free Groups*, **Proc. London Math. Soc. 48** (1984), 72-94)

Theorem 17 *Let F be a finitely generated free group and let φ be an automorphism of F. Then*

$$\operatorname{Fix}\varphi \;=\; \{a \in F \,\big|\; a\varphi = a\}$$

is a finitely generated group.

The following additional references may be useful to the interested reader.

Kovacs, L.G. and M.F. Newman, *Hanna Neumann's Problems on Varieties of Groups*, **Proc. Second Internat. Conf. Theory of Groups Canberra** (1973), 417-433.

Kurosh, A.G., *The theory of groups, 2nd edition,* **translated from the Russian by K.A. Hirsch, vols. I and II,** Chelsea Publishing Company, New York (1955).

Magnus, Wilhelm, Abraham Karrass and Donald Solitar, *Combinatorial Group Theory,* Dover Publications, Inc., New York (1976).

CHAPTER II The Weak Burnside Problem

1. Introduction

In 1902 Burnside wrote "A still undecided problem in the theory of discontinuous groups is whether the order of a group may be not finite while the order of every operation it contains is finite". He tacitly assumed that the groups involved are all finitely generated.

In fact this quotation of Burnside has now been turned into the so-called Burnside Problem, which I formulated in Chapter I.

The Burnside Problem

Let G be a finitely generated group. If for some fixed positive integer n

$$x^n = 1 \quad forall \quad x \in G,$$

is G finite?

Burnside already knew the answer for $n = 2, 3$.

Exercise 1 *Prove that Burnside's Problem has an affirmative answer when $n = 2$ or for arbitrary n when the group is abelian.*

In 1940 Sanov settled Burnside's Problem for n=4. Then in 1957 M. Hall took care of the case n=6 (see his book Marshall Hall, Jr.: *The theory of groups*, **Chelsea Publishing Company** New York, N.Y. (1976)) for detailed references). To this day the case n=5 is still unresolved!

There are other forms of this problem.

The Weak Burnside Problem

Let G be a finitely generated group. Suppose that every element of G is of finite order. Is G finite?

If V is a finite dimensional vector space, then we denote by $GL(V)$ the group of all invertible linear transformations of V. Burnside himself solved the Weak Burnside Problem for the finitely generated subgroups of $GL(V)$ when the ground field is the complex field.

In 1964 E.S. Golod showed that the answer to the Weak Burnside Problem is in the negative (see the reference cited below to Golod and Shafarevich).

Then in 1968, in a monumental piece of work, S.I. Adyan and P.S. Novikov proved that the Burnside Problem has a negative solution for every odd $n \geq 4381$, which Adyan later improved to every odd $n \geq 665$.

There is one other facet of the Burnside Problem that has attracted much attention, partly because of its connection with the theory of Lie rings.

The Restricted Burnside Problem

Let r and n be fixed positive integers. Is there a bound on the orders of the finite groups G

with r generators satisfying the condition

$$x^n = 1 \quad forall \quad x \in G \ ?$$

Here the major result is due to Kostrikin:
If n is any prime, then the Restricted Burnside Problem has a positive answer.
This has very recently been extended by Zelmanov to the case where n is an arbitrary power of a prime.

My primary objective in this chapter is to give a negative solution to the Weak Burnside Problem. The basic idea here is due to Grigorchuk, although the point of view and the exposition I shall give here is due to Gupta and Sidki. Because of this I have elected to call the groups described here the Grigorchuk-Gupta-Sidki groups.

2. The Grigorchuk-Gupta-Sidki groups

Let me recall that a group is called a p-group, where here p is a prime, if every element is of order a power of p.

Then we have the following

Theorem 1 (Grigorchuk, Gupta & Sidki) *There exists for every odd prime p a 2-generator, infinite p-group.*

We will restrict attention to the case where p=3. The Grigorchuk-Gupta-Sidki group will be a group of automorphisms of a particularly nice graph. I will take a somewhat informal approach to this graph. Later I will introduce some definitions which can be used to make a more rigorous approach to graphs possible.

The group then that we will examine is a subgroup G of the group of automorphisms of the graph X of *Fig.(II,1)*

Fig.(II,1) Graph X

Each vertex v of X is the base of another graph $X(v)$ isomorphic to X. Thus

$$X \;=\; X()\;.$$

Using this notation we can redraw X as follows in *Fig.(II,2)*.

Fig.(II,2)

Notice the labelling system: if v is any given vertex then the other vertex of the left-most edge emanating from v is labelled $v1$, the middle vertex is labelled $v2$ and the right-most vertex is labelled $v3$. Notice that if you stand at any vertex in this graph and look upwards you have exactly the same view.

The Grigorchuk-Gupta-Sidki group is generated by two elements τ and α which we define by specifying their action on the vertices of X and deducing what happens to the edges. Notice that since

$$X \;=\; X() \;\cong\; X(v)$$

for every vertex v, each automorphism β of X has associated to it a corresponding automorphism of $X(v)$ which we denote by $\beta(v)$. This is not to be confused with the image of v under β which, using algebraic notation, is properly denoted $v\beta$!!

Now for the definition of τ:

We declare τ permutes X(1), X(2) and X(3) cyclically, mapping X(1) onto its isomorphic image X(2) etc. (and leaving the base of X fixed).

We will find it convenient also, given an automorphism γ of $X(v)$ (which leaves v fixed), v some vertex of X, to continue γ to an automorphism of X. This we do by declaring that γ leaves everything outside $X(v)$ fixed. This automorphism of X we again denote by γ.

Next the definition of α:

$$\alpha \;=\; \tau(1)\;\tau(2)^{-1}\alpha(3) \;\; ! \tag{1}$$

This definition of α needs to clarified. Notice that according to (1) α on $X(1)$ is simply $\tau(1)$, on $X(2)$ it is $\tau(2)^{-1}$ and on $X(3)$ it is $\alpha(3)$! What this means is that α has been defined by 'delayed iteration'. If v is a vertex in $X(1)$ or $X(2)$ we know the effect of α. If v is a vertex in $X(3)$ we don't know yet what α does to v. But, by definition,

$$\alpha(3) \;=\; \tau(31)\;\tau(32)^{-1}\;\alpha(33)\;.$$

So if v is a vertex in $X(31)$ or in $X(32)$ we know what α does on v. Notice that

$$\alpha(v) \;=\; \tau(v1)\;\tau(v2)^{-1}\alpha(v3)\;.$$

It follows that α leaves all the vertices in the right-most set of vertices identically fixed. Eventually, then, either we find $v\alpha = v$ or else either

$$v \;=\; \underbrace{3\ldots3}_{m}1 \;\; or \;\; v \;=\; \underbrace{3\ldots3}_{m}2.$$

Hence

$$v\alpha \;=\; v\tau(\underbrace{3\ldots3}_{m}1) \;\; or \;\; v\alpha \;=\; v\tau(\underbrace{3\ldots3}_{m}2)^{-1}\;.$$

Another way of thinking about this is to note that if v is at level n, then observe that

$$\begin{aligned}
\alpha \;&=\; \tau(1)\;\tau(2)^{-1}\;\alpha(3)\\
&=\; \tau(1)\;\tau(2)^{-1}\;\tau(31)\;\tau(32)^{-1}\;\alpha(33)\\
&=\; \ldots\\
&=\; \tau(1)\;\tau(2)^{-1}\;\tau(31)\;\tau(32)^{-1}\;\ldots\;\tau(\underbrace{3\ldots3}_{n-1}1)\;\tau(\underbrace{3\ldots3}_{n-1}2)^{-1}\;\alpha(\underbrace{3\ldots3}_{n}) \qquad (2)
\end{aligned}$$

and so the effect of α on v is either to leave v fixed or else is obtained by applying the appropriate $\tau(\star)$.

As already noted, we define

$$G \;=\; gp(\alpha,\tau)\;.$$

Lemma 1 *α and τ are of order 3*

Proof Clearly $\alpha \neq 1 \neq \tau$. We note first that

$$\tau^3 \;=\; 1\;.$$

This is clear.

Next we prove that $\alpha^3 = 1$. It is enough to check that α^3 leaves every vertex fixed. This is clear from (2).

But let's give a slightly different argument. Suppose v is a vertex at level n. If $n = 0$, α leaves v fixed and therefore so does α^3. Inductively let us assume α^3 leaves every vertex at level $\leq n-1$ fixed and that $n > 0$. If v is a vertex in $X(1)$ then

$$v\ \alpha^3\ =\ v\ \tau(1)^3\ =\ v$$

and if v is a vertex in $X(2)$,

$$v\ \alpha^3\ =\ v\ \tau(2)^{-3}\ =\ v\ .$$

Finally if v is a vertex in $X(3)$,

$$v\ \alpha^3\ =\ v\ \alpha(3)^3\ .$$

But now think of v as a vertex in $X(3)$. There v is at level $n-1$. Hence the inductive assumption yields

$$v\ \alpha(3)^3\ =\ v$$

as required.

■

Lemma 2 *G is an infinite group*

Proof Let v be a vertex in X. We put

$$H(v)\ =\ gp\Big(\ \alpha(v)\ ,\ \tau(v)^{-1}\alpha(v)\tau(v)\ ,\ \tau(v)^{-2}\alpha(v)\tau(v)^2\ \Big)$$

and

$$H\ =\ H()\ =\ gp(\ \alpha\ ,\ \tau^{-1}\alpha\tau\ ,\ \tau^{-2}\alpha\tau^2\ ).$$

Now

$$H \trianglelefteq G\ . \tag{3}$$

This is clear since H is generated by all of the conjugates of α under the powers of τ.

Next we compute the forms of these conjugates of α. First

$$\alpha \;=\; \tau(1)\tau(2)^{-1}\alpha(3)\ . \tag{4}$$

It follows that

$$\tau^{-1}\alpha\tau \;=\; \alpha(1)\tau(2)\tau(3)^{-1} \tag{5}$$

and

$$\tau^{-2}\alpha\tau^{2} \;=\; \tau(1)^{-1}\alpha(2)\tau(3)\ . \tag{6}$$

These assertions can be checked by simply carrying out e.g. $\tau^{-1},\ \alpha$ and τ in this order to get (5).

Put

$$G(v) \;=\; gp\big(\ \alpha(v)\ ,\ \tau(v)\ \big)$$

where v is a vertex in X. Then $G = G() \cong G(v)$.

Next notice that

$$\tau(1),\, \alpha(1) \;\in\; G(1) \quad , \quad \tau(2),\, \alpha(2) \;\in\; G(2) \quad , \quad \tau(3),\, \alpha(3) \;\in\; G(3)$$

and that

$$gp\big(\ G(1), G(2), G(3)\ \big) \;=\; G(1) \times G(2) \times G(3)\ .$$

In particular we also note that

$$H \;\leq\; G(1) \times G(2) \times G(3)\ . \tag{7}$$

It follows that $\tau \notin H$, because by (7), every element of H leaves the vertices at level 1 fixed. So

$$\big|\, G/H \,\big| \;=\; 3\ . \tag{8}$$

Now it follows from the positioning of H (see (7)) and (4) and (5) that the projection π of H into $G(1)$ is actually onto. So if $K = \ker \pi$, the kernel of π,

$$H/K \cong G .$$

So we have a series

$$\begin{array}{l} \big|\, G/H \,\big| = 3 \\ \\ H/K \cong G \end{array} \qquad (9)$$

So G has a proper subgroup that maps onto G. This certainly means G is infinite.

We turn now to the final step in the proof. ■

Lemma 3 *Every element of G is of order a power of* 3.

Proof We put $a_i = \tau^{-i}\alpha\tau^i \quad (i = 0, 1, 2)$. Then by its very definition

$$H = gp(a_0, a_1, a_2) .$$

Notice that if $h \in H$ then h can be expressed as a Y-product where $Y = \{a_0, a_1, a_2\}$. Since ${a_i}^{-1} = {a_i}^2$ we can express every such h as a 'positive 'Y-product i.e. no negative powers occur:

$$h = y_1 y_2 \dots y_n \qquad (y_i \in Y) . \qquad (10)$$

Now an arbitrary element $g \in G$ can be written in the form (see (9))

$$g = h\tau^k \qquad (h \in H,\ 0 \le k \le 2) .$$

Suppose that, in addition we now write h in the positive form (10):

$$g = y_1 y_2 \dots y_n \tau^k \qquad (y_i \in Y,\ 0 \le k \le 2) . \qquad (11)$$

We term this representation of g a *special representation* of g and define the length $l(g)$ of such a special representation by

$$l(g) \;=\; \begin{cases} n & \text{if } k = 0 \\ n+1 & \text{if } k \neq 0 \end{cases} \tag{12}$$

Notice that $y_1 y_2 \ldots y_n$ is a positive product involving only a_0, a_1, a_2 . We express this dependence on a_0, a_1, a_2 by using functional notation:

$$h \;=\; h(a_0, a_1, a_2) \;.$$

So

$$g \;=\; h(\; a_0, a_1, a_2 \;)\tau^k \qquad\qquad (0 \leq k \leq 2) \;. \tag{13}$$

We want to prove g is of order a power of 3. This we do by induction on $l(g)$.

If $l(g) \leq 1$, this is obvious.

Suppose then that we have proved that every element of G with a special representation of length at most n is of order a power of 3 and that

$$l(g) \;=\; n+1 \qquad\qquad (n \geq 1) \tag{14}$$

with g given by (13).

We consider now two cases.

Case 1 $\qquad g \;=\; h(\; a_0, a_1, a_2 \;)\tau^k \qquad\qquad (0 < k \leq 2) \;.$

Thus here $l(h) = n$. Suppose a_0 occurs i_0 times in h, a_1 i_1 times and a_2 i_2 times. Thus

$$n \;=\; i_0 + i_1 + i_2 \;.$$

The trick is to compute

$$\begin{aligned} g^3 \;&=\; h\tau^k h\tau^k h\tau^k \\ &=\; h\tau^{3k}\tau^{-2k} h\tau^{2k}\tau^{-k} h\tau^k \\ &=\; h\tau^{-2k} h\tau^{2k}\tau^{-k} h\tau^k \;. \end{aligned}$$

Let's consider the cases $k = 1, 2$ in turn, starting with $k = 1$.
In this instance

$$g^3 \;=\; h(\,a_0, a_1, a_2\,)\; h(\,a_2, a_0, a_1\,)\; h(\,a_1, a_2, a_0\,)\;. \tag{15}$$

Now this means that a_0 occurs i_0 times in $h(\,a_0, a_1, a_2\,)$, i_1 times in $h(\,a_2, a_0, a_1\,)$ and i_2 times in $h(\,a_1, a_2, a_0\,)$, i.e.

$$n \;=\; i_0 + i_1 + i_2$$

times in all. Similarly for each of a_1 and a_2 and similarly in the case $k = 2$.

Now notice that by (4), (5) and (6)

$$\begin{aligned} a_0 &= \tau(1)\,\tau(2)^{-1}\,\alpha(3) \\ a_1 &= \alpha(1)\,\tau(2)\,\tau(3)^{-1} \\ a_2 &= \tau(1)^{-1}\,\alpha(2)\,\tau(3)\;. \end{aligned}$$

We can therefore re-express g^3 as an element in

$$G(1) \,\times\, G(2) \,\times\, G(3)$$

(see (7)). For definiteness let's consider the case $k = 1$, using (15).

Notice that the first component of g^3 is then

$$h\big(\tau(1), \alpha(1), \tau(1)^{-1}\big)\; h\big(\tau(1)^{-1}, \tau(1), \alpha(1)\big)\; h\big(\alpha(1), \tau(1)^{-1}, \tau(1)\big)\;. \tag{16}$$

Let's look at the $\tau(1)$ that replaces a_0 in forming the first component of (15). It occurs i_0 times in the first h, i_1 times in the second and i_2 times in the third, exactly as before, i.e. $\tau(1)$ occurs n times, counting its occurences stemming from a_0. Similarly $\alpha(1)$ occurs n times and finally $\tau(1)^{-1}$ occurs n times. Let's continue to focus on this first component of g^3 given in (16). We may view it as an element of $G(1)$. As such it has a corresponding form to that of g given by (13) i.e. we move all the occurences of $\tau(1)$ to the right-most-side and write the first component in the form

$$\begin{aligned} w_1\Big(\alpha(1)\,,\, \tau(1)^{-1}\alpha(1)\tau(1)\,,\, \tau(1)^{-2}\alpha(1)\tau(1)^2\Big)\,\tau(1)^m \\ = \; w_1\big(a_0(1), a_1(1), a_2(1)\big)\,\tau(1)^m\;. \end{aligned}$$

Notice that this form $w_1\big(\, a_0(1), a_1(1), a_2(1)\,\big)$ is a positive word in $a_0(1), a_1(1), a_2(1)$ of length exactly n. And since both $\tau(1)$ and $\tau(1)^{-1}$ occur n times in (16),

$$m \;=\; 0 \qquad !$$

So it follows that we can continue this argument for each one of the components of g^3, yielding

$$g^3 \;=\; w_1 w_2 w_3 \;\in\; G(1) \times G(2) \times G(3)$$

with

$$l(w_1) \;=\; l(w_2) \;=\; l(w_3) \;=\; n\ .$$

Inductively since $G \cong G(i) \quad (i = 1, 2, 3)$, we find each of w_1, w_2, w_3 is of order a power of 3. So g is also.

Case 2 $\qquad g \;=\; h(\, a_0, a_1, a_2\,)$

So again $l(g) = n + 1$ i.e. $l(h) = n + 1$. Now again we assume a_0 occurs i_0 times, $a_1 \quad i_1$ times and $a_2 \quad i_2$ times in h. So here

$$i_0 + i_1 + i_2 \;=\; n + 1\ .$$

Notice that if at least two of i_0, i_1, i_2 are zero, then h is a power of one of a_0, a_1, a_2 and hence is of order a power of 3. So we may assume that at least two of i_0, i_1, i_2 are non-zero. Now view h as an element of $G(1) \times G(2) \times G(3)$, as usual:

$$\begin{aligned} h \;&=\; h(a_0, a_1, a_2) \\ &=\; h\big(\tau(1), \alpha(1), \tau(1)^{-1}\big)\; h\big(\tau(2)^{-1}, \tau(2), \alpha(2)\big)\; h\big(\alpha(3), \tau(3)^{-1}, \tau(3)\big) \\ &=\; h_1\, h_2\, h_3 \quad . \end{aligned}$$

On re-expressing each of these components in the special form in $G(1)\,, G(2)$ and $G(3)$ we either find that

$$h_i \;=\; {h'}_i \tau(i)^k \qquad\qquad (0 < k \leq 2)$$

with $l({h'}_i) \leq n$ or else

$$h_i \;=\; {h'}_i$$

with $l(h'_i) \leq n$ again. In the first case we repeat the argument of Case 1 computing ${h_i}^3$ and deduce h_i is of order a power of 3 inductively. In the second case we can already use the inductive assumption and deduce that h_i is of order a power of 3. This completes the proof. ■

The argument given above is contained in:

N. Gupta, S. Sidki: *On the Burnside Problem for Periodic Groups* **Math.Z. 182** (1983), 385-388.

As I remarked earlier, it is based on the paper

R.I. Grigorchuk: *On the Burnside Problem for Periodic Groups* English Translation: **Functional Anal. Appl. 14** (1980), 41-43.

3. An application to associative algebras

There is an analogous problem to that of Burnside for associative algebras over a field due to Kurosh.

Kurosh's Problem

Let A be an associative algebra over a field k. Suppose that each element a is the root of some polynomial $c_0 + c_1x + \ldots + c_nx^n \quad (c_n \neq 0\,,\, n > 0)$ i.e.

$$c_0 \,+\, c_1a \,+ \ldots +\, c_na^n \;=\; 0\;.$$

Is any finitely generated subalgebra of A finite dimensional?

The first counter-example was obtained by

E.S. Golod and I.R. Shafarevich: *On towers of class fields* **Izv. Akad. Nauk SSSR Ser.Mat 28** (1964), 261-272.

The Gupta-Sidki group provides another example. In order to see why, form the group algebra $\mathbf{F}_3[G]$ where $\mathbf{F}_3$ denotes the field of three elements. So

$$\mathbf{F}_3[G] \;=\; \Big\{ \sum_{finite} c_i g_i \;\Big|\; c_i \in \mathbf{F}_3 \,,\, g_i \in G \Big\}$$

with the obvious definition of equality, coordinate-wise addition and a multiplication defined by distributivity and

$$c_i g_i \cdot c_j g_j \;=\; c_i c_j (g_i g_j) \;.$$

We claim that the so-called augmentation ideal

$$A \;=\; I(G) \;=\; \Big\{ \sum c_i g_i \;\Big|\; \sum c_i \;=\; 0 \Big\}$$

of $\mathbf{F}_3[G]$ provides a suitable example. Indeed note first that

$$\Big(\sum c_i g_i\Big)^{3^m} \;=\; \sum {c_i}^{3^m} {g_i}^{3^m} \;=\; \sum {c_i g_i}^{3^m} \;.$$

Since each g_i is of order a power of 3, by choosing m sufficiently large we can ensure that each ${g_i}^{3^m} = 1$. So

$$\Big(\sum c_i g_i\Big)^{3^m} \;=\; \sum c_i \;=\; 0 \qquad !$$

It suffices then to prove that a is finitely generated since it is clearly infinite dimensional. But we claim that A is generated by

$$\alpha \,-\, 1 \;,\; \tau \,-\, 1 \;.$$

this follows from the identity

$$xy - 1 \;=\; (x-1)(y-1) \,+\, (x-1) \,+\, (y-1) \;;$$

Finally in closing, it is clear that the Grigorchuk-Gupta-Sidki group G has a recursive presentation. I have been told G is not finitely presented, but I do not know of a proof. Perhaps one way of proving this fact is to show that $H_2(G, \mathbf{Z})$, the second homology group with coefficients in $\mathbf{Z}$ (see Chapter VI for a definition of $H_2(G, \mathbf{Z})$), viewed as a trivial G-module, is not finitely generated.

CHAPTER III Free groups, the calculus of presentations and the method of Reidemeister and Schreier

1. Frobenius' representation

Let G be a group. Then we say $G\ acts\ on\ a\ set\ Y$ if it comes equipped with a homomorphism

$$\varphi : G \longrightarrow S_Y$$

where S_Y is the symmetric group on Y i.e. the group of all permutations of Y. If $Y = \{1, 2, \ldots, n\}$ we sometimes write S_n in place of S_Y. In general a homomorphism of a group G into another group is termed a $representation\ of\ G$, on occasion. Here φ is called a $permutation\ representation$ of G. We shall have occasion also to consider $matrix\ representations$ later. A representation is called $faithful$ if it is one-to-one.

The most familiar faithful representation goes back to Cayley.

Theorem 1 (Cayley) *Let G be a group. Then the map*

$$\varrho : G \longrightarrow S_G$$

defined by

$$g \longmapsto (g\varrho : x \longmapsto xg)$$

is a faithful permutation representation of G, called the right regular representation.

The proof is an immediate application of the definition.

Next I want to describe Frobenius' version of Cayley's representation. In order to do so, we need some definitions.

Definition 1 *Let G be a group, $H \leq G$. Then a complete set of representatives of the right cosets Hg of H in G is a set R consisting of one element from each coset. The element in R coming from the coset Hg is denoted by $\overline{g}$ and is termed the representative of Hg or sometimes the representative of g. If $1 \in R$, R is termed a right transversal of H in G.*

Let us put $\delta(r,x) = rx\,(\overline{rx})^{-1}$ $(r \in R, g \in G)$. The following lemma is proved directly from the definitions.

Lemma 1

(i) $Hg = H\overline{g}$	$(g \in G)$.
(ii) $\delta(r,x) \in H$	.
(iii) $\overline{\overline{g_1}g_2} = \overline{g_1 g_2}$	$(g_1, g_2 \in G)$.
(iv) $g = \delta(g,1)\overline{g}$	$(g \in G)$.

The following theorem of Frobenius is a simple consequence of Lemma 1.

Theorem 2 (Frobenius) *Let G be a group, $H \leq G$, R a complete set of representatives of the right cosets of H in G. Then the homomorphism*

$$\gamma : G \longrightarrow S_R$$

defined by

$$g \longmapsto (g\gamma \,:\, r \longmapsto \overline{rg})$$

is a representation of G, sometimes termed a coset representation of G.

This representation has turned out to be extremely useful. I want to make one simple deduction from its existence and then I want to look at the way the regular representation can be reconstituted from a coset representation. This leads one to a very fertile area of investigation, so-called *induced representation theory* and the theory of *wreath products*. I won't go into any of the details here, but extract two facts which might be useful later.

First some more notation. Suppose G acts on a set Y where φ is the ambient representation. If $y \in Y$, then the *stabilizer of* y , denoted $stab_\varphi(y)$, is by definition,

$$stab_\varphi(y) \;=\; \{\, g \in G \;\big|\; y(g\varphi) = y \,\} \;\leq\; G \;.$$

We have the following theorem of M. Hall.

Theorem 3 (M.Hall) *Let G be a finitely generated group. Then the number of subgroups of G of a given finite index* j *is finite.*

Proof: We concoct for each subgroup H of index j in G a homomorphism

$$\varphi_{_H} \,:\, G \longrightarrow S_j$$

in such a way that

$$stab_{\varphi_{_H}}(1) \;=\; H \;.$$

We need to define $\varphi_{_H}$. To this end let us choose a complete set R of representatives of the right cosets of H in G. We choose an enumeration of the elements of R which is arbitrary except that the first element is in H:

$$R \;=\; \{\, r_1, r_2, \ldots, r_j \,\} \qquad with \qquad r_1 \,\in\, H \;.$$

Let γ be Frobenius' coset representation and define

$$\varphi_{H} \, : \, G \longrightarrow S_j$$

by

$$i(g\varphi_{H}) \; = \; k \qquad \Longleftrightarrow \qquad r_i(g\gamma) \; = \; r_k \; .$$

Notice that

$$stab_{\varphi_{H}}(1) \; = \; H \; .$$

Thus if K is a second subgroup of G of index j,

$$\varphi_{H} \; = \; \varphi_{K} \quad \Longrightarrow \quad H \; = \; K \; .$$

This means that the number of subgroups of G of index j is at most the number of homomorphisms of G into S_j. Now if G is an n-generator group, the number of such homomorphisms is at most

$$(j!)^n \; < \; \infty \; .$$

This completes the proof. ■

Now let's try to reconstitute Cayley's representation ϱ from Frobenius' γ. To this end let G be a group, $H \leq G$, R a complete set of representatives of the right cosets of H in G. Think of G as being partitioned into $\left| R \right|$ blocks with $\left| H \right|$ elements in each block:

$$G \; = \; \dot{\bigcup_{r \in R}} Hr \; \cong \; \dot{\bigcup_{r \in R}} H \times \{r\} \; = \; H \, \times \, R \; .$$

If $g \in G$ then let's think of the way in which $g\varrho$ acts on these blocks:

$$hr(g\varrho) \; = \; hrg \; = \; h \, rg \, (\overline{rg})^{-1} \, \overline{rg} \; = \; h \, \delta(r,g) \, \big(r(g\gamma)\big) \; .$$

Thinking some more allows us to view g as acting on these blocks in two stages. First, in the block $H \times \{r\}$, it right multiplies every one of the elements $h \in H$ by the fixed element $\delta(r,g)$ of H and then bodily moves the whole block to another one according to γ.

There is another way of viewing what is going on. First the functions $\delta(\star, g)$ are functions from R to H. Thus let's form

$$H^R \;=\; \{\, f \,:\, R \longrightarrow H \,\}\;.$$

H^R then can be thought of as a group of permutations of $H \times R$:

$$(h,r)f \;=\; \big(\, hf(r)\,,\, r\,\big)\;.$$

And for each $g \in G,\ g\gamma$ can be thought of as a permutation of $H \times R$:

$$(h,r)(g\gamma) \;=\; \big(\, h\,,\, r(g\gamma)\,\big)\;.$$

Here is one consequence of this approach:

Theorem 4 *Let G be a group generated by a set X, $H \leq G$ and R a right transversal of H in G. Then*

$$H \;=\; gp\big(\; \delta(r,x) \;=\; rx(\overline{rx})^{-1} \;\;\big|\;\; r \in R\,,\, x \in X\;\big)\;.$$

Proof We simply trace out for each $h \in H$ the effect of $h\varrho$ on $(1,1) \in H \times R$. First notice that

$$(h,r)(g\varrho) \;=\; \big(\; h\delta(r,g)\,,\, \overline{rg}\;\big)\;.$$

So if $h \in H$

$$(1,1)(h\varrho) \;=\; \big(\; \delta(1,h)\,,\, \overline{h}\;\big) \;=\; (h,1)\;.$$

Now express h in X-product form

$$h \;=\; {x_1}^{\varepsilon_1} \ldots {x_n}^{\varepsilon_n} \qquad\qquad (x_i \in X\,,\, \varepsilon_i = \pm 1)\;.$$

Then

$$\begin{aligned}(1,1)(h\varrho) \;&=\; (1,1)({x_1}^{\varepsilon_1}\varrho)\ldots({x_n}^{\varepsilon_n}\varrho)\\ &=\; \big(\; \delta(r_1, {x_1}^{\varepsilon_1})\ldots\delta(r_n, {x_n}^{\varepsilon_n})\,,1\big)\end{aligned}$$

where the r_i are the elements of R that arise from this computation. This proves

$$h \;=\; \delta(r_1, {x_1}^{\varepsilon_1})\ldots\delta(r_n, {x_n}^{\varepsilon_n})\;.$$

Hence

$$H \;=\; gp\big(\; \delta(r, x^{\pm 1}) \;\big|\; r \in R\,,\, x \in X \;\big)\;.$$

But

$$rx^{-1}\Big(\overline{rx^{-1}}\Big)^{-1} \;=\; \left(\overline{rx^{-1}}x\left(\overline{\overline{rx^{-1}}x}\right)^{-1}\right)^{-1}\;.$$

This completes the proof. ■

Corollary 1 *A subgroup of finite index in a finitely generated group is finitely generated.*

Exercise 1 *Prove that the intersection of two subgroups of finite index in any group is again of finite index.*

2. Semidirect products

Let

$$1 \longrightarrow A \xrightarrow{\alpha} E \xrightarrow{\beta} Q \longrightarrow 1 \tag{1}$$

be a short exact sequence of groups. We term E an extension of A by Q. If

$$1 \longrightarrow A' \xrightarrow{\alpha'} E' \xrightarrow{\beta'} Q' \longrightarrow 1$$

is another short exact sequence, we term the sequences *equivalent* if there are isomorphisms, as shown, which make the following diagram commutative:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & A & \xrightarrow{\alpha} & E & \xrightarrow{\beta} & Q & \longrightarrow & 1 \\
 & & \alpha^*\big\downarrow\wr & & \varepsilon^*\big\downarrow\wr & & \xi^*\big\downarrow\wr & & \\
1 & \longrightarrow & A' & \xrightarrow{\alpha'} & E' & \xrightarrow{\beta'} & Q' & \longrightarrow & 1
\end{array}\;.$$

Every short exact sequence (1) is equivalent to the sequence

$$1 \longrightarrow A\alpha \hookrightarrow E \longrightarrow E/A\alpha \longrightarrow 1\;.$$

We will move freely between equivalent sequences.

The sequence (1) *splits* if there exists a homomorphism

$$\eta : Q \longrightarrow E$$

such that $\eta\beta = 1$. In this case we term (1) a *split* or *splitting extension* of A by Q. Every such splitting extension (1) is equivalent to an extension

$$1 \longrightarrow \overline{A} \overset{\overline{\alpha}}{\hookrightarrow} \overline{E} \underset{\overleftarrow{\overline{\eta}}}{\overset{\overline{\beta}}{\longrightarrow}} \overline{Q} \longrightarrow 1$$

where

(i) $\overline{E} \;=\; \overline{Q}\,\overline{A}$,
(ii) $\overline{A} \trianglelefteq \overline{E}$,
(iii) $\overline{Q} \cap \overline{A} \;=\; 1$.

For simplicity of notation we simply omit all the bars. Then it follows that every element $e \in E$ can be written uniquely in the form

$$e \;=\; q\,a \qquad (q \in Q, a \in A)$$

with multiplication

$$e\,e' \;=\; q\,a\,q'a' \;=\; q\,q'a^{q'}a' \qquad (q, q' \in Q,\ a, a' \in A)\ . \tag{2}$$

Observe that the map

$$a \longmapsto a^{q'} \qquad (a \in A,\ q'\ fixed,\ q' \in Q)$$

is an automorphism $q'\varphi$ of A. The underlying map

$$\varphi : Q \longrightarrow \mathrm{Aut}\, A$$

is a homomorphism of Q into $\mathrm{Aut}\, A$ and the extension (1) can be reconstituted from $A,\ Q$ and φ.

Conversely, let A and Q be arbitrary groups,

$$\varphi : Q \longrightarrow \mathrm{Aut}\, A$$

a homomorphism. Then we can form a split extension E of A by Q as follows. Set-theoretically

$$E \;=\; Q \times A \;=\; \{\, (q,a) \;\big|\; q \in Q,\; a \in A \,\}\;.$$

We define a multiplication on E by analogy with (2):

$$(\,q,a\,)(\,q',a'\,) \;=\; (\,qq'\;,\; a(q'\varphi)\,a'\,)\;. \tag{3}$$

It is easy to see then that E is an extension of A by Q, called the *semidirect product of A by Q* and denoted by

$$E \;=\; A\;\; Q \;=\; A \,{}_{\varphi}\, Q$$

the latter to express the dependence on φ. If we identify $q \in Q$ with $(q,1),\; a \in A$ with $(1,a)$ then we find

(i) $\quad E \;=\; Q\,A\;,$

(ii) $\quad A \trianglelefteq E\;,$

(iii) $\quad Q \cap A \;=\; 1\;;$

and the multiplication in E takes the form

$$q\,a\,q'a' \;=\; q\,q'\;a\,(q'\varphi)\;a' \qquad (q,q' \in Q\,,\; a,a' \in A)\;. \tag{4}$$

We often switch from the notation (3) to the notation (4).

Examples 2 *(1) Let A be an abelian group, $Q = \langle\, q; q^2 = 1 \,\rangle$ a group of order 2. Let*

$$\varphi : Q \longrightarrow \mathrm{Aut}\, A$$

be defined by

$$q\varphi : a \longmapsto a^{-1}\;.$$

Then we can form

$$E \;=\; A\ Q\ .$$

(i) *If A is cyclic of order 4, E is the dihedral group of order 8.*

(ii) *If $A = C_{2^\infty}$ the quasicyclic group of type 2^∞ i.e. the group of all 2^n-th roots of 1 in the complex field, E is an infinite 2-group all of whose proper subgroups are either cyclic, dihedral groups or C_{2^∞}.*

Compute the upper central series of E.

(2) Let A be a free abelian group of infinite rank on

$$\ldots\ ,\ a_{-1}\ ,\ a_0\ , a_1\ ,\ \ldots$$

and let

$$Q \;=\; \langle t \rangle$$

be infinite cyclic. Let

$$\varphi : Q \longrightarrow \operatorname{Aut} A$$

be defined by

$$t\varphi : a_i \longmapsto a_{i+1} \qquad (i \in \mathbf{Z})\ .$$

Form

$$E \;=\; A\ Q\ .$$

Prove (i) *E is a 2-generator group*

and (ii) *E' is free abelian of infinite rank.*

(3) Let R be any ring with and let

$$A \;=\; R^+$$

be the additive group of R. Let Q be any subgroup of the group of units of R and let

$$\varphi : Q \longrightarrow \operatorname{Aut} A$$

be defined by

$$q \longmapsto (q\varphi : a \longmapsto aq)\ .$$

Form

$$E \;=\; A \;\; Q \;.$$

(i) *If $R = \mathbf{Z}[x, x^{-1}]$ is the group ring of the infinite cyclic group and Q=gp(x) , prove $E \cong$ the group in (2).*

(ii) *Let $R = \mathbf{Z}[\frac{1}{6}]$. Then $\frac{2}{3}$ is a unit in R. Let $Q = gp(t = \frac{2}{3})$. Is E finitely generated? Find a presentation of E.*

(4) *Let the group H act on a set Y and let the group Q act on a set X. Form*

$$A \;=\; H^X \;=\; \{\, f : X \longrightarrow H \} \;.$$

A becomes a group under coordinate-wise multiplication, and Q acts on A

$$q : f \longmapsto fq$$

where

$$f\, q(x) \;=\; f(xq^{-1}) \qquad\qquad (x \in X) \;.$$

We term the semidirect product AQ a wreath product of H by Q. Notice that AQ acts on

$$X \times Y$$

by

$$(\, x, y\,)(\, q, f\,) \;=\; \big(\, xq\,,\; y\, f(x)\,\big) \;.$$

If in place of H^X we take $H^{(X)}$ the set of all functions from X to H which are almost always 1, we get a corresponding group, the restricted wreath product of H by Q.

We denote the first wreath product by

$$\overline{W} \;=\; H \,\bar{\wr}\, Q$$

and the second by

$$W \;=\; H \wr Q \;.$$

(i) *Let G now be a group, $H \leq G$, X a right transversal of H in G. Let H act on H via the right regular representation and let γ be the Frobenius representation of G on X. Let*

$$Q \;=\; G\gamma \;.$$

Verify that Cayley's representation yields via γ a faithful representation of G in

$$\overline{W} \;=\; H \,\bar{\wr}\, Q \;.$$

Hint *Use the discussion above and that relating to the Frobenius representation.*

3. Subgroups of free groups are free

Recall that a group F is free if it has a so-called free set of generators X. So

(i) $gp(X) \;=\; F$

(ii) *every non-empty reduced X-product* ${x_1}^{\varepsilon_1} \ldots {x_n}^{\varepsilon_n} \neq 1$.

It follows that if $f \in F$, then f can be expressed as a reduced X-product

$$f \;=\; {x_1}^{\varepsilon_1} \ldots {x_n}^{\varepsilon_n}$$

in exactly one way. We term this reduced X-product the *normal form* of f, and define the *length* of f, denoted by $l(f)$, to be n:

$$l(f) \;=\; n \;.$$

If $f, g \in F$ and if

$$l(fg) \;=\; l(f) \;+\; l(g)$$

we write

$$f \,\triangle\, g$$

to express the fact that no cancellation takes place on forming the product fg i.e. the reduced X-product for fg is obtained by concatenating the reduced X-product for g with the reduced X-product for f. If

$$l(fg) \;<\; l(f) \;+\; l(g)$$

we sometimes write

$$f \underset{\sqcup}{} g$$

expressing the fact that the last letter of f cancels the first letter of g.

Examples 3 *(1) Consider the group of units of the formal power series ring in the noncommuting variables Ξ with integer coefficients. Prove that*

$$F \;=\; gp(\, 1+\xi \;\big|\; \xi \in \Xi \,)$$

is free on $\{\, 1+\xi \;\big|\; \xi \in \Xi \}$. This is a theorem of W. Magnus.

(2) Suppose that X is a set and that $G = gp(\sigma, \tau)$ is a subgroup of S_X. Furthermore suppose X_1 and X_2 are non-empty disjoint subsets of X and that

$$\begin{aligned} X_1\sigma^m \;&\subseteq\; X_2 \qquad\qquad if \quad m \neq 0 \\ X_2\tau^n \;&\subseteq\; X_1 \qquad\qquad if \quad n \neq 0 \end{aligned}$$

Prove that G is free on $\{\sigma, \tau\}$.

Hint *The trick is to verify that if*

$$w \;=\; \sigma^{m_1}\tau^{n_1} \ldots \sigma^{m_k}\tau^{n_k} \qquad (m_1, n_1, \ldots, m_k, n_k \;\neq\; 0)$$

is a reduced $\{\sigma, \tau\}$-product then

$$X_1\, w \;\subset\; X_1 \qquad and \qquad X_1\, w \;\neq\; X_1 \;.$$

The first step is to examine $X_1\sigma^m \cap X_1\sigma^{m'}$. This examination leads to the conclusion that the images of X_1 under the powers σ^m $(m \neq 0)$ are disjoint subsets of X_2. So

$$X_1\, \sigma^{m_1} \;\not\subseteq\; X_2$$

yielding

$$X_1\, w \;\not\subseteq\; X_1 \;.$$

Now let F be a free group freely generated by some set X, $H \leq F$ and S a right transversal of H in F. We term F a Schreier transversal if

$$x_1{}^{\varepsilon_1} \dots x_n{}^{\varepsilon_n} \in S \qquad implies \qquad x_1{}^{\varepsilon_1} \dots x_{n-1}{}^{\varepsilon_{n-1}} \in S$$

i.e. every "initial segment" of a representative is again a representative.

The proof then of Schreier's subgroup theorem goes as follows:

(i) There always exist Schreier transversals.

(ii) If S is a Schreier transversal then H is free on

$$Y = \{ \delta(s,x) = sx(\overline{sx})^{-1} \neq 1 \mid s \in S, x \in X \} .$$

We already know that Y generates H. It remains to check that Y freely generates H.

Proof The scheme of the proof is very simple.

(i) If $\delta(s,x) \neq 1$, then we prove

$$\delta(s,x) = s_\triangle x_\triangle (\overline{sx})^{-1}$$

and so

$$\big(\delta(s,x)\big)^{-1} = \overline{sx}_\triangle x^{-1}{}_\triangle s^{-1} .$$

(ii) If $\delta(s,x) \neq 1$, then

$$\delta(s,x) = \delta(t,y) \qquad only\ if \qquad s = t \qquad and \qquad x = y .$$

(iii) If

$$\pi = \delta(s_1,x_1)^{\varepsilon_1} \dots \delta(s_n,x_n)^{\varepsilon_n}$$

is a reduced Y-product in the symbols $\delta(s,x)$ (which by (ii) are distinct elements if the symbols are distinct) then on expanding we find

$$\pi = \bullet_\triangle x_1{}^{\varepsilon_1}{}_\triangle \bullet \dots \bullet_\triangle x_n{}^{\varepsilon_n}{}_\triangle \bullet$$

i.e. the x and x^{-1} in the middle of $\delta(s,x)$ and $\big(\delta(s,x)\big)^{-1}$ respectively never cancel on computing the reduced X-product form of π.

Let me indicate how the proof goes.

(i) Suppose $s \sqcup x$. Then

$$s \;=\; t_{\triangle} x^{-1}$$

and since S is a Schreier transversal $t \in S$ i.e. $\overline{t} = t$. Now

$$sx\left(\overline{sx}\right)^{-1} \;=\; tx^{-1}x\left(\overline{tx^{-1}x}\right)^{-1} \;=\; t\,t^{-1} \;=\; 1\ .$$

Similarly if $x \sqcup \big(\overline{sx}\big)^{-1}$ (or equivalently $\overline{sx} \sqcup x^{-1}$ – note here $\overline{\overline{sx}\,x^{-1}} = s$).

(ii) By (i) $s_{\triangle} x_{\triangle}\big(\overline{sx}\big)^{-1} = t_{\triangle} y_{\triangle}\big(\overline{ty}\big)^{-1}$. If $l(s) = l(t)$, $s = t$, $x = y$ and we are home. If $l(s) < l(t)$, sx is an initial segment of t. So $\overline{sx} = sx$ and therefore $sx\big(\overline{sx}\big)^{-1} = 1$!

(iii) Note that on computing any of the reduced Y-products

$$\big(\delta(s,x)\big)^{\pm 1}\big(\delta(t,y)\big)^{\pm 1}$$

we get the four possibilities

$$\begin{array}{l} \bullet \triangle\ x\ \triangle\ \bullet \triangle\ y\ \triangle \bullet \\ \bullet \triangle\ x\ \triangle\ \bullet \triangle\, y^{-1} \triangle \bullet \\ \bullet \triangle x^{-1} \triangle \bullet \triangle\ y\ \triangle \bullet \\ \bullet \triangle x^{-1} \triangle \bullet \triangle\, y^{-1} \triangle \bullet \end{array} .$$

This establishes the form of π.
The rest of the proof follows along the same lines. ■

We are left with the proof of the existence of Schreier transversals.

Proposition 1 *Let F be a free group on the set X, $H \leq F$. Then there exists a Schreier transversal S of H in F.*

Proof Define the length $l(Hf)$ of the right coset Hf of H in F by

$$l(Hf) \;=\; \min\{\, l(hf) \;\big|\; h \in H \,\}\,.$$

We choose the elements of S in stages. First $1 \in S$. Now we proceed by induction. Suppose representatives have been chosen for all cosets of length at most n in such a way that an initial segment of a representative is again a representative. For the right cosets of length $n+1$ we do the following: Let

$$l(Hf) \;=\; n+1\ .$$

So there exists in Hf an element $b_1 \ldots b_{n+1}$ of length $n+1$. Consider the coset

$$Hb_1 \ldots b_n\ .$$

Then

$$l(Hb_1 \ldots b_n) \;\leq\; n$$

so has a representative already, say $a_1 \ldots a_m$ $(m \leq n)$. Consider

$$a_1 \ldots a_m b_{n+1}\ .$$

Notice

$$Ha_1 \ldots a_m b_{n+1} \;=\; Hb_1 \ldots b_n b_{n+1}\ .$$

So

$$l(\, a_1 \ldots a_m b_{n+1}\,) \;=\; n+1$$

i.e. $m = n$ and in particular

$$a_1 \ldots a_n \;\triangle\; b_{n+1}\ .$$

We take

$$a_1 \ldots a_n b_{n+1}$$

to be the representative of Hf . It is clear that every initial segment of $a_1 \ldots a_n b_{n+1}$ is again a representative, as desired. ■

To sum up then, suppose the group F is free on the set X, $H \leq F$. If we choose a right transversal S of H in F closed under initial segments, i.e. a Schreier transversal, then H is freely generated by

$$Y \;=\; \{\, \delta(s,x) \;=\; sx(\overline{sx})^{-1} \;\neq\; 1 \;\big|\; s \in S\,,\; x \in X \,\}\ .$$

Examples 4 *Let F be free on $\{x, y\}$.*

(1) Define $\varphi : F \longrightarrow C_2 = \langle a; a^2 \rangle$ by $x \longmapsto a,\ y \longmapsto 1$. Let $H = \ker \varphi$. So $F/H \cong C_2$. Note $F = H \dot{\cup} Hx$. Then S is readily chosen: $S = \{1, x\}$.

$$Y \ = \ \{\ \delta(s, \xi) \ \neq \ 1 \ \big|\ s \in S\,,\ \xi \in X\ \}\ .$$

Since $\delta(1, x) = 1\,,\ \delta(1, y) = y\,,\ \delta(x, x) = x^2\,,\ \delta(x, y) = xyx^{-1}$ we get

$$Y \ = \ \{\ y\,,\ xyx^{-1}\,,\ x^2\ \}$$

and H is free on Y.

(2) Find a set of free generators for $H = gp(\ f^2 \ \big|\ f \in F\ )$. (F/H is the Klein 4-group.)

(3) Define $\varphi : F \longrightarrow C_\infty = \langle a \rangle$ by $x \longmapsto a,\ y \longmapsto 1$. Let $H = \ker \varphi$. So $F/H \cong C_\infty$. Note $F = \dot{\cup}_{i \in \mathbf{Z}} Hx^i$. Take $S = \{\ x^i \ \big|\ i \in \mathbf{Z}\}$. The set Y of free generators of H we obtain is

$$Y \ = \ \{\ x^i y x^{-i} \ \big|\ i \in \mathbf{Z}\ \}\ .$$

(4) For the commutator subgroup $F' \leq F$ we choose

$$S \ = \ \{\ x^m y^n \ \big|\ m, n \in \mathbf{Z}\ \}\ .$$

Since

$$\begin{aligned}
\delta(\ x^m y^n\,,\ x\ ) \ &= \ x^m y^n x (\overline{x^m y^n x})^{-1} \ = \ x^m y^n x (x^{m+1} y^n)^{-1} \ = \ x^m y^n x y^{-n} x^{-(m+1)}\ , \\
\delta(\ x^m y^n\,,\ y\ ) \ &= \ x^m y^n y (\overline{x^m y^n y})^{-1} \ = \ x^m y^{n+1} (x^m y^{n+1})^{-1} \ = \ 1\ ,
\end{aligned}$$

F' is free on $\{\ x^m y^n x y^{-n} x^{-(m+1)} \ \big|\ n \neq 0\ \}$.

Note Here we see that in a free group of finite rank subgroups may well be free of infinite rank.

(5) *Let φ be a homomorphism of F onto S_3 . Find free generators for the kernel H of φ.*

(6) If G is any free group prove G/G' is free abelian i.e. a direct sum of infinite cyclic groups.

Definition 2 Let $\mathcal{P}$ be a property of groups. The we say a group G is virtually $\mathcal{P}$ or has $\mathcal{P}$ virtually or is virtually a $\mathcal{P}$-group if G has a subgroup of finite index with $\mathcal{P}$.

So a virtually finite group is finite; a virtually abelian group is a finite extension of an abelian group.

One of the remarkable consequences of the theorem of Stallings and Swan alluded to in Chapter I is

Theorem 5 *A torsion-free virtually free group is free.*

Definition 3 Suppose H is a subgroup of a free group F. We term H a free factor of F if we can find a free set

$$Y \dot{\cup} Z$$

of generators of F such that H is free on Y. If K is the subgroup generated by Z we write

$$F \;=\; H \,*\, K$$

and term F the free product of H and K.

Notice K is also a free factor of F.

Definition 4 *Let G be a group acting on a set S. We say G acts transitively on S if given any pair of elements $s, t \in S$ there exists $g \in G$ such that*

$$s\,g \;=\; t\ .$$

Then it is easy to prove the following

Lemma 2 *Suppose G acts transitively on S. Let $s_0 \in S$ be any chosen element of S and for each $t \in S$ let $g \in G$ be chosen so that*

$$s_0 g \;=\; t \;.$$

Then the set R of such elements of G is a complete set of representatives of the right cosets of the stabilizer J of s_0 in G.

Proof Let $c \in G$. Consider

$$s_0 c$$

There exists $r \in R$ such that

$$s_0 c \;=\; s_0 r \;.$$

So $cr^{-1} \in J$ i.e.

$$J c \;=\; J r \;.$$

So every right coset of J in G is represented by an element of R. Moreover if $r_1, r_2 \in R$ and

$$J r_1 \;=\; J r_2$$

then

$$s_0 r_1 \;=\; s_0 r_2 \;.$$

So by the definition of R

$$r_1 \;=\; r_2 \;.$$

■

Our objective is to prove the following theorem of M. Hall (see his book that was cited in Chapter II:

Theorem 6 *Let H be a finitely generated subgroup of a finitely generated free group F. Then H is virtually a free factor of F.*

Proof Let F be free on X, R a Schreier transversal of H in F. Then H is free on

$$Y \;=\; \{\, r_1 x_1 (\overline{r_1 x_1})^{-1}, \ldots, r_n x_n (\overline{r_n x_n})^{-1} \,\} \qquad (\, n < \infty \,,\; r_i \in R \,,\; x_i \in X \,) \;.$$

Let S consist of all initial segments of the elements

$$r_1,\ \overline{r_1 x_1}, \ldots,\ r_n\ ,\ \overline{r_n x_n}\ .$$

S is then a finite subset of R.

Now for each $x \in X$ define

$$S(x)\ =\ \{\ s \in S\ \big|\ \overline{sx} \in S\ \}\ .$$

Notice that $S(x)$ may well be empty. Define

$$\varphi(x)\ :\ S(x) \longrightarrow S \qquad by \qquad s \longmapsto \overline{sx}$$

Then $\varphi(x)$ is $1-1$ and so can be continued to a permutation, again denoted by $\varphi(x)$, of S. So, allowing x to range over X, we can view this discussion as the definition of a map from X into the set of all permutations of S and hence as a homomorphism, φ say, of F into the permutation group on S. Now let $s \in S,\ x \in X$ and suppose $s_{\triangle} x \in S$. This means $s \in S(x)$ and

$$s\,\varphi(x)\ =\ (\ \overline{sx}\ =\ )\ sx\ .$$

Similarly if $s \in S,\ x \in X$ and $s_{\triangle} x^{-1} \in S$ then $sx^{-1} \in S(x)$ and

$$s\,x^{-1}\,\varphi(x)\ =\ s$$

or

$$s\,\varphi(x^{-1})\ =\ sx^{-1}\ .$$

Now suppose $t \in S$ and write t as a reduced X-product

$$t\ =\ {x_1}^{\varepsilon_1} \ldots {x_n}^{\varepsilon_n}\ .$$

Then it follows that

$$1\,\varphi(t)\ =\ 1\,\varphi\big({x_1}^{\varepsilon_1} \ldots {x_n}^{\varepsilon_n}\big)\ =\ \Big(\ldots\Big(1\,\varphi\big({x_1}^{\varepsilon_1}\big)\Big)\ldots\Big)\,\varphi\big({x_n}^{\varepsilon_n}\big)\ =\ {x_1}^{\varepsilon_1} \ldots {x_n}^{\varepsilon_n}\ =\ t\ .$$

Thus φ defines a transitive action of F on S and by the preceding lemma, S itself is a complete set of representatives of the right cosets of the stabilizer J of 1 under this action. S

is closed under initial segments and therefore is a Schreier transversal for J in F. Denote the representative of a coset Jf in S by $\widetilde{f}$. Then J is free on

$$W \;=\; \{\; sx\,\widetilde{sx}^{-1} \;\neq\; 1 \;\mid\; s \in S \;\; x \in X \;\}\;.$$

Now consider the elements of Y, e.g. $r_i x_i\,(\overline{r_i x_i})^{-1}$. Notice first, however, that if $f \in F$, then

$$\widetilde{f} \;=\; 1\,\varphi(f) \quad !$$

Let us compute then

$$\widetilde{r_i x_i} \;=\; 1\,\varphi(r_i x_i) \;=\; \big(1\varphi(r_i)\big)\varphi(x_i) \;=\; r_i\,\varphi(x_i)\;.$$

But by the very definition

$$r_i \;\in\; S(x_i)\;.$$

So

$$r_i\,\varphi(x_i) \;=\; \overline{r_i x_i}\;.$$

In other words

$$\widetilde{r_i x_i} \;=\; \overline{r_i x_i}\;.$$

So $Y \subseteq W$ which completes the proof of the theorem. ■

Exercise 2 *A free factor H of a free group F is a normal subgroup of F if and only if either $H{=}1$ or $H{=}F$.*

Corollary 1 (Schreier) *If H is a finitely generated normal subgroup of a free group F, then either $H = 1$ or H is of finite index in F (and so F is finitely generated).*

Definition 5 *Let $\mathcal{P}$ be an algebraic property of groups. We term a group G residually in $\mathcal{P}$ and write $G \in \mathcal{RP}$ (and the like) if for each $g \in G$, $g \neq 1$, there exists a normal subgroup N of G such that $g \notin N$ and $G/N \in \mathcal{P}$.*

Note $\mathcal{P} \subseteq \mathcal{RP}$;

$R(R(\mathcal{P})) = R\mathcal{P}$, i.e. R is an idempotent operator.

Exercise 3 *Prove finitely generated abelian groups are residually finite.*

Theorem 7 (F.W. Levi) *Free groups are residually finite.*

Proof It is enough to consider the case of a finitely generated free group F. Let $f \in F,\ f \neq 1$. Then $gp(f)$ is a free factor of a subgroup J of finite index j in F. Thus we can find a free set

$$f, a_1, \ldots, a_r$$

of generators of J. Now consider

$$L \ = \ gp(\, J' \,, a_1, \ldots, a_r \,, \, f^2 \,) \ .$$

Then

$$f \notin L$$

and

$$\big|\, J/L \,\big| \ = \ 2.$$

Notice that L is of index $2j$ in F. It has only finitely many conjugates in F (there are only finitely many subgroups of a given finite index in a finitely generated group). Let N be their intersection. Hence N is a normal subgroup of F of finite index and $f \notin N$. ■

We recall the

Definition 6 *A group G is termed hopfian if $G/N \cong G$ implies $N=1$.*

Levi's theorem can be used to prove that finitely generated free groups are hopfian on appealing to

Theorem 8 (A.I. Malcev) *A finitely generated residually finite group G is hopfian.*

Proof Suppose that

$$G/N \;\cong\; G\;.$$

This means that the number of subgroups of a given finite index j in G/N is the same as the number of subgroups of index j in G, which is finite by one of the theorems of M. Hall proved earlier. Let

$$J_1/N\,,\ldots,\,J_r/N$$

be the subgroups of finite index j in G/N (note each $J_i \geq N$). This means that

$$J_1\,,\ldots,\,J_r$$

are the subgroup of index j in G. Turning this around we see that this implies that every subgroup of finite index in G contains N. So

$$N \;\leq \bigcap_{L\leq G,\ [G:L]<\infty} L\;.$$

But G is residually finite. Hence the intersection of the subgroups of finite index in G is 1 i.e.

$$N \;=\; 1\;.$$ ■

Exercises 4 *(1) Prove that a finite extension of a residually finite group is residually finite.*

(2) A cyclic extension of a finitely generated residually finite group is residually finite.

Examples 5 *Let $A = \mathbf{Z}[\frac{1}{2}]^+$ i.e. the additive group of the ring $\mathbf{Z}$ of integers with $\frac{1}{2}$ adjoined. Let $B = A/gp(1)$ and let*

$$C \;=\; \{\,(l,m,\overline{n}) \;\mid\; l,m \in A\,,\ \overline{n} = n + gp(1) \in B\,\}\;.$$

Define a multiplication in C as follows:

$$(\,l,m,\overline{n}\,)\cdot(\,l',m',\overline{n'}\,) \;=\; (\,l+l'\,,\,m+m'\,,\,\overline{n+n'-ml'}\,)\;.$$

Prove

(i) *C is a group which is nilpotent of class two.*

(ii) $\zeta C \cong C_{2^\infty} \cong B$.

(iii) $C/\zeta C \cong A \times A$.

(iv) *The mapping*

$$(l, m, \overline{n}) \longmapsto (2l, \frac{m}{2}, \overline{n})$$

is an automorphism α of C of infinite order.

(v) *Let G be the semidirect product of C by an infinite cyclic group $T = \langle t \rangle$ where t acts like α on C. Prove G is finitely generated.*

(vi) *Prove that G is not a hopfian group by verifying that*

$$G/N \cong G$$

where

$$N = gp\left(\left(0, 0, \overline{\tfrac{1}{2}}\right)\right) .$$

Theorem 9 (J. Nielsen 1918) *Let F be a free group of finite rank n. Suppose F is generated by some set X of n elements. Then X freely generates F.*

Proof Suppose F is free on Y. let φ be a map from Y onto X. Then φ defines a homomorphism, say φ again, of F onto F. So

$$F/\ker\varphi \cong F .$$

But F is hopfian. Hence $\ker\varphi = 1$. Thus φ is an automorphism of F which means X freely generates F. ■

4. The calculus of presentations

Let G be a group. Recall that we write

$$G = \langle X; R \rangle \tag{5}$$

where X is a set, R a set of reduced X-products if the set X comes equipped with a map

$$\varphi \ : \ X \longrightarrow G$$

such that the extension of φ to the free group F freely generated by X is a homomorphism, again denoted by φ, with

$$\ker \varphi \ = \ gp_{_F}(R) \ .$$

The map φ, which is sometimes referred to as the $presentation\ map$ is usually suppressed. The pair $\langle X; R\rangle$ is, as usual, a $presentation$ of G. We allow $R = \emptyset$ in which case G is simply the free group on X. Each element $r \in R$ is termed a relator in G with R called a set of defining relators of G. Notice that if $r \in R$

$$r\,\varphi \ = \ 1 \ .$$

We often express this fact by writing

$$r \ = \ 1 \qquad in \quad G \ .$$

More generally if w is any reduced X-product such that

$$w\,\varphi \ = \ 1$$

we term w a relator and sometimes say

$$w \ = \ 1$$

is a relation in G. This suggests a variety of alternative notations for presentations e.g.

$$G \ = \ \langle \ X \ ; \ \{ \, r = 1 \ \big| \ r \in R \, \} \ \rangle$$

and so on. In all instances such notations only make sense once they are recast in the form (5).

Examples 6 *(1) Let $G = \{\pm 1\} \subseteq \mathbf{Q}^{\bullet}$, the multiplicative group of non-zero rational numbers. Then*

$$G \ = \ \langle \, a \, ; \, a^2 = 1 \, \rangle \ .$$

Here $X = \{a\}$, $\varphi : a \longmapsto -1$.

(2) $G = \mathbf{Z}^+$ *, the additive group of integers. Then*

$$G \;=\; \langle a \rangle \ .$$

Here $X = \{a\}$, $R = \emptyset$ *and* $\varphi : a \longmapsto 1$ *(or* $\varphi : a \longmapsto -1$ *).*

(3) $G = \mathbf{Z}[x]^+$*, the additive group of the ring of all polynomials in* x *with integer coefficients. Then*

$$G \;=\; \langle\ a_0, a_1, \ldots\ ;\ \{\,[a_i, a_j] \,=\, 1 \;\big|\; i, j \geq 0\,\}\ \rangle\ .$$

(4) $G = S_3$ *, the symmetric group of degree* 3*. Then*

$$G \;=\; \langle\ \sigma, \tau\ ;\ \sigma^3 = \tau^2 = \sigma^\tau \sigma = 1\ \rangle\ .$$

(5) Every finite group has a finite presentation which is given by its multiplication table.

(6) Let G *be the semidirect product of an infinite cyclic group* $A = \langle a \rangle$ *by a group* $T = \langle t; t^2 = 1 \rangle$ *of order two where* t *acts on* A *by inversion. Then*

$$G \;=\; \langle\ a, t\ ;\ t^2 = 1\ ,\ a^t = a^{-1}\ \rangle\ .$$

(7) Find a presentation for the direct product of two groups.

(8) Prove that an extension of one finitely presented group by another is finitely presented.

(9) Let

$$G \;=\; \langle\ a, t\ ;\ a^t = a^2\ \rangle\ .$$

Prove that G *is the semidirect product of*

$$A \;=\; \{\ \frac{l}{2^m}\ \;\big|\; l, m \in \mathbf{Z}\ \}$$

(thought of as a subgroup of the additive group $\mathbf{Q}^+$ of rational numbers) by an infinite cyclic group $T = \langle t \rangle$ where t acts on A by multiplication by 2.

(10) Let

$$G \;=\; \langle\, X \,\dot\cup\, \{y\} \;;\; y = 1 \,\rangle \;.$$

Verify G is free on X.

We have tacitly taken for granted the construction of a group with a given presentation. In more detail, suppose

$$\langle X; R \rangle$$

is a pair where X is a set and R is a subset of reduced X-products. Then there exists a group G such that

$$G \;=\; \langle X; R \rangle \;. \tag{6}$$

Indeed take F to be the free group on X and

$$G \;=\; F/gp_{_F}(R) \;.$$

The presentation map in (6) is simply

$$\varphi \;:\; x \longmapsto x\, gp_{_F}(R) \qquad (x \in X) \;.$$

Definition 7 *Suppose $G = \langle X; R \rangle$ and that w is a relator in G. So $w\varphi = 1$ i.e. $w \in \ker \varphi$ where φ denotes the presentation map. We term w a consequence of R. Notice*

$$w \;=\; \prod_i {f_i}^{-1} {r_i}^{\pm 1} f_i \qquad (f_i \in F,\ \textit{the free group on } X,\ r_i \in R) \;.$$

So every relation w=1 in G is a "consequence" of the defining relations of G.

Of course a given group can well have lots of different presentations e.g. here is a presentation of a free group:

$$G \;=\; \langle\, x_1, x_2, \ldots \;;\; \{x_{f(i)} \;=\; 1 \;\mid\; i = 1, 2, \ldots\} \,\rangle \tag{7}$$

where f is a function from the positive integers into the positive integers. If f is recursive then (7) is a recursive presentation; and if the complement of the range of f is not a recursively enumerable subset of the positive integers, then (7) is a recursive presentation of a free group (incidentally necessarily of infinite rank) with an insoluble word problem. Of course a free group of countably infinite rank has a recursive presentation with a solvable word problem. So two recursive presentations of a group can, from an algorithmic standpoint, be radically different.

We shall show that if we restrict our attention to finite presentations this behaviour cannot occur i.e. if a group G is given by two finite presentations then either both presentations have a solvable word problem or neither does. This means that having a solvable word problem is an invariant of finite presentations i.e. having a solvable word problem is an algebraic property of finite presentations of a given group.

This needs a little clarification.

Definition 8 *Let $\langle X; R\rangle$ be a presentation where $X = \{x_1, x_2, \ldots\}$ is a countable, possibly infinite, set. Then we say $\langle X; R\rangle$ has a solvable word problem if $gp_{_F}(R)$ is a recursive subset of the free group F on X. In other words we can not only effectively list the elements of $gp_{_F}(R)$ but also those of $F - gp_{_F}(R)$. Incidentally the elements of $F - gp_{_F}(R)$ are sometimes termed irrelators.*

Lemma 3 *Let $G = \langle X; R\rangle$ be a recursive presentation of G. Then the consequences of R form a recursively enumerable set.*

Proof By assumption

$$X \;=\; \{\, x_1, x_2, \ldots \,\}\ .$$

And R is a recursively enumerable subset of the free group F on X. So we can list R and hence all products of conjugates of R and their inverses i.e. $gp_{_F}(R)$ is a recursively enumerable subset. ■

This preoccupation with presentations of this kind is no idle pursuit – as I have already noted they play a critical role in the subgroup structure of finitely presented groups.

Exercises 5 *(1) Prove that the additive group* $\mathbf{Q}^+$ *of rational numbers has a recursive presentation with a solvable word problem.*

(2) Does $\mathbf{Q}^+$ *have a recursive presentation with an unsolvable word problem?*

(3) Prove that the restricted direct product of all the symmetric groups of finite degree has a recursive presentation with a solvable word problem.

5. The calculus of presentations (continued)

There are four simple ways of going from one presentation of a group G to another, called Tietze transformations, which we formulate in the form of a proposition.

Proposition 2 *Let* G *be a group. Then the following hold:*

$T1$ *If* $G = \langle X; R\rangle$ *then* $G = \langle\, X \dot\cup Y\,;\, R \cup \{\, yw(\underset{\sim}{x})^{-1} \,\big|\, y \in Y\}\,\rangle$ *where here* $w(\underset{\sim}{x})$ *is a reduced* X*-product corresponding to* $y \in Y$ *i.e. different* y*'s may well have different* $w(\underset{\sim}{x})$*'s attached to them.*

$T1'$ *If* $G = \langle\, X \dot\cup Y\,;\, R \cup \{\, yw(\underset{\sim}{x})^{-1} \,\big|\, y \in Y\}\,\rangle$ *then* $G = \langle X; R\rangle$ *.*

$T2$ *If* $G = \langle X; R\rangle$ *and* S *is a set of consequences of* R*, then* $G = \langle X; R \cup S\rangle$ *.*

$T2'$ *If* $G = \langle X; R \cup S\rangle$ *and* S *is a set of consequences of* R*, then* $G = \langle X; R\rangle$ *.*

$T1$ and $T1'$ are called *Tietze transformations of type 1*;
$T2$ and $T2'$ are called *Tietze transformations of type 2.*

Once one interprets these transformations, the proofs are easy. E.g. consider $T1$: We have

$$G \;=\; \langle\, X; R\,\rangle\ .$$

So X comes with its presentation map φ. We infer, although this information is not given explicitly, that the presentation map φ^+ in $\langle\, X \cup Y\,;\, R \cup \{\, yw(\underset{\sim}{x})^{-1} \,\big|\, y \in Y\}\,\rangle$ is φ on X and

that

$$y\,\varphi^+ \;=\; w(\underset{\sim}{x\varphi}) \quad !$$

Our objective is to prove that φ^+ is onto and that the kernel of φ^+ is

$$\ker\varphi^+ \;=\; gp_{_{F^+}}(\,R \cup \{\, yw(\underset{\sim}{x})^{-1} \;\big|\; y \in Y\}\rangle\;, \tag{8}$$

where now

$$F \;=\; \langle X\rangle \quad and \quad F^+ \;=\; \langle X \cup Y\rangle\;.$$

It is obvious that φ^+ is onto. To prove (8) we observe first that

$$F^+ \; is \; free \; on \; X \cup \widetilde{Y}\;,$$

where

$$\widetilde{Y} \;=\; \{\, yw(\underset{\sim}{x})^{-1} \;\big|\; y \in Y\,\}\;.$$

We define then φ^+ in stages:

$$F^+ \xrightarrow{\;\chi\;} F \xrightarrow{\;\varphi\;} G$$

where

$$\chi\,\big|\,X \;=\; id\;, \quad \chi\,\big|\,\widetilde{Y} \;=\; \text{the trivial map.}$$

Now

$$\varphi^+ \;=\; \chi\varphi\;.$$

So

$$\ker\varphi^+ \;=\; \chi^{-1}\big(\varphi^{-1}(1)\big) \;=\; \chi^{-1}\big(gp_{_F}(R)\big) \;=\; gp_{_{F^+}}\big(\,R \cup \{\, yw(\underset{\sim}{x})^{-1} \,\big|\, y \in Y\}\,\big)\;. \qquad \blacksquare$$

Theorem 10 (Tietze) *Let*

$$G \;=\; \langle\, X; R\,\rangle \; and \; also \; G \;=\; \langle\, Y; S\,\rangle\;.$$

Then the first presentation can be transformed into the second by Tietze transformations.

Proof Let $\varphi,\ \psi$ be the presentation maps involved in the two presentations. Then

$$y\,\psi \;=\; w(\underset{\sim}{x\varphi}) \quad \text{foreach} \quad y \in Y\;.$$

So, by $T1$,

$$G \;=\; \langle\, X\dot{\cup}Y \,;\, R \cup \{\, yw(\underset{\sim}{x})^{-1} \,\big|\, y \in Y\}\,\rangle \;.$$

Now each $s \in S$ is a relator in this presentation. So, by $T2$,

$$G \;=\; \langle\, X\dot{\cup}Y \,;\, R \cup \{\, yw(\underset{\sim}{x})^{-1} \,\big|\, y \in Y\} \,\cup\, S\,\rangle \;.$$

Again

$$x\,\varphi \;=\; v(\underset{\sim}{y\psi}) \quad \text{foreach} \quad x \,\in\, X \;.$$

So, by $T2$,

$$G \;=\; \langle\, X\dot{\cup}Y \,;\, R \,\cup\, S \,\cup\, \{\, yw(\underset{\sim}{x})^{-1} \,\big|\, y \in Y\} \,\cup\, \{\, xv(\underset{\sim}{y})^{-1} \,\big|\, x \in X\}\,\rangle \;.$$

We want to get rid of X. If we write $R = \{r(\underset{\sim}{x}) \,\big|\, r \in R\}$, then, by $T2$,

$$G \;=\; \Big\langle\, X\,\dot{\cup}\,Y \,;\, \{r(\underset{\sim}{x}) \,\big|\, r \in R\} \cup S \cup \{\, yw(\underset{\sim}{x})^{-1} \,\big|\, y \in Y\} \cup \{\, xv(\underset{\sim}{y})^{-1} \,\big|\, x \in X\} \\ \cup \{r\big(\,\underset{\sim}{v(y)}\,\big) \,\big|\, r \in R\} \cup \{\, yw\big(\,\underset{\sim}{v(y)}\,\big)^{-1} \,\big|\, y \in Y\}\,\Big\rangle$$

and, by $T2'$,

$$G \;=\; \langle\, X\dot{\cup}Y \,;\, S \cup \{\, xv(\underset{\sim}{y})^{-1} \,\big|\, x \in X\} \cup \{r\big(\,\underset{\sim}{v(y)}\,\big) \,\big|\, r \in R\} \cup \{\, yw\big(\,\underset{\sim}{v(y)}\,\big)^{-1} \,\big|\, y \in Y\}\,\rangle \;.$$

By $T1'$, we can throw away X:

$$G \;=\; \langle\, Y \,;\, S \cup \{r\big(\,\underset{\sim}{v(y)}\,\big) \,\big|\, r \in R\} \cup \{\, yw\big(\,\underset{\sim}{v(y)}\,\big)^{-1} \,\big|\, y \in Y\}\,\rangle \;.$$

Finally, by $T2'$,

$$G \;=\; \langle\, Y ; S\,\rangle \;.$$

■

It follows immediately from the proof of this theorem that we have also proved the

Lemma 4 *Let $\langle X; R\rangle$, $\langle Y; S\rangle$ be finite presentations of the group G. Then $\langle X; R\rangle$ can be transformed into $\langle Y; S\rangle$ by a finite number of Tietze transformations where in each instance the transformation involved "adds" or "subtracts" a single generator and a single relator.*

Now it is easy to prove – and the proof is left to the reader – that the following holds:

Lemma 5 *Suppose $G = \langle X; R \rangle$ is a recursive presentation of the group G and that $G = \langle X'; R' \rangle$ is a second presentation of G obtained from the first by a single Tietze transformation which adds or subtracts either a single generator or a single relator. Then $\langle X'; R' \rangle$ is again a recursive presentation of G. Moreover $\langle X; R \rangle$ has a solvable word problem if and only if $\langle X'; R' \rangle$ does.*

Hence we find, on combining the previous two lemmas, that we have proved the

Theorem 11 *Two finite presentations of a group G either both have a solvable word problem or neither one does.*

Exercise 6 *Prove that if G has a finite presentation in which the word problem is solvable in linear time, then every finite presentation of G has this property.*

Theorem 12 (B.H. Neumann) *Suppose the group G has a finite presentation. Then every presentation of G on finitely many generators has a presentation on these generators with only finitely many of the given relators i.e. if*

$$G \;=\; \langle\, y_1, \ldots, y_m \;;\; s_1, s_2, \ldots \,\rangle \qquad (m < \infty)$$

then

$$G \;=\; \langle\, y_1, \ldots, y_m \;;\; s_1, \ldots, s_n \,\rangle \qquad (n < \infty)\ .$$

The following simple lemma is the key to the proof of Neumann's theorem.

Lemma 6 *Suppose*

$$G \;=\; \langle\, x_1, \ldots, x_m \;;\; r_1, \ldots, r_l \,\rangle \qquad (m < \infty, l < \infty)$$

and also

$$G \;=\; \langle\, x_1, \ldots, x_m \;;\; s_1, s_2, \ldots \,\rangle\ .$$

Then

$$G \;=\; \langle\, x_1, \ldots, x_m \;;\; s_1, \ldots, s_n \,\rangle \qquad (n < \infty)\ .$$

Proof Let φ be the underlying presentation map, F the free group on $x_1, \ldots, x_m$. Then

$$gp_{_F}(\, r_1, \ldots, r_l \,) \;=\; \ker \varphi \;=\; gp_{_F}(\, s_1, s_2, \ldots \,) \; .$$

So each of r_i can be expressed as a product of conjugates of the s_j and their inverses. Since only finitely many s_j come into play

$$gp_{_F}(\, r_1, \ldots, r_l \,) \;=\; gp_{_F}(\, s_1, \ldots, s_n \,)$$

for some $n < \infty$.■

We are now in a position to prove Neumann's theorem.

Proof We use Tietze transformations to reduce the theorem to Lemma 6. Suppose

$$G \;=\; \langle\, x_1, \ldots, x_k \,;\, r_1, \ldots, r_l \,\rangle \qquad\qquad (k < \infty, l < \infty) \; .$$

Then if

$$G \;=\; \langle\, y_1, \ldots, y_m \,;\, s_1, s_2, \ldots \rangle$$

we add then y_i to the generators for G and then remove the x_i :

$$G \;=\; \langle\, x_1, \ldots, x_k, y_1, \ldots, y_m \,;\, \{r_i(\underset{\sim}{x}) \,\big|\, i = 1, \ldots, l\} \cup \{y_i w(\underset{\sim}{x})^{-1} \,\big|\, i = 1, \ldots, m\} \,\rangle$$

$$= \langle\, x_1, \ldots, x_k, y_1, \ldots, y_m \,;\, \{r_i(\underset{\sim}{x}) \,\big|\, i = 1, \ldots, l\} \cup \{y_i w(\underset{\sim}{x})^{-1} \,\big|\, i = 1, \ldots, m\} \cup \{x_i v(\underset{\sim}{y})^{-1} \,\big|\, i = 1, \ldots, k\} \,\rangle$$

$$= \langle\, x_1, \ldots, x_k, y_1, \ldots, y_m \,;\, \{r_i\,(\underset{\sim}{v(y)}) \,\big|\, i = 1, \ldots, l\} \cup \{y_i w\,(\underset{\sim}{v(y)})^{-1} \,\big|\, i = 1, \ldots, m\} \cup \{x_i v(\underset{\sim}{y})^{-1} \,\big|\, i = 1, \ldots, k\} \,\rangle$$

$$= \langle\, y_1, \ldots, y_m \,;\, \{r_i\,(\underset{\sim}{v(y)}) \,\big|\, i = 1, \ldots, l\} \cup \{y_i w\,(\underset{\sim}{v(y)})^{-1} \,\big|\, i = 1, \ldots, m\} \,\rangle \; .$$

Now apply Lemma 6. ■

Notation Let $N \trianglelefteq G$. Then

$$d_{_G}(N) \;=\; \min\Big\{\; \big|\, X \,\big| \;\Big|\; gp_{_G}(X) \;=\; N \;\Big\}\;.$$

Thus another way of putting Neumann's theorem is as follows: Suppose G is a finitely generated group and F is a finitely generated free group such that

$$G \;\cong\; F/K\;.$$

Then G is finitely presented if and only if $d_{_F}(K)$ is finite. Hence the following lemma holds:

Lemma 7 *Let G be a finitely presented group and suppose*

$$G \;\cong\; H/N$$

where H is finitely generated. Then

$$d_{_H}(N) \;<\; \infty\;.$$

So we find

Corollary 1 *Let H be a finitely generated group, N a subgroup of the center of H which is not finitely generated. Then H/N is not finitely presented.*

The point here is that $d_{_H}(N)$ is simply the minimum number $d(N)$ of generators of N.

Exercise 7 (P. Hall) *Let H be the following subgroup of* GL*(3,*$\mathbf{R}$*):*

$$H \;=\; gp\left(\quad \tau \;=\; \begin{pmatrix} 1 & 0 & 0 \\ 0 & \pi & 0 \\ 0 & 0 & 1 \end{pmatrix}\;,\quad \alpha \;=\; \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \right)\;.$$

Prove

(i) ζH *is free abelian of infinite rank;*

(ii) $H/\zeta H \cong C_\infty \wr C_\infty = W$.

Hence deduce that W *is not finitely presented.*

Next we record another useful fact:

Theorem 13 (W. Dyck) *Suppose*

$$G = \langle X; R \rangle$$

and

$$H = \langle X; R \cup S \rangle .$$

If γ *and* ϑ *are the respective presentation maps, then*

$$x\gamma \longmapsto x\vartheta \qquad (x \in X)$$

defines a homomorphisms of G *onto* H.

The proof is left to the reader.

Exercises 8 *(1) Let* $G = \langle x, y\,;\, x^2 = y^2 = 1 \rangle$.

(i) Prove G *maps onto* C_2.

(ii) Prove G *is infinite.*

(2) Let $\langle x_1, \ldots, x_m; r_1, \ldots, r_n \rangle$ *be a finite presentation of* G. *If* $m > n$ *prove* G *has an infinite cyclic factor group.*

(3) If $A = \langle X; R \rangle$, $B = \langle Y; S \rangle$ *we define the free product* G *of* A *and* B *which we denote by* $A * B$ *by means of presentations, as follows:*

$$G = \langle X \dot{\cup} Y; R \cup S \rangle .$$

Prove

(i) A *and* B *embed in* G.

(ii) *If* $a \in A,\ a \neq 1$, *and* $b \in B,\ b \neq 1$, *then* $[a, b] \neq 1$.
(Hint: Map G *onto* $A \wr B$.*)*

(iii) *If* A *and* B *are free, so is* $A * B$.

(iv) *If* $A * B$ *is free, so are* A *and* B.

6. The Reidemeister – Schreier method

Suppose that

$$G \;=\; \langle\, X; R \,\rangle \tag{9}$$

and that

$$H \;\leq\; G\;. \tag{10}$$

Now (9) comes with an underlying presentation map γ, say. So if F is the free group on X, then γ induces an isomorphism

$$\gamma_* \;:\; F/K \xrightarrow{\sim} G \tag{11}$$

where

$$K \;=\; gp_{_F}(R)\;. \tag{12}$$

Notice then that

$$\gamma_*{}^{-1}(H) \;=\; E/K \tag{13}$$

where E is a subgroup of F. So (13) really is a presentation of H in the making. Indeed let T be a Schreier transversal of E in F. Then E is free on Y, where

$$Y \;=\; \{\; \delta(t,x)\;\left(= tx(\overline{tx})^{-1}\right)\;\neq 1 \;\;\Big|\;\; t\in T,\; x\in X\;\}\;. \tag{14}$$

Now (13) tells us that

$$\gamma_*\,\big|\,E \;:\; E/K \xrightarrow{\sim} H\;. \tag{15}$$

So, by (14),

$$\gamma\,\big|\,Y \quad \text{is a presentation map of } H. \tag{16}$$

Indeed, by (15),

$$H \;=\; \langle\, Y; S \,\rangle$$

where we have to properly interpret S !

Now, by (12),

$$K \;=\; gp_{_F}(R) \;=\; gp_{_E}\big(\{\, trt^{-1} \;\big|\; t\in T,\, r\in R\,\}\big)\;.$$

But the elements trt^{-1} come to us as X-products, not Y-products. However

$$K \ \leq \ E$$

and so each trt^{-1} can be expressed as a reduced Y-product. Let us denote this expression for trt^{-1} by $\varrho(trt^{-1})$. Then what we mean by S above is precisely this set of all rewritten X-products:

$$S \ = \ \{ \ \varrho(trt^{-1}) \ \big| \ t \in T, \ r \in R \ \} \ .$$

To repeat then, we have a presentation of H:

$$H \ = \ \langle \ \{ \ \delta(t,x) \ \neq 1 \ \big| \ t \in T, \ x \in X \ \} \ ; \ \{ \ \varrho(trt^{-1}) \ \big| \ t \in T, \ r \in R \ \} \ \rangle \ . \tag{17}$$

Examples 7 *Let* $G = \langle \, b, u \, ; \, ubu^{-1} = b^2 \, \rangle$.
G maps onto C_∞ by W. Dyck:

$$C_\infty \ = \ \langle \, b, u \, ; \, ubu^{-1} = b^2, \, b = 1 \, \rangle \ .$$

Let H be the kernel of this map. Then G/H is infinite cyclic with generator uH. So $G/H = \dot{\cup}_{n \in \mathbf{Z}} \, u^n H$. Thus if we go back to the ambient free groups E and F in the discussion of the Reidemeister-Schreier method we find:

(i) $T \ = \ \{ \, u^n \ \big| \ n \in \mathbf{Z} \, \}$
(ii) $Y \ = \ \{ \, u^n b u^{-n} \ (= b_n) \ \big| \ n \in \mathbf{Z} \, \}$
(iii) $S \ = \ \{ \, \varrho\big(u^n (ubu^{-1}b^{-2}) u^{-n}\big) \ \big| \ n \in \mathbf{Z} \, \} \ = \ \{ \, b_{n+1} {b_n}^{-2} \ \big| \ n \in \mathbf{Z} \, \}$.

Thus

$$H \ = \ \langle \, \ldots, b_{-1}, b_0, b_1, \ldots \ ; \ \ldots, b_0 = {b_{-1}}^2, b_1 = {b_0}^2, b_2 = {b_1}^2, \ldots \, \rangle \ .$$

By Tietze transformations of type 1 we have

$$H \ = \ \langle \, \ldots, b_{-1}, b_0 \ ; \ \ldots \ b_{-1} = {b_{-2}}^2, b_0 = {b_{-1}}^2 \, \rangle \ .$$

Now let's map H into $\mathbf{Q}^+$ as follows:

$$b_n \longmapsto 2^n \qquad\qquad n = 0, -1, -2, \ldots \ .$$

By W. Dyck's theorem this defines a homomorphism μ of H into $\mathbf{Q}^+$. Indeed the image of H is simply

$$D \;=\; \left\{ \frac{l}{2^n} \;\middle|\; l, n \in \mathbf{Z} \right\} .$$

It is easy to see that μ is an isomorphism because D is a union of infinite cyclic groups! Notice

$$D \;=\; \mathbf{Z}\left[\frac{1}{2}\right]^{+} .$$

Exercises 9 *(1) Let*

$$G \;=\; \langle\, a, b, c, d \,;\, aba^{-1}b^{-1}cdc^{-1}d^{-1} = 1 \,\rangle \ .$$

Find a presentation for

$$H_n \;=\; gp_{_G}(\, a^n, b, c, d \,) \qquad\qquad n = 1, 2, \ldots$$

(Hint: H_n has a presentation of the form

$$H_n \;=\; \langle\, a_1, b_1, \ldots, a_g, b_g \;;\; \prod_{i=1}^{g} a_i b_i {a_i}^{-1} {b_i}^{-1} = 1 \,\rangle \ .$$

Find g.)

(2) Let G be as in (1) and let

$$H \;=\; gp_{_G}(\, b, c, d \,) \ .$$

Prove H is free.

(3) Let

$$G \;=\; \langle\, a, b \,;\, a^3 = b^3 = (ab)^3 = 1 \,\rangle \ .$$

Prove G' is free abelian of rank two and hence that G is infinite.

7. Generalized free products

Suppose

$$\big(G_i \;=\; \langle\, X_i \,;\, R_i \,\rangle \big)_{i\in I}$$

is an indexed family of groups G_i equipped with presentations, and suppose H is another group equipped with monomorphisms

$$\varphi_i \;:\; H \longrightarrow G_i \qquad\qquad (i \in I)\ .$$

Then we term the group G defined by the presentation

$$G \;=\; \langle\, \dot\cup_{i\in I} X_i \,;\, \cup_{i\in I} R_i \,\cup\, \{\, h\varphi_i h^{-1}\varphi_j \;\big|\; h\in H, i,j\in I \,\}\,\rangle \tag{18}$$

the generalized free product of the G_i amalgamating H. We write

$$G \;=\; \mathop{\ast}_{\substack{H\\ i\in I}} G_i\ .$$

If $H = 1$, then G is termed the free product of the G_i and we write

$$G \;=\; \mathop{\ast}_{i\in I} G_i\ .$$

If $I = \{1,\dots,n\}$ is finite we sometimes denote G by

$$G \;=\; G_1 \underset{H}{*} G_2 \underset{H}{*} \dots \underset{H}{*} G_n$$

or if $H = 1$ by

$$G \;=\; G_1 * G_2 * \dots * G_n$$

When $n = 2$, this reduces to

$$G \;=\; G_1 \underset{H}{*} G_2$$

or again, as we have already noted before, when $H = 1$

$$G \;=\; G_1 * G_2\ .$$

We assume that in (18) each $h\varphi_i$ is expressed as an X_i-product so that (18) actually looks like a presentation. According to W. Dyck's theorem there are canonical homomorphisms μ_i

of each G_i to G. It turns out that the μ_i are monomorphisms and that if we identify G_i with $G_i\mu_i$, then $h\varphi_i = h\varphi_j$ for all $h \in H,\ i,j \in I$. So we can identify H with any one of its images $H\varphi_i$ (which we have already identified with $H\varphi_i\mu_i$) and it then turns out that

$$G_i \cap G_j \ = \ H \qquad (i,j \in I,\ i \neq j)\ ,$$

i.e. the G_i intersect in precisely H. Notice that

$$G \ = \ gp\big(\bigcup_{i\in I} G_i\big)\ .$$

It follows that every element $g \in G$ can be expressed in the form

$$g \ = \ y_1 \dots y_n h \qquad (n \geq 0) \tag{19}$$

$$\begin{aligned} &where \quad (i) \quad y_j \in G_{i_j} - H \quad and \quad i_j \neq i_{j+1} \quad for \quad j = 1,\dots,n-1\ ; \\ &\qquad\quad (ii) \quad h \in H\ . \end{aligned}$$

In particular then if $g \notin H$, then (19) can be rewritten as

$$g \ = \ z_1 \dots z_n \qquad (n \geq 1) \tag{20}$$

$$where \quad (iii) \quad z_j \in G_{i_j} - H \quad and \quad i_j \neq i_{j+1} \quad for \quad j = 1,\dots,n-1\ .$$

Theorem 14 (O. Schreier) *Suppose* $G = \mathop{\ast}_{\substack{H\\ i\in I}} G_i$ *. Then*

(a) *the* G_i *embed into* G*.*

(b) $G_i \cap G_j = H \qquad (i,j \in I,\ i \neq j)$ *.*

(c) *every product of the form (20) satisfying (iii) is* $\neq 1$*.*

Moreover if conversely G *is a group with a subgroup* H *and an indexed family of subgroups* $(G_i)_{i\in I}$ *such that* $G = gp\big(\bigcup_{i\in I} G_i\big)$ *and also* $G_i \cap G_j = H$ $(i,j \in I,\ i \neq j)$ *then*

$$G = \mathop{\ast}_{\substack{H\\ i\in I}} G_i$$

if and only if every product of the form (20) satisfying (iii) is $\neq 1$ *in* G*. (For a proof see e.g. A.G. Kurosh, Vol.2).*

Corollary 1 *Let* $G = \ast\!\!\underset{i\in I}{}_{H} G_i$ *and suppose* $F_i \leq G_i \quad (i \in I), \quad K \leq H$ *and that*

$$F_i \cap F_j \;=\; K \qquad\qquad (i, j \in I,\ i \neq j)\ .$$

Then

$$F \;=\; gp\Big(\bigcup_{i\in I} F_i\Big) \;=\; \ast\!\!\underset{i\in I}{}_{K} F_i\ .$$

In particular for $K = 1$

$$F \;=\; \ast\!\!\underset{i\in I}{} F_i\ .$$

Furthermore if each F_i *is infinite cyclic,* F *is free.*

This corollary explains the existence of many free subgroups in generalized free products.

The importance of generalized free products lies in the fact that they not only occur frequently in a variety of different contexts, but that they can be used to construct important classes of groups. (See the book by Kurosh cited at the end of Chapter1 for some additional references.)

CHAPTER IV Recursively presented groups, word problems and some applications of the Reidemeister–Schreier method

1. Recursively presented groups

The following lemma is due to G. Higman.

Lemma 1 *A finitely generated subgroup of a finitely generated recursively presented group is recursively presented.*

Proof Let $G = \langle X; R \rangle$ be a recursive presentation of the group G with X finite. So

$$G \cong F/K$$

where F is free on X and $K = gp_F(R)$. As we already noted before, K is a recursively enumerable subset of F. Now let H be a finitely generated subgroup of G. Then

$$H \cong EK/K$$

where E is a finitely generated subgroup of F. So

$$H \cong E/(E \cap K) .$$

Now E is a recursively enumerable subset of F and hence so too is $E \cap K$. Indeed if we list E we find that we can produce a sublisting of the elements of $E \cap K$ i.e. $E \cap K$ is a recursively enumerable subset of E. ■

Corollary 1 *A finitely generated subgroup of a finitely presented group is recursively presented.*

Lemma 2 *Let G be a finitely generated subgroup of $\mathrm{GL}(n,K)$, where K is any commutative field and n is any positive integer. Then G has a recursive presentation with a solvable word problem. In particular, G is recursively presented.*

Proof Since G is finitely generated, it can be viewed as a group of matrices over a finitely generated subfield L of K. Now L can be embedded in the algebraic closure U of

$$P(x_1, x_2, \ldots)$$

the field of fractions of the polynomial ring in infinitely many variables $x_1, x_2, \ldots$ over the prime field P of K. Now according to a theorem of Rabin U is a *computable field* i.e. we can effectively compute in U.

Since G is a finitely generated subgroup of $\mathrm{GL}(n, U)$ we simply list all elements of G and check to see which of them are 1. This yields then a recursive presentation of G with a solvable word problem. ■

In fact we have proved the

Lemma 3 *Let U be a countable computable field. Then $\mathrm{GL}(n,U)$ is recursively presented.*

Of course $\mathrm{GL}(n, U)$ is nothing but $\mathrm{Aut}\, V$, where V is an n-dimensional vector space over U. A similar result, due to Baumslag, Cannonito and Miller, holds also for automorphism groups of finitely presented groups.

Theorem 1 *Let G be a finitely presented group. Then $\mathrm{Aut}\, G$ is recursively presented. If G has a solvable word problem then $\mathrm{Aut}\, G$ has a presentation with a solvable word problem.*

Proof Suppose

$$G \;=\; \langle\, x_1, \ldots, x_m \,;\, r_1(x_1, \ldots, x_m) = 1, \ldots, r_n(x_1, \ldots, x_m) = 1 \,\rangle \qquad (m, n \leq \infty)\ .$$

We first list all automorphisms of G. To do so, notice that if $\alpha \in \mathrm{Aut}\, G$,

$$x_i\alpha \;=\; w_i(x_1, \ldots, x_m) \qquad \text{for suitable words } w_i \qquad (i = 1, \ldots, m) \tag{1}$$

Now α is a homomorphism. So

$$r_j\big(\, w_1(x_1, \ldots, x_m), \ldots, w_m(x_1, \ldots, x_m)\,\big) \;=\; 1 \qquad (j = 1, \ldots, n)\ . \tag{2}$$

Suppose $\beta \;:\; G \longrightarrow G$ is the inverse of α. Then

$$x_i\beta \;=\; v_i(x_1, \ldots, x_m) \qquad \text{for suitable words } v_i \qquad (i = 1, \ldots, m)\ . \tag{3}$$

Again

$$r_j\big(\, v_1(x_1, \ldots, x_m), \ldots, v_m(x_1, \ldots, x_m)\,\big) \;=\; 1 \qquad (j = 1, \ldots, n)\ . \tag{4}$$

Moreover since α and β are inverses:

$$\left\{\begin{array}{ll} v_j\big(\, w_1(x_1, \ldots, x_m), \ldots, w_m(x_1, \ldots, x_m)\,\big) \;=\; x_i & (i = 1, \ldots, m)\ , \\ w_j\big(\, v_1(x_1, \ldots, x_m), \ldots, v_m(x_1, \ldots, x_m)\,\big) \;=\; x_i & (i = 1, \ldots, m)\ . \end{array}\right\} \tag{5}$$

Conversely if $(w_1, \ldots, w_m)$ and $(v_1, \ldots, v_m)$ are pairs of m-tuples of words in $x_1, \ldots, x_m$ satisfying (2), (4), (5) then the maps given in (1) and (3) are automorphisms of G which are inverses. In any event we can recursively enumerate $\mathrm{Aut}\, G$. We take this entire set, viz. $\mathrm{Aut}\, G$, as a set of generators of $\mathrm{Aut}\, G$. Now since G is finitely presented the set of such pairs is recursively enumerable and if G has a solvable word problem this set of pairs is even recursive.

Words in the above set of generators of $\mathrm{Aut}\, G$ act on the generators $x_1, \ldots, x_m$ of G and a given word is 1 if and only if it acts on each x_i like the identity. In more detail, let w be such a word. Thus

$$x_i w \;=\; u_i(x_1, \ldots, x_m) \qquad\qquad (i = 1, \ldots, m)\ .$$

So $w = 1$ in $\mathrm{Aut}\, G$ if and only if

$$u_i(x_1, \ldots, x_m) \;=\; x_i \qquad in \;\; G \qquad\qquad (i = 1, \ldots, m)\;. \tag{6}$$

But the set of such equations is recursively enumerable. So $\mathrm{Aut}\, G$ is recursively presented. If in addition G has a solvable word problem we actually can decide whether or not (6) holds. So $\mathrm{Aut}\, G$ has a solvable word problem. ■

2. Some word problems

Here we consider two classes of groups which have solvable word problem. The first of them goes back to work of McKinsey:

Theorem 2 *A finitely presented residually finite group has a solvable word problem.*

Proof Suppose G is a finitely presented residually finite group given by the finite presentation

$$G \;=\; \langle\, x_1, \ldots, x_m \,;\, r_1(x_1, \ldots, x_m) = 1, \ldots, r_n(x_1, \ldots, x_m) = 1 \,\rangle\;.$$

We can enumerate the homomorphisms of G to all finite quotients of G. Indeed for each symmetric group S_l , $l = 1, 2, \ldots$, take every m-tuple $(\overline{x}_1, \ldots, \overline{x}_m) \in (S_l)^m$ and check whether it satisfies

$$r_j(\overline{x}_1, \ldots, \overline{x}_m) \;=\; 1 \qquad\qquad (j = 1, \ldots, n)\;,$$

in which case we have a homomorphism

$$\varphi \;:\; G \longrightarrow gp(\overline{x}_1, \ldots, \overline{x}_m) \;\leq\; S_l\;, \qquad x_i \longmapsto \overline{x}_i \qquad\qquad (i = 1, \ldots, m)\;.$$

At the same time we can enumerate all consequences of the given defining relators of G.

So if w is any reduced $\{x_1, \ldots, x_m\}$-product either we will find $w = 1$ or else, since G is residually finite, that $w\varphi \neq 1$ for some homomorphism φ of the kind mentioned above. This solves the word problem for G. ■

This algorithm is very complicated.

Next a theorem that goes back to Kuznetsov:

Theorem 3 *A finitely generated recursively presented simple group G has a solvable word problem. (Note that by definition $G \neq 1$.)*

Proof Suppose G is given by the recursive presentation

$$G \;=\; \langle\, x_1, \ldots, x_m \,;\, r_1, r_2, \ldots \,\rangle \qquad\qquad (m < \infty)\ .$$

To determine whether a reduced $\{x_1, \ldots, x_m\}$-product w is 1 in G enumerate the consequences of $r_1, r_2, \ldots$. At the same time enumerate the consequences of $w, r_1, r_2, \ldots$. If w appears in the first list, $w = 1$ in G. If all of $x_1, \ldots, x_m$ appear in the second list, then $w \neq 1$ in G. This algorithm solves the word problem for G. ■

Note This algorithm solves the word problem for a class of groups we know very little about. In fact this algorithm, like the one above, is quite complex.

3. Groups with free subgroups

In 1972 J.Tits (*Free subgroups in linear groups*, **J. Algebra 20** (1972), 250-270) proved the following

Theorem 4 *A finitely presented group of matrices over a commutative field is either virtually solvable or contains a free subgroup of rank two.*

This has led to what has now become known as the *Tits alternative* in accordance with the following

Definition 1 *A group satisfies the Tits alternative if either it is virtually solvable or contains a free subgroup of rank two.*

Unlike matrix groups, finitely generated groups need not satisfy the Tits alternative – the Grigorchuk-Gupta-Sidki group, for instance, is such an example. Finitely presented examples are harder to find, but they do exist: Groups of piecewise linear homeomorphisms of the real line by Matthew G. Brin & Craig C. Squier, **Invent. math. 79** (1985), 485-498 and C.H. Houghton (unpublished). Here is one of the Brin, Squier examples: Let G be generated by the following two permutations $\tau,\ \sigma$ of the real line:

$$\tau \ : \ x \longmapsto x+1$$

$$\sigma \ : \ x \longmapsto \begin{cases} x & (-\infty < x \leq 0) \\ 2x & (0 \leq x \leq 1) \\ x+1 & (1 \leq x < \infty) \end{cases}$$

Then G turns out to be a finitely generated group which does not satisfy the Tits alternative.

The question therefore arises as to which finitely presented groups satisfy the Tits alternative. I want to dicuss next some answers to this question which fall out of the Reidemeister-Schreier method.

First let me recall that if

$$G \ = \ \langle\, X ; R \,\rangle \ , \ H \ = \ \langle\, Y ; S \,\rangle$$

and if C is a group equipped with two monomorphisms

$$\vartheta \ : \ C \longrightarrow G \quad , \quad \varphi \ : \ C \longrightarrow H$$

then we term

$$K \ = \ \langle\, X \dot{\cup} Y \,;\, R \cup S \cup \{ c\vartheta c^{-1}\varphi \,\big|\, c \in C \,\} \,\rangle$$

the generalized free product of G and H amalgamating C and denote it by writing

$$K \ = \ G \underset{C}{*} H \ .$$

The canonical homomorphisms of G and H into K are monomorphisms and we identify G and H with their images in K then

$$G \cap H \ = \ C \ .$$

The point for us here is that we have the following

Lemma 4 *If $C \neq G$ and $C \neq H$ and C has index at least three in one or other of G and H, then $G \underset{C}{*} H$ contains a free subgroup of rank two.*

The proof of this lemma is left as an exercise.

Next I want to introduce the notion of an HNN extension, taking some its properties for granted, for the moment (see Chapter VI for more details).

Definition 2 *Let*

$$B \;=\; \langle\, X; R \,\rangle$$

be a group given by a presentation and suppose U and V are subgroups of B equipped with an isomorphism

$$\tau \;:\; U \xrightarrow{\sim} V \;.$$

Then we term

$$E \;=\; \langle\, X, t\,;\, R \cup \{ tut^{-1} = u\tau \,\big|\, u \in U \} \,\rangle \qquad (where\ t \notin X)$$

an HNN extension of B with stable letter t, associated subgroups U and V and associating isomorphism τ. It turns out that

(i) B embeds in E.

(ii) t is of infinite order in E and U and V are conjugate in E via t.

We term B the base group of E and sometimes write

$$E \;=\; \langle\, B, t\,;\, tUt^{-1} = V \,\rangle \;.$$

E will be termed a degenerate HNN extension if either U=B or V=B.

Definition 3 *A group G is termed indicable if there exists a homomorphism of G onto the infinite cyclic group.*

Notice then that a finitely generated group G is indicable if and only if G_{ab} is infinite.

We need one additional piece of notation. To this end suppose X is a subset of some group and that

$$g \;=\; {x_1}^{\varepsilon_1} \ldots {x_n}^{\varepsilon_n} \qquad\qquad (x_i \in X,\ \varepsilon_i = \pm 1)$$

is an X-product. Then we define for $x \in X$

$$\exp_x g \;=\; \sum_{x_i = x} \varepsilon_i$$

and term $\exp_x g$ the exponent sum of x in the X-product g.

Here is a version of the Tits alternative due to Bieri & Strebel, 1978:

Theorem 5 (Bieri & Strebel) *Suppose that G is a finitely presented indicable group. Then either G is a degenerate HNN extension with a finitely generated base or else G contains a free subgroup of rank two.*

Proof Let μ be a homomorphism of G onto the infinite cyclic group. So if $H = \ker \mu,\ G/H$ is infinite cyclic on say Ht $(t \in G)$. Now G is finitely generated. So we can find a set of generators for G of the form

$$t, a_1, \ldots, a_m \qquad\qquad (a_i \in H) \qquad\qquad\qquad (m < \infty)\ .$$

And since G is finitely presented, by Neumann's theorem we can present G finitely using these generators:

$$G \;=\; \langle\, t, a_1, \ldots, a_m \,;\, r_1, \ldots, r_n \,\rangle \qquad\qquad\qquad (n < \infty)\ .$$

It follows that

$$H \;=\; gp_G(a_1, \ldots, a_m)$$

and that

$$\exp_t r_k \;=\; 0 \qquad\qquad (k = 1, \ldots, n)\ .$$

Thus $G = \dot{\bigcup}_{i \in \mathbf{Z}} Ht^i$ and so we can use $\{t^i \,\big|\, i \in \mathbf{Z}\}$ as a transversal for H in G and apply the Reidemeister-Schreier process to obtain a presentation for H. To this end put

$$t^i a_j t^{-i} \;=\; a_{j,i} \qquad\qquad (j = 1, \ldots, m,\ i \in \mathbf{Z})$$

and put the rewrite

$$\varrho(t^i r_k t^{-i}) \;=\; r_{k,i} \qquad\qquad (k = 1, \ldots, n,\ i \in \mathbf{Z}) \ .$$

We need to look more closely at the rewrites $\varrho(r_k)$ of the r_k. Since we lose nothing on replacing each r_k by a conjugate of itself by a power of t we may assume then that for $k = 1, \ldots, n$,

$$\varrho(r_k) \;=\; r_{k,0} \;=\; r_k\big(\, a_{1,0}, \ldots, a_{1,\lambda}, \ldots, a_{m,0}, \ldots, a_{m,\lambda}\,\big) \ .$$

Here we use λ to denote a positive integer, chosen once and for all. Our notation is functional and is designed to reflect the fact that each r_k is some product of the generators listed without carrying with it the further implication that all of the generators listed actually appear! It follows, in any case, that

$$r_{k,i} \;=\; r_k(\, a_{1,i}, \ldots, a_{1,\lambda+i}, \ldots, a_{m,i}, \ldots, a_{m,\lambda+i}\,)$$

where $k = 1, \ldots, n,\ i \in \mathbf{Z}$. Hence if we put

$$Y \;=\; \{\ a_{j,i} \ \big|\ j = 1, \ldots, m,\ i \in \mathbf{Z}\ \}$$

and

$$R \;=\; \{\ r_{k,i} \ \big|\ k = 1, \ldots, n,\ i \in \mathbf{Z}\ \}$$

then

$$H \;=\; \langle\, Y; R \,\rangle \ .$$

Put

$$H^+ \;=\; gp(\ a_{j,i} \ \big|\ j = 1, \ldots, m,\ i \geq 0)$$

and

$$H^- \;=\; gp(\ a_{j,i} \ \big|\ j = 1, \ldots, m,\ i \leq \lambda) \ .$$

Finally we put

$$U \;=\; gp(\ a_{j,i} \ \big|\ j = 1, \ldots, m,\ 0 \leq i \leq \lambda) \ .$$

So U is a finitely generated subgroup of both H^+ and H^-. We now form the generalized free product $\widetilde{H}$ of H^+ and H^- amalgamating U:

$$\widetilde{H} \;=\; H^+ \underset{U}{*}\ H^- \ .$$

In order to avoid confusion we present $\widetilde{H}$ using different letters to those previously used. First we present H^+ and H^-:

$$H^+ \;=\; \langle\, \{\, x_{j,i} \;\big|\; j = 1, \ldots, m,\; i \geq 0 \,\}\,; \\ \{\, r_k(x_{1,i}, \ldots, x_{1,\lambda+i}, \ldots, x_{m,i}, \ldots, x_{m,\lambda+i}) \;\big|\; k = 1, \ldots, n,\; i \geq 0 \,\} \cup S^+ \,\rangle$$

where S^+ is a set of additional relations, if needed, to define H^+ in the manner indicated;

$$H^- \;=\; \langle\, \{\, y_{j,i} \;\big|\; j = 1, \ldots, m,\; i \leq \lambda \,\}\,; \\ \{\, r_k(y_{1,i}, \ldots, y_{1,\lambda+i}, \ldots, y_{m,i}, \ldots, y_{m,\lambda+i}) \;\big|\; k = 1, \ldots, n,\; i \leq 0 \,\} \cup S^- \,\rangle$$

where S^- is defined similarly to S^+. So, adopting the obvious notation

$$H^+ \;=\; \langle\, X; R^+ \cup S^+ \,\rangle\,, \qquad H^- \;=\; \langle\, Y; R^- \cup S^- \,\rangle\,.$$

Hence

$$\widetilde{H} \;=\; \langle\, X \cup Y; R^+ \cup S^+ \cup R^- \cup S^- \cup \{x_{j,i} = y_{j,i} \,|\, j = 1, \ldots, m,\, 0 \leq i \leq \lambda\} \,\rangle\,.$$

The point of all of this is that we claim

$$\widetilde{H} \;\cong\; H\,. \tag{7}$$

First we concoct a homomorphism $\widetilde{\sigma}$ of $\widetilde{H}$ onto H. The thing to notice is that all of the relations we have used to define $\widetilde{H}$ go over into relations in H. More precisely, suppose we define

$$x_{j,i}\widetilde{\sigma} \;=\; a_{j,i} \qquad (j = 1, \ldots, m,\; i \geq 0)$$
$$y_{j,i}\widetilde{\sigma} \;=\; a_{j,i} \qquad (j = 1, \ldots, m,\; i \leq \lambda)$$

then $\widetilde{\sigma}$ maps the kernel of the presentation map of $\widetilde{H}$ onto the identity of H. So $\widetilde{\sigma}$ induces a homomorphism of $\widetilde{H}$ onto H – this is essentially a variation of W. Dyck's theorem using different sets of generators. And conversely the map $\sigma : H \longrightarrow \widetilde{H}$ defined by

$$a_{j,i}\sigma \;=\; x_{j,i} \qquad (j = 1, \ldots, m,\; i \geq 0)$$
$$a_{j,i}\sigma \;=\; y_{j,i} \qquad (j = 1, \ldots, m,\; i < 0)$$

similarly defines a homomorphism of H into $\widetilde{H}$. These two homomorphisms are inverses of each other. So we have proved (7) and essentially the theorem itself.

To see why we notice that we now know that (cf (7))

$$H \;=\; H^+ \underset{U}{*} H^- \;.$$

There are a number of possibilities. First if $U \neq H^+,\ U \neq H^-$ and is of index at least three in one of $H^+,\ H^-$ then H (and therefore G) contains a free subgroup of rank two. If U is of index two in both H^+ and H^- then they are both finitely generated. Hence H is also and G is a degenerate HNN extension in which H is not only the base group but the pair of associated subgroups as well. Finally if $U = H^+$ or $U = H^-$ then again G is a degenerate HNN extension with base group U (check). We note, for later use, also that if either H^+ or H^- is finitely generated, then again G is an HNN extension with a finitely generated base. ■

The technique involved in this proof can be exploited to yield the next theorem, which is due to Baumslag and Shalen.

Theorem 6 *Let G be a finitely presented indicable group which is not a degenerate HNN extension with a finitely generated base. Then G is virtually a non-trivial generalized free product of two finitely generated groups where the amalgamated subgroup is of infinite index in one factor and of arbitrarily large index in the other.*

Thus most finitely presented groups are virtually generalized free products of two finitely generated groups.

Proof The proof partly mimics and makes use of the proof of the Bieri-Strebel theorem just given. Thus we assume the notation used there. In particular then

$$G \;=\; \langle\, t, a_1, \ldots, a_m \,;\, r_1, \ldots, r_n \,\rangle$$

where as before

$$\exp_t r_k \;=\; 0 \qquad\qquad (k = 1, \ldots, n)$$

and G/H is infinite cyclic on Ht where

$$H \;=\; gp_G(\, a_1, \ldots, a_m \,) \;.$$

Again we assume that

$$\varrho(r_k) \;=\; r_k(\, a_{1,0}, \ldots, a_{1,\lambda}, \ldots, a_{m,0}, \ldots, a_{m,\lambda}\,)$$

for some fixed integer $\lambda > 0$. Now we put

$$H_l \;=\; gp_{_G}(\, t^l, a_1, \ldots, a_m\,)$$

where we assume only that l is a very large positive integer. Now, by assumption, G is not a degenerate HNN extension with a finitely generated base. Hence if we put

$$C \;=\; gp\big(\, a_{j,0}, \ldots, a_{j,\lambda-1}\,;\, a_{j,l-\lambda}, \ldots, a_{j,l-1}\ (j = 1, \ldots, m)\big)$$

then it follows that C is of infinite index in H^+ since H^+ is not finitely generated. Thus if we put

$$H_l{}^+ \;=\; gp\big(\, a_{j,0}, \ldots, a_{j,l-1}\ (j = 1, \ldots, m)\big)$$

and make sure l is large enough then the index of C in $H_l{}^+$ can be made as large as we please.

Our next objective is to compute a presentation for $H_l{}^+$. We use $1, t, \ldots, t^{l-1}$ as a Schreier transversal for H_l in G. Then the generators of H_l are simply

$$\{t^l\} \,\cup\, \{\, a_{j,i} \;\big|\; j = 1, \ldots, m,\ 0 \le i \le l-1\,\}\ .$$

The relators of H_l have to be examined a little more closely. They take the form

$$\begin{aligned}
r_{k,0} \quad &= \; r_k(\, a_{1,0}, \ldots, a_{1,\lambda}, \ldots, a_{m,0}, \ldots, a_{m,\lambda}\,) \\
&\vdots \\
r_{k,l-\lambda-1} \quad &= \; r_k(\, a_{1,l-\lambda-1}, \ldots, a_{1,l-1}, \ldots, a_{m,l-\lambda-1}, \ldots, a_{m,l-1}\,) \\
r_{k,l-\lambda} \quad &= \; r_k(\, a_{1,l-\lambda}, \ldots, a_{1,l-1}, t^l a_{1,0} t^{-l}, \ldots, a_{m,l-\lambda}, \ldots, a_{m,l-1}, t^l a_{m,0} t^{-l}\,) \\
&\vdots \\
r_{k,l-1} \quad &= \; r_k(\, a_{1,l-1}, t^l a_{1,0} t^{-l}, \ldots, t^l a_{1,\lambda-1} t^{-l}, \ldots, a_{m,l-1}, t^l a_{m,0} t^{-l}, \ldots, t^l a_{m,\lambda-1} t^{-l}\,)\ .
\end{aligned}$$

Put

$$H_l{}^- \;=\; gp\big(\, t^l\,;\, a_{j,0}, \ldots, a_{j,\lambda-1}\,;\, a_{j,l-\lambda}, \ldots, a_{j,l-1}\ (j = 1, \ldots, m)\big)\ .$$

So $H_l{}^-$ is a $(1 + 2m\lambda)$-generator group. Notice that, exactly as before in the proof of the Bieri-Strebel theorem

$$H_l \;=\; H_l{}^+ \underset{C}{*} H_l{}^-$$

where C is the $2m\lambda$-generator group given above. Now t is of infinite order modulo H. Hence C is of infinite index in $H_l{}^-$. This proves the theorem. ■

We can use the same kind of argument as we have just described together with an idea of B. Baumslag & S.J. Pride to prove the next

Theorem 7 *Let G be a finitely presented group and suppose that*

$$W \;=\; C_h \wr C_\infty \qquad\qquad (h \geq 2)\ ,$$

the wreath product of a cyclic group of order h (possibly infinite) and an infinite cyclic group, is a quotient of G. Then G contains a subgroup H of finite index which maps onto the free product

$$K \;=\; C_\infty * C_h\ .$$

Notice that if we consider the normal closure J in K of the infinite cyclic factor, then by the Reidemeister-Schreier method, J turns out to be a free group of rank h. Since J is of index $h \geq 2$ in K it follows that we have proved the

Corollary 2 *If G satifies the hypothesis of the theorem, then it contains a subgroup of finite index which maps onto the free group of rank two.*

I shall say more about this in a little while, after giving the proof of the theorem.

Proof To begin with notice that W can be generated by two elements τ and α which have the following properties:

(i) α is of order h and τ is of infinite order;

(ii) if we put

$$\alpha_i \;=\; \tau^i \alpha \tau^{-i} \qquad\qquad (i \in \mathbf{Z})\ ,$$

then $A = gp_{_W}(\alpha)$ is the direct product of the cyclic groups $A_i = gp(\alpha_i)$ of order h:

$$A \;=\; \prod_{i \in \mathbf{Z}} A_i \ .$$

We are now in a position to prove the theorem. By hypothesis there exists a surjective homomorphism

$$\mu \;:\; G \longrightarrow W \ .$$

Let t and a_1 be pre-images in G of the elements τ and α in W. We can, using Neumann's theorem, supplement t and a_1 with a finite set $a_2, \ldots, a_m \quad (a_2, \ldots, a_m \in \ker \mu)$ and present G finitely on this set of generators:

$$G \;=\; \langle\, t, a_1, \ldots, a_m \;;\; r_1, \ldots, r_n \,\rangle \ .$$

Notice that

$$\exp_t r_k \;=\; 0 \qquad\qquad (k = 1, \ldots, n) \ .$$

As in the proof of the previous theorem and using the same notation as that employed in that proof, decompose $H_l = gp_{_G}(t^l, a_1, \ldots, a_m)$:

$$H_l \;=\; {H_l}^- \underset{C}{*} {H_l}^+ \ .$$

Now if we map G onto W via μ, then H_l maps onto $gp_{_W}(\tau^l, \alpha)$ and ${H_l}^+$ maps onto $B_l = gp(\alpha_0, \ldots, \alpha_{l-1}) = A_0 \times \ldots \times A_{l-1}$. Notice that $a_{1,i}$ maps onto $\alpha_i \quad (0 \leq i \leq l-1)$. Now assume l is large as compared to λ and add the relations

$$a_{i,j} \;=\; 1 \qquad\qquad (j = 1, \ldots, m,\ 0 \leq i \leq \lambda - 1,\ l - \lambda \leq i \leq l - 1) \tag{8}$$

to H_l. Now observe that the resultant quotient group, say $\overline{H}_l$, of H_l is defined by the relators

$$r_{k,i} \qquad\qquad (k = 1, \ldots, n,\ 0 \leq i \leq l - 1) \ .$$

The only occurences of t^l in each of these relators arise as

$$t^l a_{j,i} t^{-l} \qquad (j = 1, \ldots, m,\ 0 \leq i \leq \lambda - 1,\ l - \lambda \leq i \leq l - 1) \ .$$

This means that the addition of the relations (8) to H_l gives rise to a presentation for $\overline{H}_l$ of the form (cf. the proof of the previous theorem):

$$\begin{aligned}\overline{H}_l \;=\; \langle\, & t^l, a_{j,i} \quad (j=1,\dots,m,\ 0\le i\le l-1)\,; \\ & a_{j,i}=1 \quad (j=1,\dots,m,\ 0\le i\le \lambda-1,\ l-\lambda\le i\le l-1) \\ & r_k(a_{1,0},\dots,a_{1,\lambda},\dots,a_{m,0},\dots,a_{m,\lambda})=1 \\ & \vdots \\ & r_k(a_{1,l-\lambda-1},\dots,a_{1,l-1},\dots,a_{m,l-\lambda-1},\dots,a_{m,l-1})=1 \\ & r_k(a_{1,l-\lambda},\dots,a_{1,l-1},1,\dots,a_{m,l-\lambda},\dots,a_{m,l-1},1)=1 \\ & \vdots \\ & r_k(a_{1,l-1},1,\dots,1,\dots,a_{m,l-1},1,\dots,1)=1\ \rangle\ .\end{aligned}$$

Thus t^l does not occur in any of the relators. So

$$\overline{H}_l \;=\; \langle t^l\rangle * gp\big(a_{j,i} \quad (j=1,\dots,m,\ \ 0\le i\le l-1)\big)\ .$$

Here $\langle t^l\rangle$ is the image of ${H_l}^-$ and $gp\big(a_{j,i} \quad (j=1,\dots,m,\ 0\le i\le l-1)\big)$ is the image of ${H_l}^+$. We denote these groups by

$$\overline{H}_l{}^- \qquad and \qquad \overline{H}_l{}^+\ .$$

Thus $\overline{H}_l{}^-$ is infinite cyclic and $\overline{H}_l{}^+$ maps via μ again onto

$$C_l \;=\; B_l/gp\big(\,a_{j,i}\mu \quad (j=1,\dots,m,\ 0\le i\le\lambda-1,\ l-\lambda\le i\le l-1)\big)\ .$$

Now since l is large relative to λ and B_l is the direct product of l cyclic groups of order h, factoring out a 2λ-generator subgroup of B_l cannot affect B_l too much. Indeed it follows from the basis theorem for finitely generated abelian groups that, denoting the images of the α_i in C_l again by α_i,

$$C_l \;=\; gp(\alpha_f)\ \times\ E_l$$

where α_f is of order $h \quad (0\le f\le l-1)$. Putting this another way, there is a homomorphism of $\overline{H}_l{}^+$ onto a cyclic group of order h which maps $\alpha_{1,f}$ onto the generator of this cyclic group. So $\overline{H}_l$ itself maps onto the free product of an infinite cyclic group and a cyclic group of order h. This completes the proof of the theorem. ■

This brings me to the last result I want to deduce that I have already alluded to before, due to B. Baumslag & S.J. Pride (**J. London Math. Soc. (2) 17** (1978), 425-426). It is the first in a series of three papers – the second by the same authors (**Math. Z. 167** (1979), 279-281) and the last by R. Stohr (**Math. Z. 182** (1983), 45-47). Their results all follow, incidentally, from the theorem I have just proved.

The first of the B. Baumslag & S.J. Pride results is:

Theorem 8 *Let G be a group given by $m+1$ generators and n relations $(m, n < \infty)$. If*

$$(m+1) - n \geq 2$$

then G contains a subgroup of finite index which maps onto a free group of rank two.

The proof of this theorem follows from what we have already proved and the following

Lemma 5 *Let G satify the above hypothesis. Then given any prime p there exists a surjective homomorphism*

$$\mu : G \longrightarrow W = C_p \wr C_\infty .$$

Proof We proceed as follows: First we present G in the form

$$G = \langle\, t, a_1, \ldots, a_m \,;\, r_1, \ldots, r_n \,\rangle$$

where as usual

$$\exp_t r_k = 0 \qquad\qquad (k = 1, \ldots, n) .$$

Let p be any given prime. Let

$$N = gp_{_G}(\, a_1, \ldots, a_m \,)$$

and let

$$M = N/(N'N^p) .$$

Here N' is the derived group of N and N^p denotes the subgroup of N generated by the p-th powers of elements of N. Now M is an abelian group of exponent p. We write M additively

and turn it into a left module over the group ring $\mathbf{F}_p[t,t^{-1}]$ of the infinite cyclic group on t over the field $\mathbf{F}_p$ of p elements:

$$\left(\sum_i c_i t^i\right)\left(aN'N^p\right) \;=\; \sum_i \left(t^i a t^{-i}\right)^{c_i} N'N^p \;.$$

View M as a module on m generators (the images of the a_j in M) subject to n module relations (the images of the r_k written in $\mathbf{F}_p[t,t^{-1}]$-module form in terms of the generators of M I have just described). Now the ring $R = \mathbf{F}_p[t,t^{-1}]$ is a principal ideal domain. So every finitely generated R-module is a direct sum of cyclic modules. Since $n < m$ one of these cyclic modules is free. Therefore M has a quotient which is free on one generator. The submodules of M corresepond exactly to the normal subgroups of G contained in N and containing $N'N^p$. So this translates into the existence of a normal subgroup L of G contained in N such that L contains $N'N^p$ with N/L, viewed as an R-module, free on one generator, say aL. This simply means that modulo L the conjugates of a under the powers of t are independent elements of the vector space N/L ! In other words

$$G/L \;=\; gp(\,aL, tL\,) \;=\; gp(aL) \wr gp(tL)\;!$$

This completes the proof of the lemma. ■

Incidentally the same kinds of arguments yield also:

Theorem 9 *Let G be a group defined by a single relation and at least three generators. Then G is virtually an amalgamated product of the form*

$$H^+ \underset{U}{*} H^-$$

where U is of infinite index in H^+ and also in H^- and all of H^+, H^- and U are finitely presented.

(The books by Magnus, Karrass and Solitar and by Lyndon and Schupp cited in Chapter 1 are a good general reference for this chapter. See also the survey by Strebel, cited in Chapter I.)

CHAPTER V Affine algebraic sets and the representation theory of finitely generated groups

1. Background

In the next few lectures I want to develop only the very beginnings of what one might call geometric representation theory and then give two applications of this theory to combinatorial group theory.

Before I begin, however, I want to give you an idea as to how this geometric representation theory comes into play.

Let us suppose then that G is a given finitely generated group. We consider the set $\mathrm{R}(G)$ of all representations of G in $\mathrm{SL}(2, \mathbf{C})$. This set $\mathrm{R}(G)$ carries with it the stucture of an affine algebraic set (see the book by Robin Hartshorne: $Algebraic\ Geometry$, **Graduate Texts in Mathematics 52**, Springer-Verlag, New York (1977), for a more detailed discussion of algebraic geometry). If G has sufficiently many representations, then we can find so-called curves in $\mathrm{R}(G)$. In 1983, Culler & Shalen in an important paper in the Annals of Mathematics, showed

how such a curve of representations can be used to produce a canonical representation

$$\gamma \ : \ G \longrightarrow \mathrm{SL}(2,F)$$

where F is a finite algebraic extension of the field $\mathbf{C}(x)$ of rational functions over $\mathbf{C}$ in a single variable. Such a field F has a so-called discrete valuation. There is a theory due to Bass, Serre and Tits which shows that such an $\mathrm{SL}(2,F)$ acts as a group of automorphisms of a tree that is associated to $\mathrm{SL}(2,F)$. Hence G also acts on a tree. The Bass-Serre part of this theory (see Chapter VII) yields a so-called graph product decomposition of G. This means that G can be reconstituted from a select set of its subgroups by using HNN extensions and amalgamated products, allowing for a detailed study of G. In fact Culler & Shalen applied this approach in the case where G is the fundamental group of a 3-manifold and obtained important new results about such groups as well as new proofs of earlier theorems of Thurston. This technique promises to be an important tool also in combinatorial group theory. I want only to consider here the geometric representation theory. In the last part of the course I will describe the Bass-Serre theory and and also the way in which $\mathrm{SL}(2,F)$ acts on a tree.

2. Some basic algebraic geometry

Let k be a fixed algebraically closed field, n a positive integer and $\mathbf{A}^n_k = \mathbf{A}^n$ the set of all n-tuples of elements of k:

$$\mathbf{A}^n \ = \ \{ \ (a_1, \ldots, a_n) \ \mid \ a_j \in k \ \} \ .$$

We sometimes denote $\mathbf{A}^n$ simply by $\mathbf{A}$, which is usually referred to as affine n-space and its elements are thenreferred to as points.

Let

$$A \ = \ k[T_1, \ldots, T_n]$$

be the polynomial algebra over k in the variables $T_1, \ldots, T_n$. An algebra B which is isomorphic to a quotient of A is termed finitely generated – note here $n < \infty$. It follows that the algebra B can be generated by n elements, say $t_1, \ldots, t_n$. We express this fact by writing

$$B \ = \ k[t_1, \ldots, t_n] \ .$$

Notice that these generators $t_1, \ldots, t_n$ are not necessarily algebraically independent. All algebras dicussed here will be associative k-algebras with a multiplicative identity 1. We shall need two theorems of Hilbert. The first of these is

Theorem 1 (The Hilbert Basis Theorem) *Let B be a finitely generated commutative k-algebra. Then every ideal of B is finitely generated (as an ideal).*

It follows that such k-algebras satisfy the ascending chain condition for ideals.

Now let S be a subset of A. We define

$$Z(S) \;=\; \{\, (a_1, \ldots, a_n) \in \mathbf{A} \;\big|\; f(a_1, \ldots, a_n) = 0 \quad \text{for all} \quad f \in S \,\} \;.$$

$Z(S)$ is termed the *zero set* or *set of zeroes of S*.

Definition 1 *A subset X of $\mathbf{A}$ is termed an affine algebraic set if $X = Z(S)$ for some subset $S \subseteq A$.*

We say X is the *affine algebraic set defined by S*. Notice that if $\mathcal{A}$ is the ideal of A generated by S, then by the Hilbert Basis Theorem $\mathcal{A}$ is finitely generated, say by $f_1, \ldots, f_r$. Thus it follows that

$$Z(S) \;=\; Z(\mathcal{A}) \;=\; Z(f_1, \ldots, f_r) \;.$$

Thus

Lemma 1 *Every affine algebraic set is defined by a finite set.*

Next we record some simple facts about affine algebraic sets.

Lemma 2

(i) *$\emptyset$ and $\mathbf{A}$ are affine algebraic sets.*

(ii) *Then union of two affine algebraic sets is again an affine algebraic set. More precisely if $\mathcal{A}_1$ and $\mathcal{A}_2$ are ideals of A then*

$$Z(\mathcal{A}_1) \cup Z(\mathcal{A}_2) \;=\; Z(\mathcal{A}_1 \cap \mathcal{A}_2) \;=\; Z(\mathcal{A}_1 \mathcal{A}_2) \;.$$

(iii) *The intersection of an arbitrary number of affine algebraic sets is again an affine algebraic set. More precisely if* $\{\mathcal{A}_i \,\big|\, i \in I\}$ *is an indexed family of ideals of* A *then*

$$\bigcap_{i\in I} Z(\mathcal{A}_i) \;=\; Z\Big(\bigcup_{i\in I} \mathcal{A}_i\Big) \;.$$

(iv) *If* $S_1 \subseteq S_2 \subseteq A$ *then* $Z(S_1) \supseteq Z(S_2)$ *.*

The proof of Lemma 2 is left to the listener. Only (ii) needs a little thought – if necessary one can use (iv) to help in the proof.

Lemma 2 can be used to put a topology on $\mathbf{A}$. We take the closed sets in this topology to be the affine algebraic sets. The resultant topology on $\mathbf{A}$ is termed the Zariski topology.

Exercises 1 *(1) Prove Lemma 2.*

(2) Observe that
(i) $S^{n-1} \;=\; \{\, (a_1, \ldots, a_n) \in \mathbf{A} \;\big|\; \sum_{j=1}^{n} a_j^2 = 1 \,\}$
(ii) $H \;=\; \{\, (x, y) \in \mathbf{A}^2 \;\big|\; xy = 1 \,\}$
are affine algebraic sets.

(3) Let M=M*(n,k) be the set of all* $n \times n$ *matrices over* k*. Identify* M *with* $\mathbf{A}^{n^2}$ *by simply writing down the rows of each matrix one after the other.*

(4) Prove that $\mathrm{SL}(n,k) \subseteq \mathbf{A}^{n^2}$ *and* $\mathrm{GL}(n,k) \subseteq \mathbf{A}^{n^2+1}$ *are affine algebraic sets.*

(5) Prove that if $k^\bullet$ *is the multiplicative group of* k*, then*

$$\underbrace{k^\bullet \times \ldots \times k^\bullet}_{l} \;\subseteq\; \mathbf{A}^{2l}$$

is an affine algebraic set.

(6) Prove that if X *and* Y *are affine algebraic sets, then so is* $X \times Y$*.*

Notice that the Zariski topology is not a particularly nice one. For instance the closed sets in $\mathbf{A}^1$ are the finite sets and $\mathbf{A}^1$. So the topology is not even Hausdorff.

Definition 2 *Let G be a group. Then a representation of G in $\mathrm{SL}(n,k)$ is a homomorphism*

$$\varrho \ : \ G \longrightarrow \mathrm{SL}(n,k) \ .$$

Notice that if G is finitely generated, say by $g_1, \ldots, g_m$, then ϱ is completely determined by its effect on $g_1, \ldots, g_m$. So if $\mathrm{R}(G,n)$ is the set of all representations of G in $\mathrm{SL}(n,k)$ we can parametrize $\mathbf{R}(G,n)$ by points in $\mathbf{A}^{n^2 m}$. More precisely we associate with ϱ the point $\big(\varrho(g_1), \ldots, \varrho(g_m)\big)$:

$$\varrho \longmapsto \big(\varrho(g_1), \ldots, \varrho(g_m)\big) \ .$$

Lemma 3 *Let G be a group with $g_1, \ldots, g_m$ a finite set of generators of G. Then $\mathrm{R}(G,n)$ carries with it the stucture of an affine algebraic set.*

Proof Let

$$G \ = \ \langle \, g_1, \ldots, g_m \, ; \, r_1 = 1, \ldots \, \rangle$$

be a presentation for G on the given generators. Let $\mathrm{M}_1, \ldots, \mathrm{M}_m$ be $n \times n$ matrices over k. Then a point

$$u \ = \ \big(\mathrm{M}_1, \ldots, \mathrm{M}_m\big) \ \in \ \mathbf{A}^{n^2 m} \tag{1}$$

corresponds to a representation ϱ of G in $\mathrm{SL}(n,k)$, i.e. the mapping defined by

$$\varrho \ : \ g_l \longmapsto \mathrm{M}_l \qquad (l = 1, \ldots, m)$$

is a representation of G in $\mathrm{SL}(n,k)$, if and only if

$$\det \mathrm{M}_l \ = \ 1 \qquad (l = 1, \ldots, m) \quad \text{and} \quad r_h\big(\mathrm{M}_1, \ldots, \mathrm{M}_m\big) \ = \ 1 \qquad (h = 1, \ldots) \ .$$

Notice $\det \mathrm{M}, \ \mathrm{M}$ any $n \times n$ matrix, is a polynomial in the coefficients of M. Now if $\det \mathrm{M} = 1$ then M^{-1} can be expressed in the usual way in terms of the coefficients of M. In fact each of

the coefficients of M^{-1} is a polynomial in the coefficients of M. Thus if we assume $\det \mathrm{M}_l = 1$ for $l = 1, \ldots, m$, then $r_h\big(\mathrm{M}_1, \ldots, \mathrm{M}_m\big)$ is a matrix whose (i,j)-entry is a polynomial $r_h(i,j)$ in the coefficients of the M_l. Thus $r_h\big(\mathrm{M}_1, \ldots, \mathrm{M}_m\big) = 1$ if and only if the polynomials $r_h(i,j) = \delta_{ij}$ (the Kronecker delta). So u (given by (1)) corresponds to a representation of G in $\mathrm{SL}(n,k)$ if and only if

$$\det \mathrm{M}_l \;=\; 1 \qquad (l = 1, \ldots, m) \quad \textit{and} \quad r_h(i,j) \;=\; \delta_{ij} \qquad (h = 1, \ldots)\ . \tag{2}$$

Thus $\mathrm{R}(G,n)$ is the set of zeroes of a (possibly infinite) set of polynomials which arise from a set of defining relations for G written in terms of the given generators.

Exercises 2 *(1) Suppose G is free on $g_1, \ldots, g_m$. Then*

$$\mathrm{R}(G,n) \;=\; \mathrm{SL}(n,k)^m\ .$$

(2) Let G be a group given by a single defining relation. Then $\mathrm{R}(G,n)$ can be defined by $m + n^2$ equations, where m is the number of generators of G.

(3) If M is any square matrix, the trace $\mathrm{tr}\,\mathrm{M}$ of M is the sum of its diagonal entries. Verify that

(i) $\mathrm{tr}\,\big(\mathrm{M}_1\mathrm{M}_2\big) \;=\; \mathrm{tr}\,\big(\mathrm{M}_2\mathrm{M}_1\big)$
where M_1 is an $n \times m$ matrix and M_2 an $m \times n$ matrix.

(ii) $\mathrm{tr}\,\big(\mathrm{T}^{-1}\mathrm{M}\mathrm{T}\big) \;=\; \mathrm{tr}\,\mathrm{M}$ *for $\mathrm{M} \in \mathrm{M}(n,k)$ and $\mathrm{T} \in \mathrm{GL}(n,k)$.*

(iii) if $\det(t1 - \mathrm{M}) \;=\; f(t)$ *is the characteristic polynomial of $\mathrm{M} \in \mathrm{M}(n,k)$ and if*

$$f(t) \;=\; t^n + c_{n-1}t^{n-1} + \ldots + c_0$$

then $c_0 = (-1)^n \det \mathrm{M}$ and $c_{n-1} = -\mathrm{tr}\,\mathrm{M}$.

(iv) $\mathrm{M} \in \mathrm{SL}(2, \mathbf{C})$ *is of order $e > 2$ if and only if*

$$\mathrm{tr}\,\mathrm{M} \;=\; \omega + \omega^{-1}$$

where ω is a primitive e-th root of 1.

[Hint: Use Jordan normal forms and note that if

$$\lambda \;=\; \omega + \omega^{-1}$$

then the roots of the equation $x^2 - \lambda x + 1 = 0$ *are* ω *and* ω^{-1}.]

(4) Suppose that G is a group defined by a single defining relator which is a proper power greater than 2:

$$G \;=\; \langle\, g_1, \ldots, g_m \,;\, \big(r(g_1, \ldots, g_m)\big)^e = 1 \,\rangle \;.$$

Prove that $\mathrm{R}(G,2)$ *can be defined by* $m+1$ *equations.*

On the face of it the affine algebraic set $\mathrm{R}(G,n)$ appears to depend on the generators chosen for G. This is not the case however. In order to prove the invariance of $\mathrm{R}(G,n)$ we need the notion of a *morphism* from one affine algebraic set to another. We will prepare ourselves for the introduction of this and other notions in the next two sections.

3. More basic algebraic geometry

Let Y be a subset of $\mathbf{A}$. Then we define

$$I(Y) \;=\; \{\; f \in A \;\big|\; f(a_1, \ldots, a_n) = 0 \quad \text{for all} \quad (a_1, \ldots, a_n) \in Y \;\} \;.$$

$I(Y)$ is clearly an ideal of A and is termed the *ideal of* Y. In order to better understand this function I we will need the other theorem of Hilbert that I mentioned earlier.

Theorem 2 (Hilbert's Nullstellensatz) *Let* k *be an algebraically closed field,* $\mathcal{A}$ *an ideal of* $A = k[T_1, \ldots, T_n]$. *If* $f \in A$ *vanishes on* $Z(\mathcal{A})$ [*i.e.* $f \in I\big(Z(\mathcal{A})\big)$] *then a positive power of* f *lies in* $\mathcal{A}$.

Notice that if $\mathcal{A} \neq A$, then $Z(\mathcal{A}) \neq \emptyset$ is one of the consequences of Theorem 2. For if $Z(\mathcal{A}) = \emptyset$ then the polynomial 1 vanishes on $Z(\mathcal{A})$ and hence $1 \in \mathcal{A}$ i.e. $\mathcal{A} = A$.

Recall that if B is a commutative k-algebra then an element $b \in B$ is termed *nilpotent* if $b^r = 0$ for some positive integer r. If $\mathcal{B}$ is an ideal of B then the *radical of* $\mathcal{B}$, denoted $\sqrt{\mathcal{B}}$,

is defined by

$$\sqrt{\mathcal{B}} \;=\; \{\, b \in B \;\big|\; b^r \in \mathcal{B} \quad \text{for some} \quad r > 0 \,\}\ .$$

The radical $\sqrt{\mathcal{B}}$ of $\mathcal{B}$ is again an ideal of B and $B/\sqrt{\mathcal{B}}$ has no non-zero nilpotent elements. We need two definitions for later use.

Definition 3 *An ideal $\mathcal{B}$ of a commutative k-algebra B is termed a radical ideal if $\sqrt{\mathcal{B}} = \mathcal{B}$.*

Definition 4 *A finitely generated commutative k-algebra is termed an affine algebra if it contains no non-zero nilpotent elements.*

The function I has properties corresponding to those of Z.

Lemma 4 *Let $Y,\ Y_1,\ Y_2$ be subsets of $\mathbf{A}$. Then the following hold:*
(i) *If $Y_1 \subseteq Y_2$, then $I(Y_1) \supseteq I(Y_2)$.*
(ii) *$I(Y_1 \cup Y_2) \;=\; I(Y_1) \cap I(Y_2)$.*
(iii) *If $\mathcal{A}$ is any ideal of A, then $I\,Z(\mathcal{A}) = \sqrt{\mathcal{A}}$.*
(iv) *$Z\,I(Y) = \overline{Y}$, the closure of Y in the Zariski topology on $\mathbf{A}$.*

The proof of Lemma 4 is straightforward and is left to the reader.

On allying part of Lemma 2 with Lemma 4 it follows that we have proved

Lemma 5 *Let $\mathbf{X}$ be the set of affine algebraic sets in $\mathbf{A}$ and $\mathbf{R}$ the set of radical ideals of A. Then, restricting I to $\mathbf{X}$ and Z to $\mathbf{R}$,*

$$Z\,I \;=\; 1_{\mathbf{X}}\ , \qquad I\,Z \;=\; 1_{\mathbf{R}}\ .$$

Since both Z and I are inclusion reversing it follows that the maximal ideals of A correspond via Z to the minimal affine algebraic sets in $\mathbf{A}$.

Notice that the minimal affine algebraic sets in $\mathbf{A}$ are simply points. Thus if $\mathcal{M}$ is a maximal ideal in A then

$$\begin{aligned}\mathcal{M} &= I\,Z(\mathcal{M}) = I\,\{(a_1,\ldots,a_n)\}\\ &= \text{the ideal generated by } T_1-a_1,\ldots,T_n-a_n\ .\end{aligned}$$

Thus the maximal ideals of A all have this form. Notice also that if $x=(a_1,\ldots,a_n)$ then

$$\widehat{x}\ :\ A\longrightarrow k \text{ defined by } f\longmapsto f(x)$$

is a homomorphism of A onto k with kernel $\mathcal{M}$. So $\mathbf{A}$ is, in a sense, completely determined by the maximal ideals of A. There is, as we shall see shortly, a similar result for affine algebraic sets in general.

4. Useful notions from topology

We recall first the

Definition 5 *A topological space X is termed irreducible if*
(i) $X\neq\emptyset$ *and*
(ii) *X is not the union of two proper closed subsets.*

Then the following holds:

Lemma 6 *(i) A topological space $X\neq\emptyset$ is irreducible if and only if every non-empty open set U is dense in X i.e. $\overline{U}=X$.*
(ii) If Y is a subspace of a topological space then Y is irreducible if and only if $\overline{Y}$ is irreducible.
(iii) The closure of a point in a topological space X is irreducible.
(iv) If $\varphi\ :\ X\longrightarrow Y$ is a continuous map and X is irreducible, then so is $\varphi(X)$.

The proof of Lemma 6 is straightforward and is left to the reader.

Now let X be a topological space. Then by Zorn's Lemma every irreducible subspace is contained in a maximal one. By Lemma 6 (ii) a maximal irreducible subspace is closed.

Definition 6 *Let X be a topological space. Then the maximal irreducible subspaces of X are termed the irreducible components of X.*

So by Lemma 6 (iii) every topological space is the union of its irreducible components.

Definition 7 *A topological space X is termed Noetherian if its open sets satisfy the ascending chain condition i.e. every properly ascending chain of open sets is finite.*

Note that X is Noetherian if and only if it satisfies the descending chain condition for closed sets.

Lemma 7 *Every affine algebraic set X is Noetherian.*

Proof Every properly descending chain of closed subspaces of X gives rise, via I, to a properly ascending chain of ideals in A. By the Hilbert Basis Theorem this latter chain is always finite. ■

Lemma 8 *Let X be a non-empty Noetherian topological space. Then X has only finitely many irreducible components, say $X_1, \ldots, X_m$ and*

$$X \;=\; X_1 \cup \ldots \cup X_m \;.$$

Proof We already know that every topological space is the union of its irreducible components (which we know also are closed). In order to prove $m < \infty$, let $\mathcal{S}$ be the set of all closed subspaces of X which are the union of finitely many irreducible subspaces of X. If $X \notin \mathcal{S}$ let Y be a minimal closed non-empty subspace of X which is not in $\mathcal{S}$. Then Y is certainly not irreducible. Hence $Y = Y_1 \cup Y_2$ where $Y_1,\ Y_2$ are proper closed subspaces of Y. By the

minimality of Y, each of $Y_1,\ Y_2$ can be written as a union of finitely many irreducible subspaces of X. Hence so can Y. This contradiction proves the lemma. ■

It is time to identify the irreducible affine algebraic sets.

Lemma 9 *An affine algebraic set X in $\mathbf{A}$ is irreducible if and only if $I(X)$ is a prime ideal in A.*

Proof Suppose $I(X)$ is prime and that $X = X_1 \cup X_2$ is a decomposition of $X \neq \emptyset$ into two proper closed subsets. Then

$$I(X) \ = \ I(X_1) \cap I(X_2) \ .$$

But $I(X)$ is a prime ideal in A. So either $I(X_1) \subseteq I(X)$ or $I(X_2) \subseteq I(X)$. So, applying Z, we find either $X_1 = X$ or $X_2 = X$, a contradiction.

On the other hand suppose X is irreducible. We want to prove $I(X)$ is prime. $I(X) \neq A$ since $X \neq \emptyset$. Suppose

$$f_1 f_2 \ \in \ I(X) \qquad\qquad (f_1, f_2 \in A) \ .$$

Then

$$X \ = \ Z\, I(X) \ \subseteq \ Z(f_1 f_2) \ = \ Z(f_1) \cup Z(f_2) \ .$$

Hence

$$X \ = \ \big(X \cap Z(f_1)\big) \ \cup \ \big(X \cap Z(f_2)\big)$$

is a decomposition of X into two closed sets. By irreducibility, either

$$X \ \subseteq \ Z(f_1) \qquad \text{or} \qquad X \ \subseteq \ Z(f_2)$$

and so either

$$f_1 \ \in \ I(X) \qquad \text{or} \qquad f_2 \ \in \ I(X) \ .$$

Hence $I(X)$ is prime as claimed. ■

Exercises 3 *(1) Suppose the topological space X has only finitely many irreducible components, say $X_1, \ldots, X_m$. Then*

$$X \ = \ X_1 \cup \ldots \cup X_m \ .$$

Prove that if

$$X \;=\; Y_1 \cup \ldots \cup Y_h \qquad (h < \infty)$$

where the Y_i are closed irreducible sets and $Y_i \not\subseteq Y_j$ if $i \neq j$, then $h = m$ and the Y's can be renumbered so that $Y_i = X_i$ for $i = 1, \ldots, m$.

(2) Prove that if $\mathcal{B}$ is any radical ideal in any finitely generated commutative k-algebra B then

(i) there are only finitely many minimal prime ideals containing $\mathcal{B}$;

(ii) $\mathcal{B}$ is the intersection of these minimal prime ideals.

[*Hint: Prove (2) first for $A = k[T_1, \ldots, T_n]$ and then use the third isomorphism theorem.*]

We come now to an important definition.

Definition 8 *Let X be a non-empty topological space. Term a series*

$$X_0 \subset X_1 \subset \ldots \subset X_m$$

of distinct irreducible closed subspaces of X an irreducible chain of length m. Then we define the dimension of X, denoted $\dim X$, by

$$\dim X \;=\; \textit{supremum of lengths of irreducible chains in } X \;.$$

We note some simple consequences of the definition.

Lemma 10 *Let X be a finite dimensional irreducible topological space. Then the following hold:*

(i) If $\dim X = 0$ then the closure of every 1-point set is X. So if X is T_1 i.e. every 1-point set is closed then X is a space with a single point.

(ii) If $\dim X > 0$ and Y is a closed proper subspace of X, then $\dim Y < \dim X$.

Proof We verify (ii), which is almost obvious. Indeed let

$$Y_0 \;\subset\; Y_1 \;\subset\; \ldots \;\subset\; Y_m$$

be an irreducible chain in Y of length m. Since Y is closed this is an irreducible chain in X. But then

$$Y_0 \subset Y_1 \subset \ldots \subset Y_m \subset X$$

is an irreducible chain in X of length $m+1$. So this proves $\dim Y < \infty$ and also that $\dim Y < \dim X$. ■

Exercise 4 *If X is a topological space of finite dimension m, prove that every subspace Y of X is of finite dimension at most m.*

5. Morphisms

Let X be an affine algebraic set contained in $\mathbf{A}$.

Definition 9 *A map*

$$\mu : X \longrightarrow k$$

is termed a polynomial function if there exists a polynomial $f = f(T_1, \ldots, T_n) \in A$ such that

$$\mu(a_1, \ldots, a_n) = f(a_1, \ldots, a_n) \qquad \big((a_1, \ldots, a_n) \in X \big) .$$

Thus the polynomial functions on X are simply the polynomials in A restricted to X

The set of such functions is denoted by $k[X]$ and becomes a k-algebra using scalar multiplication and coordinate-wise addition and multiplication of functions.

Definition 10 *$k[X]$ is termed the coordinate algebra of X.*

Notice that for each $f \in A$ we have the polynomial function f_X, which is f restricted to X.

The map

$$f \longmapsto f_X \qquad\qquad (f \in A)$$

is a homomorphism of A onto $k[X]$. The kernel of this homomorphism is simply $I(X)$. So

Lemma 11 $k[X] \;\cong\; A/I(X)$.

Thus the coordinate algebras of affine algebraic sets are affine algebras since $I(X)$ is a radical ideal of A. Let t_i be the function $T_i \,\big|\, X$. Then

$$k[X] \;=\; k[t_1, \ldots, t_n] \ .$$

The functions t_i are termed the i-th coordinate functions of $X \quad (i = 1, \ldots, n)$. So

$$t_i(a_1, \ldots, a_n) \;=\; a_i \qquad\qquad (i = 1, \ldots, n) \ .$$

Exercises 5 *Verify the following:*
(1) $k[\mathbf{A}] \;=\; A$.
(2) If X is a 1-point set, $k[X] \;\cong\; k$.
(3) Let X and Y be affine algebraic sets. Prove (remember $X \times Y$ is affine algebraic) that

$$k[X \times Y] \;\cong\; k[X] \underset{k}{\otimes} k[Y] \ .$$

By Lemma 9 an affine algebraic set X is irreducible if and only if $I(X)$ is prime. We reformulate this remark as

Lemma 12 *An affine algebraic set X is irreducible if and only if $k[X]$ is a domain.*

Definition 11 *Let $X \subseteq \mathbf{A}^m,\ Y \subseteq \mathbf{A}^n$ be affine algebraic sets. Then the map*

$$\varphi \;:\; X \longrightarrow Y$$

is termed a morphism from X to Y if there exist $f_1, \ldots, f_n \in k[X]$ such that

$$\varphi(a_1, \ldots, a_m) \;=\; (\, f_1(a_1, \ldots, a_m), \ldots, f_n(a_1, \ldots, a_m)\,)$$

for all $(a_1, \ldots, a_m) \in X$.

Definition 12 *We term two affine algebraic sets* X *and* Y *isomorphic if there exist morphisms*

$$\varphi \ : \ X \longrightarrow Y \ , \qquad \gamma \ : \ Y \longrightarrow X$$

such that

$$\gamma\varphi \ = \ 1_X \ , \qquad \varphi\gamma \ = \ 1_Y \ .$$

Note A morphism $\varphi : X \longrightarrow Y$ from one affine algebraic set to another can be viewed as a (possibly singular) polynomial change of coordinates and $\varphi(X)$ then is "X re-expressed in terms of these new coordinates".

Lemma 13 *Let* $X \subseteq \mathbf{A}^m$, $Y \subseteq \mathbf{A}^n$ *be affine algebraic sets. A morphism* $\varphi : X \longrightarrow Y$ *is a continuous map in the Zariski topology.*

Proof Suppose $\varphi(a_1, \ldots, a_m) = (f_1(a_1, \ldots, a_m), \ldots, f_n(a_1, \ldots, a_m))$ with the usual notation. We have only to prove that if Z is a closed set in Y, then $\varphi^{-1}(Z)$ is closed in X. Now if $g_1, \ldots, g_h$ are polynomials which define Z then the polynomials

$$g_i(f_1, \ldots, f_n) \qquad (i = 1, \ldots, h)$$

define $\varphi^{-1}(Z)$. ■

Finally we come to the proof of the

Lemma 14 *Let* G *be a given finitely generated group,* d *a positive integer. Suppose*

$$G \ = \ gp(g_1, \ldots, g_m) \qquad (m < \infty)$$

and that also

$$G \ = \ gp(h_1, \ldots, h_n) \qquad (n < \infty) \ .$$

Then the respective affine algebraic sets $\mathrm{R}(G,d)$, $\widetilde{\mathrm{R}}(G,d)$ *associated with* G *are isomorphic.*

Proof Notice that

$$h_i \;=\; w_i(g_1,\ldots,g_m) \qquad\qquad (i=1,\ldots,n)$$

and that

$$g_j \;=\; v_j(h_1,\ldots,h_n) \qquad\qquad (j=1,\ldots,m)\ .$$

$\mathrm{R}(G,d)$ is parametrized in $\mathbf{A}^{d^2m}$ by the affine algebraic set

$$X \;=\; \{\,(\varrho(g_1),\ldots,\varrho(g_m)) \;\big|\; \varrho : G \longrightarrow \mathrm{SL}(d,k) \quad \text{a representation}\,\}$$

and $\widetilde{\mathrm{R}}(G,d)$ is parametrized in $\mathbf{A}^{d^2n}$ by the affine algebraic set

$$Y \;=\; \{\,(\varrho(h_1),\ldots,\varrho(h_n)) \;\big|\; \varrho : G \longrightarrow \mathrm{SL}(d,k) \quad \text{a representation}\,\}\ .$$

Then $w_1,\ldots,w_n$ define a morphism

$$\varphi \;:\; X \longrightarrow Y$$

by

$$\big(\varrho(g_1),\ldots,\varrho(g_m)\big) \longmapsto \big(w_1\big(\varrho(g_1),\ldots,\varrho(g_m)\big),\ldots,w_n\big(\varrho(g_1),\ldots,\varrho(g_m)\big)\big)$$
$$\big(\;=\; \big(\varrho(h_1),\ldots,\varrho(h_n)\big)\big)$$

and a similar remark holds for $v_1,\ldots,v_m$. These maps are clearly inverses and so, using the obvious notation,

$$X \;\cong\; Y$$

as claimed. ■

The affine algebraic sets together with the morphisms between them form a category, the category of affine algebraic sets which we denote for the moment by $\mathcal{C}$. Similarly the affine k-algebras together with the k-algebra homomorphisms between them form a second category, the category of affine k-algebras which we denote, again for the moment, by $\mathcal{D}$. The following theorem then holds.

Theorem 3 *The categories* $\mathcal{C}$ *and* $\mathcal{D}$ *are equivalent.*

Proof The equivalence between $\mathcal{C}$ and $\mathcal{D}$ is essentially defined by the following contravariant functor $\mathcal{F} : \mathcal{C} \longrightarrow \mathcal{D}$. First we define $\mathcal{F}$ on objects:

$$\mathcal{F}(\mathcal{X}) \;=\; k[X] \;.$$

Second on morphisms: if $\varphi : X \longrightarrow Y$, then

$$\mathcal{F}(\varphi) \;=\; \varphi^\star \;:\; k[Y] \longrightarrow k[X]$$

where

$$\varphi^\star(f) \;=\; f\varphi \;.$$

It is easy to check that $\varphi^\star$ is a k-homomorphism of k-algebras, that

$$\mathcal{F}(\psi\varphi) \;=\; \mathcal{F}(\varphi)\mathcal{F}(\psi) \;, \qquad \mathcal{F}(\infty) \;=\; \infty \;.$$

The corresponding functor $\mathcal{G}$ from $\mathcal{D}$ to $\mathcal{C}$ involves choosing for each affine k-algebra B a finite set $b_1, \ldots, b_m$ of generators:

$$B \;=\; k[b_1, \ldots, b_m] \;.$$

This choice is made separately for each affine k-algebra. Thus "incompatible" choices may well be made for isomorphic k-algebras, but once a choice is made, we stick to it.

Now let

$$\sigma \;:\; k[T_1, \ldots, T_m] \longrightarrow B$$

be the homomorphism defined by

$$\sigma \;:\; T_i \longrightarrow b_i \qquad\qquad (i = 1, \ldots, m) \;.$$

Let

$$\mathcal{B} \;=\; \ker \sigma \;.$$

Then we define our functor $\mathcal{G} : \mathcal{D} \longrightarrow \mathcal{C}$ on objects as follows:

$$\mathcal{G}(\mathcal{B}) \;=\; \mathcal{X}$$

where

$$X \;=\; Z(\mathcal{B}) \qquad (\subseteq \mathbf{A}^m) \;.$$

Notice that

$$k[X] \;\cong\; k[T_1,\ldots,T_m]/\mathcal{B} \;\cong\; B\ .$$

An element $b \in B$ can then be viewed as a polynomial function on X; indeed using the above isomorphisms b_i defines the i-th coordinate function on X.

Now we need to define $\mathcal{G}$ on k-algebra homomorphisms. Let then

$$\vartheta \;:\; B \longrightarrow C$$

be a homomorphism from one affine k-algebra into another. Both B and C come equipped with finite sets of generators:

$$B \;=\; k[b_1,\ldots,b_m]\ , \qquad C \;=\; k[c_1,\ldots,c_n]\ .$$

So

$$\vartheta(b_i) \;=\; w_i(c_1,\ldots,c_n) \qquad\qquad (i=1,\ldots,m)\ .$$

Notice

$$\mathcal{G}(B) \;=\; X \;\subseteq\; \mathbf{A}^m\ , \qquad \mathcal{G}(C) \;=\; Y \;\subseteq\; \mathbf{A}^n\ .$$

Define

$$\mathcal{G}(\vartheta) \;:\; Y \longrightarrow X$$

by

$$\mathcal{G}(\vartheta)(a_1,\ldots,a_n) \;=\; (\,w_1(a_1,\ldots,a_n),\ldots,w_m(a_1,\ldots,a_n)\,)\ .$$

The point we made before is appropriate here – each $w_i(c_1,\ldots,c_n)$ can be viewed as an element of $k[Y]$.

Then one checks that $\mathcal{G}$ is a contravariant functor from $\mathcal{D}$ to $\mathcal{C}$ and that

$$\mathcal{G}\mathcal{F} \;\simeq\; 1_{\mathcal{C}}\ , \qquad \mathcal{F}\mathcal{G} \;\simeq\; 1_{\mathcal{D}}\ .$$

■

There are a couple of consequences of the proof that I want to draw attention to.

Corollary 1 *Let X and Y be affine algebraic sets and $\varphi : X \longrightarrow Y$ a dominant morphism i.e. $\varphi(X)$ is dense in Y. Then*

$$\varphi^\star : k[Y] \longrightarrow k[X]$$

is a monomorphism.

Corollary 2 *Let $\vartheta : B \longrightarrow C$ be a monomorphism from the affine k-algebra B to the affine k-algebra C. Then*

$$\mathcal{G}(\vartheta) : \mathcal{G}(\mathcal{C}) \longrightarrow \mathcal{G}(\mathcal{B})$$

is a dominant morphism.

We shall make frequent use of Corollary 2, especially in the case where ϑ is actually an inclusion. Indeed we note here for later ease of exposition the following special case of Corollary 2, which follows immediately from what has already been noted.

Corollary 3 *Let $\vartheta : B \hookrightarrow C$ be an inclusion of affine k-algebras and suppose*

$$B = k[s_1, \ldots, s_m] , \qquad C = k[s_1, \ldots, s_m, t_1, \ldots, t_n] .$$

Suppose $\mathcal{G}(B) = X \subseteq \mathbf{A}^m$, $\mathcal{G}(C) = Y \subseteq \mathbf{A}^{m+n}$. If

$$X_0 = \{ (z_1, \ldots, z_m) \mid (z_1, \ldots, z_m, a_1, \ldots, a_n) \in Y \text{ for some choice of } a_1, \ldots, a_n \in k \}$$

then

$$X = \overline{X_0} .$$

We shall use Corollary 2 to manufacture the appropriate varieties that are our basic concern.

If $\varphi : X \longrightarrow Y$ is a dominant morphism of affine algebraic sets, we sometimes refer to Y as a *quotient affine algebraic set*. Our objective then is to construct such quotients, using Corollary 2.

Exercise 6 *Illustrate Corollary 3 by means of the inclusion*

$$\vartheta \ : \ k[T] \hookrightarrow k[T, T^{-1}] \ .$$

6. Dimension

We begin with a

Definition 13 *An affine algebraic set X is termed an affine variety if $I(X)$ is a prime ideal.*

Notice that X is an affine variety if and only if $k[X]$ is a domain.

Exercises 7 *(1) Let $f \in k[T_1, \ldots, T_n] = A$ the polynomial algebra in n variables. Then A is a unique factorization domain. Suppose $f \neq 0$. Then Z(f) is an affine variety if and only if f is prime.*

(2) Now let $n = d^2$ and label the T_l $(l = 1, \ldots, n)$ as T_{ij} $(i, j = 1, \ldots, d)$. Let

$$f \ = \ \det(T_{ij}) \ .$$

So f is a polynomial of degree d. Prove that $f - 1$ is prime and hence that SL*(d,k) is an affine variety.*

(3) Prove that if X and Y are affine varieties, so is $X \times Y$.

Now if $X \subseteq \mathbf{A}$ is an affine algebraic set then $\dim X$ is the length m of the longest chain

$$X_0 \ \subset \ X_1 \ \subset \ \ldots \ \subset \ X_m$$

of distinct irreducible closed subspaces of X. Applying the operator I yields then a chain

$$\mathcal{X}_0 \supset \mathcal{X}_1 \supset \ldots \supset \mathcal{X}_m \tag{3}$$

of prime ideals of $k[X]$. So $\dim X$ is what is termed the *Krull dimension* of the affine k-algebra $k[X]$.

We take for granted some facts about the Krull dimension.

Theorem 4 *Suppose that the affine k-algebra B is a domain. Then*
(i) $\dim B$ *is the transcendence degree over k of the field of fractions of B; hence* $\dim B < \infty$.
(ii) if $b_1, \ldots, b_q \in B$ *and if* $\mathcal{B}$ *is a minimal prime ideal of B containing* $b_1, \ldots, b_q$ *– so by assumption* $\mathcal{B} \neq B$ *– then*

$$\dim(B/\mathcal{B}) \geq \dim B - q \ .$$

Examples *(1)* $\dim \mathbf{A}^n = \dim k[\mathbf{A}^n] = \dim k[T_1, \ldots, T_n] = n$.

(2) $\dim \mathrm{SL}(d,k) = d^2 - 1$.
$k[\mathrm{SL}(d,k)] = k[T_{11}, \ldots, T_{dd}]/\big(\det(T_{ij}) - 1\big)$.

Since the ideal generated by the polynomial $\det(T_{ij}) - 1$ *is prime, Theorem 4, (ii) applies.*

(3) Let X and Y be affine varieties of dimension m and n respectively. Then

$$\dim(X \times Y) = m + n \qquad (= \dim X + \dim Y) \ .$$

Since X is an affine variety, k[X] is a domain. We denote by k(X) its field of fractions. Now $X \times Y$ is again an affine variety since

$$k[X \times Y] \cong k[X] \underset{k}{\otimes} k[Y]$$

is a domain. We claim that the field of fractions of $k[X] \underset{k}{\otimes} k[Y]$ *is simply* $k(X) \underset{k}{\otimes} k(Y)$. *Let* $u_1, \ldots, u_m$, $v_1, \ldots, v_n$ *be transcendece bases for k(X) over k, k(Y) over k, respectively. Then*

$$u_1 \otimes 1, \ldots, u_m \otimes 1, \quad 1 \otimes v_1, \ldots, 1 \otimes v_n$$

is a transcendence basis for $k(X) \underset{k}{\otimes} k(Y)$ *over* k.

(4) Let F *be the free group on* q *free generators. Then* R*(F,d) is an affine variety of dimension* $q(d^2-1)$. *We have already seen that*

$$\mathrm{R}(F,d) \;=\; \underbrace{\mathrm{SL}(d,k) \times \ldots \times \mathrm{SL}(d,k)}_{q} \;.$$

So

$$\dim \mathrm{R}(F,d) \;=\; q(d^2-1) \;.$$

Note in particular that

$$\dim \mathrm{R}(F,2) \;=\; 3q \;.$$

(5) Let G *be a group defined by* q *generators and* n *defining relations. Then*

$$\dim \mathrm{R}(G,d) \;\geq\; (q-n)(d^2-1) \;.$$

So if $q > n$, *then* $\dim \mathrm{R}(G,d) > 0$ *for* $d \geq 2$.

We compute $\dim \mathrm{R}(G,d)$ *by using (ii) of Theorem 4. Thus notice that*

$$\dim \mathrm{R}(G,d) \;=\; \dim k[\mathrm{R}(G,d)] \;,$$

and that if $f_1, \ldots, f_l$ *define* R*(G,d) then*

$$k[\mathrm{R}(G,d)] \;\cong\; k[\mathbf{A}^{qd^2}]/\sqrt{(f_1,\ldots,f_l)} \;.$$

Let $\mathcal{B}$ *be a minimal prime ideal containing* $\sqrt{(f_1,\ldots,f_l)}$. *Since* $\mathrm{R}(G,d)$ *always contains the trivial representation, i.e. the representation that maps every element of* G *to the identity matrix,* $\mathrm{R}(G,d) \neq \emptyset$. *So* $\sqrt{(f_1,\ldots,f_l)} \neq k[\mathbf{A}^{qd^2}]$. *Hence* $\mathcal{B}$ *exists and so we need only estimate* l *and then apply Theorem 4, (ii). Suppose then that* $\varrho \in \mathrm{R}(G,d)$ *and that in the parametrization of* R*(G,d)*

$$\varrho \longmapsto (\mathrm{M}_1, \ldots, \mathrm{M}_q) \;.$$

In order to ensure ϱ is a representation of G in SL*(d,k) we need first to make sure that*

$$\det \mathrm{M}_i \;=\; 1 \qquad\qquad (i = 1, \ldots, q)\ .$$

This requires q polynomial equations. In addition we need

$$r_j(\mathrm{M}_1, \ldots, \mathrm{M}_q) \;=\; 1 \qquad\qquad (j = 1, \ldots, n)$$

for each of the n defining relations for G. On the face of it this entails d^2 equations ensuring that the coefficients of the matrix $r_j(\mathrm{M}_1, \ldots, \mathrm{M}_q)$ are either 0 or 1. Since the determinant of $r_j(\mathrm{M}_1, \ldots, \mathrm{M}_q)$ is 1 we need only $d^2 - 1$ of these equations – specifying the off-diagonal entries are 0 together with all excepting one of the diagonal entries are 1 suffices. So

$$\dim \mathrm{R}(G, d) \;\geq\; qd^2 - q - n(d^2 - 1) \;=\; (q - n)(d^2 - 1)\ .$$

(6) Let

$$G \;=\; \langle\, g_1, \ldots, g_q \,;\, \big(r(g_1, \ldots, g_q)\big)^e \;=\; 1 \,\rangle$$

where r is a non-trivial cyclically reduced $\{g_1, \ldots, g_q\}$- product and $e > 2$.

Assume that there exists a representation of G in SL *(2,$\mathbf{C}$) such that the image of $r(g_1, \ldots, g_q)$ is of order e. Verify*

$$\dim \mathrm{R}(G, 2) \;\geq\; 3q - 1\ .$$

Suppose now

$$\varphi \;:\; X \longrightarrow Y$$

is a morphism of affine algebraic sets.

Definition 14 *The fibres of φ are the closed sets $\varphi^{-1}(y) \quad (y \in Y)$.*

The following theorem is a useful tool in computing dimensions.

Theorem 5 *Let $\varphi : X \longrightarrow Y$ be a morphism of affine varieties. If $y \in \varphi(X)$ then each of the irreducible components Z of $\varphi^{-1}(y)$ has dimension at least $\dim X - \dim Y$:*

$$\dim Z \ \geq \ \dim X \ - \dim Y \ .$$

We shall have occasion to use Theorem 5 later.

7. Representations of the free group of rank two in $SL(2, \mathbf{C})$

I want now to look at a particular example, in some detail, namely the affine algebraic set $\mathrm{R}(F, 2)$ of the free group F of rank 2 on a and b in $\mathrm{SL}(2, \mathbf{C})$.

We already know that

$$\mathrm{R}(F, 2) \ = \ \mathrm{SL}(2, \mathbf{C}) \times \mathrm{SL}(2, \mathbf{C}) \ .$$

So

$$\dim \mathrm{R}(F, 2) \ = \ 6 \ .$$

We consider now the map

$$\varphi \ : \ \mathrm{R}(F, 2) \longrightarrow \mathbf{C}^3 \quad \textit{given by} \quad \varrho \longmapsto \big(\mathrm{tr}\varrho(a), \mathrm{tr}\varrho(b), \mathrm{tr}\varrho(ab)\big) \ .$$

Thinking of φ as a map on the affine algebraic set representing $\mathrm{R}(F, 2)$ it is clear that φ is a polynomial map:

$$\begin{aligned} &\varphi(a_{11}, a_{12}, a_{21}, a_{22}, b_{11}, b_{12}, b_{21}, b_{22}) \\ &\qquad = \ (\, a_{11} + a_{22} \,,\, b_{11} + b_{22} \,,\, a_{11}b_{11} + a_{12}b_{21} + a_{21}b_{12} + a_{22}b_{22} \,) \end{aligned}$$

i.e. φ is a morphism from the affine algebraic set $\mathrm{R}(F, 2)$ to the affine algebraic set $\mathbf{C}^3$.

Lemma 15 *φ is onto.*

Proof Let $(z_1, z_2, z_3) \in \mathbf{C}^3$ and consider the quadratic equations

$$\lambda^2 - z_1\lambda + 1 \;=\; 0 \qquad \text{and} \qquad \mu^2 - z_2\mu + 1 \;=\; 0$$

for λ and μ. Notice that by Vieta

$$\lambda + \lambda^{-1} \;=\; z_1 \qquad \text{and} \qquad \mu + \mu^{-1} \;=\; z_2 \;.$$

Put

$$\mathrm{A} \;=\; \begin{pmatrix} \lambda & 0 \\ z & \lambda^{-1} \end{pmatrix} \qquad \text{and} \qquad \mathrm{B} \;=\; \begin{pmatrix} \mu & 1 \\ 0 & \mu^{-1} \end{pmatrix}$$

with z still to be determined. Then

$$\operatorname{tr}\mathrm{A} \;=\; z_1 \qquad \text{and} \qquad \operatorname{tr}\mathrm{B} \;=\; z_2 \;.$$

Notice also that $\det \mathrm{A} = 1 = \det \mathrm{B}$. Now compute

$$\mathrm{AB} \;=\; \begin{pmatrix} \lambda\mu & \lambda \\ z\mu & z + \lambda^{-1}\mu^{-1} \end{pmatrix}$$

So

$$\operatorname{tr}(\mathrm{AB}) \;=\; \lambda\mu + \lambda^{-1}\mu^{-1} + z \;.$$

Now put

$$z \;=\; z_3 - \lambda\mu - \lambda^{-1}\mu^{-1} \;.$$

Let $\varrho \in \mathrm{R}(F, 2)$ then be defined by

$$\varrho(a) \;=\; \mathrm{A} \qquad \text{and} \qquad \varrho(b) \;=\; \mathrm{B} \;.$$

Then

$$\varphi(\varrho) \;=\; (z_1, z_2, z_3) \;.$$

■

So this easily gives us

Theorem 6 *Let*

$$G \;=\; \langle\, a, b \,;\, a^l = b^m = (ab)^n = 1 \,\rangle \qquad\qquad (l, m, n > 1) \;.$$

Then a is of order l, b is of order m and ab is of order n.

In order to verify Theorem 6 we need the following

Lemma 16 *Let* M *be of order* 2 *in* $\mathrm{SL}(2,\mathbf{C})$. *Then* $\mathrm{M} = -1$.

Proof M is conjugate to a matrix

$$\mathrm{N} \;=\; \begin{pmatrix} x & 0 \\ y & x^{-1} \end{pmatrix} .$$

Thus

$$\mathrm{N}^2 \;=\; \begin{pmatrix} x^2 & 0 \\ y(x+x^{-1}) & x^{-2} \end{pmatrix} .$$

Hence $x = \pm 1$. If $x = 1$ we get a contradiction: $\mathrm{N} = 1$ or N is of infinite order. So $x = -1$. But $y(x+x^{-1}) = 0$. So $y = 0$. Thus M is conjugate to an element in the center of $\mathrm{SL}(2,\mathbf{C})$. So $\mathrm{M} = -1$. In fact

$$\zeta\big(\mathrm{SL}(2,\mathbf{C})\big) \;=\; \{\pm 1\} .$$ ■

Now to the proof of Theorem 6: Consider the group

$$H \;=\; \langle\, \alpha, \beta \,;\, \alpha^{2l} = \beta^{2m} = (\alpha\beta)^{2n} = 1 \,\rangle .$$

Then there exists a representation ϱ of H in $\mathrm{SL}(2,\mathbf{C})$ such that

$$\mathrm{tr}\,\varrho(\alpha) \;=\; \lambda + \lambda^{-1}, \qquad \mathrm{tr}\,\varrho(\beta) \;=\; \mu + \mu^{-1}, \qquad \mathrm{tr}\,\varrho(\alpha\beta) \;=\; \nu + \nu^{-1}$$

where

$$\lambda,\ \mu,\ \nu$$

are primitive $2l$-th, $2m$-th and $2n$-th roots of 1. So by one of the exercises, $\varrho(\alpha)$ is of order $2l$, $\varrho(\beta)$ is of order $2m$, $\varrho(\alpha\beta)$ is of order $2n$. So it follows from Lemma 16 that the images of $\varrho(\alpha)$, $\varrho(\beta)$ and $\varrho(\alpha\beta)$ are of orders l, m and n in $\mathrm{PSL}(2,\mathbf{C}) = \mathrm{SL}(2,\mathbf{C})/\{\pm 1\}$. This then provides us with a representation of G itself in $\mathrm{PSL}(2,\mathbf{C})$ in which the images of a, b and ab have the right order. So a, b and ab do too, in G. ■

I want to look again at Lemma 15. First let's observe that the coordinate algebra of $\mathbf{A} = \mathbf{C^3}$ is $\mathbf{C}[x, y, z]$, the polynomial algebra over $\mathbf{C}$ in three independent variables. Indeed we can take x, y, z to be the coordinate functions

$$x(z_1, z_2, z_3) \;=\; z_1, \qquad y(z_1, z_2, z_3) \;=\; z_2, \qquad z(z_1, z_2, z_3) \;=\; z_3 \;.$$

Now the map

$$\varphi \;:\; \mathrm{R}(F, 2) \longrightarrow \mathbf{A}$$

is onto. Hence the homomorphism

$$\varphi^\star \;:\; \mathbf{C}[\mathbf{A}] \longrightarrow \mathbf{C}[\mathrm{R}(F, 2)]$$

is a monomorphism. Observe that

$$\varphi^\star(x)(\varrho) \;=\; \mathrm{tr}\, \varrho(a) \qquad \text{etc.}$$

The next thing I would like to point out is that the image of $\mathbf{C}[\mathbf{A}]$ under $\varphi^\star$ contains also the polynomial functions f_w where

$$f_w(\varrho) \;=\; \mathrm{tr}\, \varrho(w) \qquad\qquad (w \in F) \;.$$

Notice $\varphi^\star(x) = f_a$ etc.

Lemma 17 *If $w \in F$ then $f_w \in \mathbf{C}[f_a, f_b, f_{ab}]$* $\qquad \big(= \varphi^\star \mathbf{C}[\mathbf{A}]\big).$

What this means is that f_w is a polynomial in $f_a,\ f_b$ and f_{ab} i.e. there exists a unique polynomial p_w in these variables such that

$$\mathrm{tr}\, \varrho(w) \;=\; p_w\big(\, \mathrm{tr}\, \varrho(a),\ \mathrm{tr}\, \varrho(b),\ \mathrm{tr}\, \varrho(ab)\, \big) \;.$$

In order to prove Lemma 17 we need the following "trace identities" for $\mathrm{SL}(2, \mathbf{C})$:

Lemma 18

(i) $\quad \mathrm{tr}\, 1 \;=\; 2 \;.$

(ii) $\quad \mathrm{tr}\,(\mathrm{AB}) \;=\; \mathrm{tr}\, \mathrm{A}\ \mathrm{tr}\, \mathrm{B} \;-\; \mathrm{tr}\,(\mathrm{AB}^{-1}).$

Proof By Cayley-Hamilton we have for $B \in \mathrm{SL}(2, \mathbf{C})$

$$\mathrm{B}^2 - (\mathrm{tr}\,\mathrm{B}) \cdot \mathrm{B} + 1 \;=\; 0 \;.$$

Hence

$$\mathrm{B} + \mathrm{B}^{-1} \;=\; (\mathrm{tr}\,\mathrm{B}) \cdot 1$$

(which we also could have verified directly). Now compute

$$\begin{aligned} \mathrm{tr}\,(\mathrm{AB}) + \mathrm{tr}\,(\mathrm{AB}^{-1}) \;&=\; \mathrm{tr}\,\big(\mathrm{A}(\mathrm{B} + \mathrm{B}^{-1})\big) \\ &=\; \mathrm{tr}\,\big(\mathrm{A} \cdot (\mathrm{tr}\,\mathrm{B}) \cdot 1\big) \\ &=\; \mathrm{tr}\,\mathrm{A}\ \mathrm{tr}\,\mathrm{B}\;. \end{aligned}$$

■

Remarks (i) Inserting A$= 1$ in Lemma 18 yields $\mathrm{tr}\,\mathrm{B}^{-1} = \mathrm{tr}\,\mathrm{B}$.
(ii) Lemma 18, (ii) appears in formula (7) of R. Fricke, F.Klein: Vorlesungen über die Theorie der automorphen Functionen, Band 1; Leipzig: Teubner 1897, p. 338.

Lemma 17 follows from Lemma 18 by induction.

Exercise 8 *Compute* f_w *for* $w = aba^{-1}b^{-1}$.

If we denote $\varphi^\star(x)$ again by x, $\varphi^\star(y)$ by y and $\varphi^\star(z)$ by z then

$$f_w \;=\; p_w(x, y, z)$$

is a polynomial in three independent variables x, y, z . Let

$$\lambda \;=\; \alpha + \alpha^{-1}\;, \qquad \mu \;=\; \beta + \beta^{-1}$$

be complex numbers with

$$\alpha^2 \;\neq\; 1\;, \qquad \beta^2 \;\neq\; 1\;.$$

T. Jorgensen has proved that if

$$w \;=\; a^{r_1} b^{s_1} \ldots a^{r_k} b^{s_k} \qquad (k \geq 1,\ r_i, s_i > 0)$$

then

$$q_w(z) \;=\; p_w(\lambda, \mu, z)$$

is a polynomial of degree k and that the coefficient of z^k is given by the formula

$$\prod_{i=1}^{k} \left(\frac{\alpha^{r_i} - \alpha^{-r_i}}{\alpha - \alpha^{-1}} \right) \left(\frac{\beta^{s_i} - \beta^{-s_i}}{\beta - \beta^{-1}} \right) \;.$$

Exercise 9 *Prove Jorgensen's formula.*

Jorgensen's formula makes it relatively easy to deduce the following

Lemma 19 *Let F be a free group on $a_1, \ldots, a_m$. Then the following hold:*
(i) If $w \in F,\ w \neq 1$, there exists a homomorphism

$$\varrho \;:\; F \longrightarrow \mathrm{SL}(2, \mathbf{C})$$

such that $\varrho(w) \neq 1$ i.e. F is residually a subgroup of $\mathrm{SL}(2, \mathbf{C})$.
(ii) If $w \in F,\ w \neq 1$, and if $n > 2$ is a given integer then there exists a representation ϱ of F in $\mathrm{SL}(2, \mathbf{C})$ such that $\varrho(w)$ is of order n.

The proof of Lemma 19 is not difficult once one observes that F can be embedded in a free group of rank two. It is left to the reader as an exercise.

It follows then from Lemma 19 that we have proved the following corollary which is a special case of a theorem of Magnus, Karrass and Solitar.

Corollary 4 *Suppose*

$$G \;=\; \langle\, a_1, \ldots, a_m \,;\, w^n = 1 \,\rangle \qquad\qquad (n > 1)$$

is a group with a single defining relation, where $w \neq 1$ in the free group on $a_1, \ldots, a_m$. Then $w \neq 1$ in G; indeed w is of order n in G.

The argument is analogous to that used in the proof of Theorem 6 and is left to the reader.

Lemma 19 has other uses. For instance it can be used, under special circumstances, to deduce a strengthened form of the following celebrated theorem of W. Magnus:

Theorem 7 (W. Magnus 1932) *Let*

$$G \;=\; \langle\, a_1, \ldots, a_m \;;\; r = 1 \,\rangle$$

be a group defined by a single relation. Suppose r is cyclically reduced and involves the generator a_1. Then

$$gp(a_2, \ldots, a_m)$$

is a free subgroup of G freely generated by $a_2, \ldots, a_m$.

Theorem 7 is sometimes referred to as the Freiheitssatz.

8. Affine algebraic sets of characters

I should point out that it is, of course, impossible to obtain any definitive information about a finitely generated group G from its affine algebraic set $\mathrm{R}(G, d)$ of representations in $\mathrm{SL}(d, k)$ without knowing something about G. We have already seen how successfully this tactic works in the case of a free group of rank two where knowledge gained led to some non-trivial results about various classes of groups. However we really did not use any of the theory we have been developing, only some of the ideas behind the theory. In point of fact we used not $\mathrm{R}(G, d)$ but a quotient variety of $\mathrm{R}(G, d)$ [with G free of rank two, $d = 2$ and $k = \mathbf{C}$], namely $\mathbf{C}^3$. We need to define and better understand what this quotient is in general. This is our next objective. The full Culler-Shalen approach employing the Bass-Serre-Tits theory can then be applied. This approach is still little understood and little utilized. But it is certain to be used more in time to come. I want to describe one purely group-theoretic application of this method. But first let me give a definition.

Definition 15 *Let G be a group given by the finite presentation*

$$G \;=\; \langle\, x_1, \ldots, x_m \,;\, r_1, \ldots, r_n \,\rangle\ .$$

Then the deficiency of this presentation for G is denoted, somewhat ambiguously, by $defG$ and defined by $defG = m - n$.

Let us denote the subgroup of a group G generated by its squares by G^2. Then G/G^2 is an abelian group all of whose elements have order dividing 2. Hence it can be viewed as a vector space over the field of two elements. We denote the dimension of this vector space by $\dim(G/G^2)$.

The following theorem can be proved by the Culler-Shalen approach.

Theorem 8 *Let G be a group given by a finite presentation of deficiency $\operatorname{def} G = \delta$. Suppose*

$$3\delta - 3 > \dim(G/G^2) .$$

Then G is an amalgamated product where the amalgamated subgroup is of infinite index in one factor and of index at least two in the other.

As a consequence of Theorem 8 we find

Corollary 5 *Let G be a group defined by a single defining relation with at least four generators. Then G is an amalgamated product of the kind described in Theorem 8.*

Recently, in fact after these lectures had been given, Baumslag and Shalen proved that any finitely presented group, with a presentation of deficiency at least 2, can be decomposed as an amalgamated product of two groups where the amalgamated subgroup is of index at least 2 in one factor and of index at least 3 in the other.

We need a bunch of definitions in order to be able to describe a quotient $\mathrm{X}(G,d)$ of $\mathrm{R}(G,d)$ which we term the *affine algebraic set of characters of G in* $\mathrm{SL}(d,k)$. Here, as usual, G is a finitely generated group.

Definition 16 *Let ϱ, σ be representations of G in $\mathrm{SL}(d,k)$. Then we term ϱ and σ*

equivalent if there exists a matrix $\mathrm{T} \in \mathrm{SL}(d,k)$ *such that*

$$\varrho(g) \;=\; \mathrm{T}\sigma(g)\mathrm{T}^{-1} \qquad\qquad (all\ g \in G)\ .$$

In general it is not possible to parametrize the equivalence classes of equivalent representations of G by the points of an affine algebraic set. However, the semi-simple representations can so be parametrized. This is our objective. We need to recall some related definitions and notions. To this end, let G be a group, V a finite dimensional vector space over k, $\mathrm{SL}(V)$ the group of all invertible linear transformations of V of determinant 1 and ϱ a representation of G in V i.e. a homomorphism from G into $\mathrm{SL}(V)$. We say V *affords the representation* ϱ. If now $k[G]$ denotes the group algebra of G over k, then a $k[G]$-module V can be viewed as a vector space V together with a representation ϱ of G in V. Then two representations ϱ and σ of G in V and W are *equivalent* if the corresponding $k[G]$-modules V and W are isomorphic. A representation ϱ of G in V is termed *irreducible* if V is a simple $k[G]$-module i.e. has no non-trivial submodules and $V \neq 0$. Then according to Schur's Lemma, the k-algebra $\mathrm{End}\, V$ is a division ring. Since k is algebraically closed it follows that $\mathrm{End}\, V$ is simply k. V is termed *semi-simple* if it is a direct sum of simple $k[G]$-modules, in which case the corresponding representation is termed *semi-simple*.

Now suppose ϱ is a representation of G in V. Then think of V as a $k[G]$-module and let

$$0 \;=\; V_0 \;<\; \ldots \;<\; V_l \;=\; V$$

be a composition series for V. Put

$$W_i \;=\; V_i/V_{i-1} \qquad\qquad (i = 1, \ldots, l)\ .$$

By the Jordan-Hölder theorem the $k[G]$-modules W_i are unique up to isomorphism. Put

$$W \;=\; W_1 \oplus \ldots \oplus W_l\ .$$

Then W is again a $k[G]$-module, indeed a semi-simple $k[G]$-module. Let ϱ^{ss} denote the underlying representation of G in W. Then ϱ^{ss} is unique up to equivalence.

Suppose next that G is a finitely generated group, V a finite dimensional vector space of dimension d over k. If we fix a basis for V we have already seen how to view the set $\mathrm{R}(G,d)$ of all representations of G in V as an affine algebraic set. As before then we have the coordinate algebra $k[\mathrm{R}(G,d)]$ of $\mathrm{R}(G,d)$. For each $g \in G$ define

$$\widehat{g} \ : \ \mathrm{R}(G,d) \longrightarrow \mathrm{SL}(V)$$

by

$$\widehat{g}(\varrho) \ = \ \varrho(g) \ .$$

Since we have fixed a basis for V, we can view $\mathrm{SL}(V)$ in terms of this basis. Then $\widehat{g}$ is readily seen to be a morphism of affine algebraic sets i.e. is defined as usual by polynomial functions. Next we define, for $i = 0, \ldots, d-1$ and $g \in G$, the functions

$$f_g^i \ : \ \mathrm{R}(G,d) \longrightarrow k$$

as follows:

$$f_g^i(\varrho) \ = \ \pm \ \text{coefficient of the degree}\, i - \text{term in the characteristic polynomial of}\, \varrho(g) \ .$$

The remark above about $\widehat{g}$ shows that

$$f_g^i \ \in \ k[\mathrm{R}(G,d)] \qquad\qquad (i = 0, \ldots, d-1, \ g \in G) \ . \tag{4}$$

Let C be the k-subalgebra of $k[\mathrm{R}(G,d)]$ generated by these functions given by (4). We have the following theorem of Procesi:

Theorem 9 *C is an affine k-algebra.*

Let $\mathrm{X}(G,d)$ be the affine algebraic set defined by C via the categorical equivalence between affine algebraic sets and affine k-algebras and let

$$p \ : \ \mathrm{R}(G,d) \longrightarrow \mathrm{X}(G,d)$$

be the canonical projection of $\mathrm{R}(G,d)$ to $\mathrm{X}(G,d)$ that comes from the inclusion

$$C \hookrightarrow k[\mathrm{R}(G,d)] \ .$$

Definition 17 $\mathrm{X}(G,d)$ *is termed the affine algebraic set of characters of* G *in* $\mathrm{SL}(V)$.

The nature of $\mathrm{X}(G,d)$ is clarified by the next theorem, which is also due to Procesi.

Theorem 10 *(i)* $p : \mathrm{R}(G,d) \longrightarrow \mathrm{X}(G,d)$ *is onto.*
(ii) $p(\varrho) = p(\varrho^{ss})$.
(iii) If ϱ *and* σ *are semi-simple representations of* G *in* V *then* $p(\varrho) = p(\sigma)$ *if and only if* ϱ *and* σ *are equivalent.*
(iv) If ϱ *is irreducible, then* $p^{-1}\big(p(\varrho)\big)$ *is the equivalence class of representations of* G *in* V *equivalent to* ϱ.

It follows from Theorem 10 that $\mathrm{X}(G,d)$ can be thought of as an affine algebraic set which parametrizes the equivalence classes of equivalent semi-simple representations of G in V.

Notice that in the case $d = 2$, the functions $f_g^i \quad (i = 0, 1)$ are particularly easy to describe:

$$\begin{aligned} f_g^0(\varrho) &= \det \varrho(g) = 1 \ ; \\ f_g^1(\varrho) &= \mathrm{tr}\, \varrho(g) \ . \end{aligned}$$

Let

$$\chi_\varrho : G \longrightarrow k \quad \text{be defined by} \quad g \longmapsto \mathrm{tr}\, \varrho(g) \ .$$

We term χ_ϱ the *character* of ϱ.

Now our affine k-algebra C is generated by the functions

$$f_{g_1}^1, \ldots, f_{g_l}^1 \qquad\qquad (l < \infty)$$

for some choice of elements $g_1, \ldots, g_l \in G$ and

$$p(\varrho) = \big(\mathrm{tr}\, \varrho(g_1), \ldots, \mathrm{tr}\, \varrho(g_l) \big) \ .$$

This means that if ϱ and σ are semi-simple representations of G then $\chi_\varrho = \chi_\sigma$ if and only if ϱ and σ are equivalent. So $\mathrm{X}(G,2)$ parametrizes the characters of the semi-simple representations of G.

Exercise 10 *If G has an irreducible representation in V, a vector space of dimension d, prove that*

$$\dim \mathrm{X}(G,d) \ \geq \ (m-n)(d^2-1) \ - \ (d^2-1)$$

if G has a presentation on m generators and n defining relations. Hence deduce that if G is free of rank m, that

$$\dim \mathrm{X}(G,d) \ \geq \ (m-1)(d^2-1) \ .$$

In order to delve deeper into this theory we need more information about HNN extensions and generalized free products. This is our next objective.

CHAPTER VI Generalized free products and HNN extensions

1. Applications

Recall that if a group G is an amalgamated product

$$G \;=\; A \underset{H}{*} B$$

then

(i) $G \;=\; gp(A \cup B)$, where A and B are subgroups of G;
(ii) $A \cap B \;=\; H$;
(iii) every "strictly alternating" $A \cup B$-product

$$x_1 \ldots x_n \;\neq\; 1 \qquad\qquad (n > 0)$$

(so here $x_i \in (A - H) \cup (B - H)$ and if $x_i \in A$ then $x_{i+1} \notin A$ and if $x_i \in B$ then $x_{i+1} \notin B \quad (i = 1, \ldots, n-1)$).

The very definition of such an amalgamated product ensures that G has the following universal mapping property: for every group X and every pair of homomorphisms

$$\alpha \;:\; A \longrightarrow X \;, \qquad \beta \;:\; B \longrightarrow X$$

such that

$$\alpha \,\big|\, H \;=\; \beta \,\big|\, H$$

there exists a homomorphism

$$\mu \;:\; G \longrightarrow X$$

which agrees with α on A and β on B.

In this chapter I want to give some examples of the way in which generalized free products can be used.

In the early 1940s H. Hopf asked whether a finitely generated free group can be isomorphic to any of its proper factor groups. He later did prove that this is impossible, but he left open the corresponding question for finitely generated groups. We start out here with Graham Higman's answer to this question (G.Higman : *A finitely related group with group with an isomorphic proper factor group*, **J. London Math. Soc. 26**, 59-61 (1951)).

Theorem 1 (G. Higman 1951) *There exists a finitely presented group G which is isomorphic to one of its proper factor groups.*

Proof Let

$$A \;=\; \langle\, a, s \,;\, a^s = a^2 \,\rangle \;, \qquad B \;=\; \langle\, b, t \,;\, b^t = b^2 \,\rangle \;.$$

Recall that A and B are simply semidirect products of the dyadic fractions, i.e. the subgroup of $\mathbf{Q}$ consisting of all rational numbers of the form $\frac{l}{2^m}$, by an infinite cyclic group where the infinite cyclic group acts by multiplication by 2. In particular then a and b are of infinite order. So we can form the amalgamated product

$$G \;=\; \{\; A * B \,;\, a = b \;\}$$

(using the obvious notation). So we have

$$H \;=\; gp(a) \qquad \big(\;=\; gp(b)\big) \;.$$

Let

$$\alpha \;:\; A \longrightarrow G$$

be defined by

$$\alpha \ : \ a \longmapsto a^2 \ , \ s \longmapsto s$$

(i.e. α is conjugation by s followed by the inclusion of A in G).
Similarly define

$$\beta \ : \ B \longrightarrow G$$

by

$$\beta \ : \ b \longmapsto b^2 \ , \ t \longmapsto t \ .$$

Notice α and β agree on H:

$$\alpha \ : \ a \longmapsto a^2 \ , \ \beta \ : \ a \, (= \, b \,) \longmapsto a^2 \ .$$

So they can be extended to a homomorphism

$$\mu \ : \ G \longrightarrow G \ .$$

Since

$$G\mu \ \ a^2, s, t \ ,$$

it follows that

$$G\mu \ \ sa^2 s^{-1} \ = \ a \ .$$

Thus

$$G\mu \ = \ G \ .$$

Now consider the element

$$g \ = \ \underbrace{sas^{-1}} \underbrace{tb^{-1}t^{-1}} \ .$$

If we gather sas^{-1} together and $tb^{-1}t^{-1}$ together, g becomes a strictly alternating $A \cup B$-product. So

$$g \ \neq \ 1 \ .$$

But observe that

$$g\mu \ = \ sa^2 s^{-1} tb^{-2} t^{-1} \ = \ ab^{-1} \ = \ 1 \ .$$

So

$$G/\ker \mu \ \cong \ G$$

with $\ker \mu \neq 1$. ■

Next let me turn to a theorem of G. Higman, B.H. Neumann and Hanna Neumann proved in 1949.

Theorem 2 *Every countable group can be embedded in a 2-generator group.*

Proof Let G be a countable group. We enumerate the elements of G as an infinite sequence, using repetitions of the elements of G if needed:

$$G \;=\; \{\, g_0 = 1\,,\, g_1, g_2, \dots \,\}$$

Let now

$$U \;=\; \langle\, u, v \,\rangle \qquad B \;=\; \langle\, a, b \,\rangle$$

be two free groups of rank two. Notice that the elements

$$u\,,\; vuv^{-1}\,,\; v^2uv^{-2}\,,\; \dots$$

freely generate a free subgroup of U and similarly for

$$a\,,\; bab^{-1}\,,\; b^2ab^{-2}\,,\; \dots$$

in B. Now let A be the free product of G and U:

$$A \;=\; G * U\ .$$

Notice that the elements

$$g_0u\,,\; g_1vuv^{-1}\,,\; g_2v^2uv^{-2}\,,\; \dots$$

freely generate a free subgroup of A. To see this observe that every non-empty reduced product of these elements is $\neq 1$ since its projection onto U has this property. Put

$$H \;=\; gp(\, g_0u\,,\; g_1vuv^{-1}\,,\; g_2v^2uv^{-2}\,,\; \dots)$$

and

$$K \;=\; gp(\, a\,, bab^{-1}\,,\; b^2ab^{-2}\,,\; \dots)\ .$$

Now H and K are both free of countably infinite rank. So we can form the amalgamated product

$$P \;=\; \{\, A * B\,;\, H = K \,\}$$

where the equality $H = K$ is defined by identifying the elements

$$g_i v^i u v^{-i} \qquad with \qquad b^i a b^{-i} \qquad (i = 0, 1, 2, \ldots) \ .$$

So A and B can be viewed as subgroups of P and

$$g_0 u = a \ , \ g_1 v u v^{-1} = b a b^{-1} \ , \ g_2 v^2 u v^{-2} = b^2 a b^{-2} \ , \ \ldots$$

in P. But this means

$$P \ = \ gp(u, v, a, b) \ .$$

Hence

$$P \ = \ gp(v, a, b)$$

since $g_0 = 1$ and therefore $u = a$.

Now observe that v and a freely generate a free group of rank two, a and b freely generate a free group of rank two. Form the HNN extension E with base P, associated subgroups $gp(v, a), \ gp(a, b)$ and associating isomorphism

$$\varphi \ : \ v \longmapsto a \ , \ a \longmapsto b$$

and stable letter t:

$$E \ = \ \langle \ P, t \, ; \, t v t^{-1} = a \, , \, t a t^{-1} = b \ \rangle \ .$$

Notice

$$E \ = \ gp(t, v)$$

is the desired 2-generator group. ■

2. Back to basics

We need to think again about amalgamated products and HNN extensions.

Suppose then that

$$G \ = \ A \underset{H}{*} B$$

is an amalgamated product. Here we adopt the point of view as before that

$$A \leq G\ ,\ B \leq G\ ,\ A \cap B = H\ .$$

Let us choose a left transversal S of H in A and a left transversal T of H in B. Then every element $g \in G$ can be expressed in the form

$$g = u_1 \dots u_n h \qquad (n \geq 0) \tag{1}$$

where

$$u_i \in (S \cup T) - \{1\} \qquad (i = 1, \dots, n)\ , \qquad h \in H$$

and successive u's come from different transversals. We define the length $l(g)$ of g by

$$l(g) = n$$

and term (1) the *normal form* of g. The following lemma justifies these notions.

Lemma 1 *Let*

$$G = A \underset{H}{*} B\ .$$

Then the following hold:
(i) If $g \in G - H$ is written as a strictly alternating $A \cup B$-product

$$g = x_1 \dots x_n \qquad (n > 0) \tag{2}$$

then n depends only on g, i.e. any two such representations for g have the same number of factors.
(ii) If S and T are left transversals of H in A and B respectively, then the normal form (1) for $g \in G$ is unique.
(iii) If g is given by (2), then $l(g) = n$.
(iv) $l(g) = 0$ if and only if $g \in H$.

The proof of Lemma 1 rests on the fact that in an amalgamated product, strictly alternating products of elements are $\neq 1$ and is left to the listener.

Similar remarks hold also in the case of an amalgamated product with more than two factors. Returning to the amalgamated product G of Lemma 1, let us term the strictly alternating

product (2) *cyclically reduced* if either $n = 1$ or if $n > 1$, if x_1 and x_n come from different factors A, B. Since this is a property of g itself we say g *is cyclically reduced.*

Lemma 2 *Let*

$$G \;=\; A \underset{H}{*} B \;.$$

Then every element of G which is cyclically reduced and of length at least two is of infinite order.

Proof Let $g \in G$ be cyclically reduced and of length at least two. Then g can be written in strictly alternating form

$$g \;=\; x_1 \dots x_n \qquad\qquad (n \geq 2) \;.$$

Then for every $m > 0$,

$$g^m \;=\; x_1 \dots x_n x_1 \dots x_n \dots x_1 \dots x_n \;. \tag{3}$$

Now g is cyclically reduced. Hence x_1 and x_n lie in different factors. It follows from (3) that g^m is also cyclically reduced and of length $mn \geq 2$. So

$$g^m \;\neq 1$$

(indeed g^m does not lie even in H). ■

Corollary 1 *In an amalgamated product every element of finite order is conjugate to an element in one of the factors. Hence an amalgamated product of torsion-free groups is torsion-free.*

We recall now Corollary 4 of Chapter III as Lemma 3 and, for completeness, give a proof of it.

Lemma 3 *Let* $G = \underset{\substack{H \\ i \in I}}{\ast} G_i$, *let* $F_i \leq G_i$ $(i \in I)$ *be such that*

$$F_i \cap H \;=\; K \;=\; F_j \cap H \qquad\qquad (i, j \in I) \;.$$

Then

$$P \;=\; gp\left(\bigcup_{i \in I} F_i\right) \;=\; \underset{\substack{K \\ i \in I}}{\ast} F_i \;.$$

Proof Every strictly alternating $\bigcup_{i\in I} F_i$-product (relative to K) is $\neq 1$ in P because this is true in G! ■

It is time to turn our attention to HNN extensions, a special case of which was introduced in Chapter IV. To this end let $B = \langle\, X; R \,\rangle$ be a presentation of a given group B. We then define an HNN extension E with base B, associated subgroups $H_i,\ K_i \quad (i \in I)$, associating isomorphisms $\varphi_i : H_i \xrightarrow{\sim} K_i \quad (i \in I)$ and stable letters $t_i \quad (i \in I)$ to be the group

$$E \;=\; \langle\, X \dot\cup \{t_i \,\big|\, i \in I\} \;;\; R \cup \{t_i h t_i^{-1} (h\varphi_i)^{-1} \,\big|\, h \in H_i,\, i \in I\} \,\rangle \;.$$

As we have already noted previously, groups given by generators and defining relations can be difficult to unravel. Our objective now is to show that E can be reasonably well understood by finiding an isomorphic copy of E in a suitably chosen amalgamated product. To this end let

$$U \;=\; B * \langle u_i \,\big|\, i \in I\rangle \quad , \quad V \;=\; B * \langle v_i \,\big|\, i \in I\rangle \;.$$

Observe that the subgroup C of U generated by B together with the conjugates $u_i H_i u_i^{-1}$ of the $H_i \quad (i \in I)$ is their free product:

$$C \;=\; gp\big(\, B \,,\, u_i H_i u_i^{-1}\, (i \in I)\,\big) \;=\; B * \mathop{\Large *}_{i\in I} u_i H_i u_i^{-1} \;.$$

Similarly in V we find

$$D \;=\; gp\big(\, B \,,\, v_i^{-1} K_i v_i\, (i \in I)\,\big) \;=\; B * \mathop{\Large *}_{i\in I} v_i^{-1} K_i v_i \;.$$

There is an obvious isomorphism

$$\varphi \;:\; C \xrightarrow{\sim} D$$

which is the identity on B and maps $u_i H_i u_i^{-1}$ onto $v_i^{-1} K_i v_i$ as prescribed by φ_i. So we can form the generalized free product

$$G \;=\; \{\, U * V \,;\, C \overset{\varphi}{=} D \,\}$$

using φ to identify C and D. We understand G well since it is an amalgamated product. Now this means, in particular, that B embeds into G and if we set

$$\widetilde{t_i} \;=\; v_i u_i \qquad\qquad (i \in I)$$

then

$$\widetilde{t}_i h \widetilde{t}_i^{-1} \;=\; h\varphi_i \qquad (h \in H_i, i \in I)\ .$$

We claim that E is isomorphic to the subgroup $\widetilde{E}$ of G defined by

$$\widetilde{E} \;=\; gp\big(\, B\,,\, \widetilde{t}_i\ (i \in I)\,\big)\ .$$

This is easy enough to check in a number of ways. For example, by W. Dyck we are assured of a homomorphism

$$\alpha \;:\; E \longrightarrow \widetilde{E}$$

mapping B identically to B and t_i to $\widetilde{t}_i$ $(i \in I)$. On the other hand we can define a homomorphism γ of G onto E by first defining γ on U and V as follows:

$$\gamma \,\big|\, B \;=\; \mathrm{id}\ , \qquad \gamma(u_i) \;=\; 1 \quad (i \in I)\ ,$$

$$\gamma \,\big|\, B \;=\; \mathrm{id}\ , \qquad \gamma(v_i) \;=\; t_i \quad (i \in I)\ .$$

One checks that γ has the same effect on C and D to verify it continues to a homomorphism of G onto E. Notice that if we put

$$\beta \;=\; \gamma \,\big|\, \widetilde{E}$$

then α and β are mutually inverse, as required. This then permits us to deduce what we have already assumed about HNN extensions, plus a little more, which we record here as

Theorem 3 *Let*

$$E \;=\; \langle\, B\,, t_i\ (i \in I)\ ;\ t_i H_i t_i^{-1} = K_i\ (i \in I)\,\rangle$$

be an HNN extension. Then
(i) the canonical homomorphism of B into E is a monomorphism;
(ii) if

$$w \;=\; b_0 t_{j_1}^{\varepsilon_1} b_1 t_{j_2}^{\varepsilon_2} \ldots t_{j_n}^{\varepsilon_n} b_n \qquad\qquad (n > 0)$$

where

$$j_1, \ldots, j_n \;\in\; I\ , \qquad b_0, \ldots, b_n \;\in\; B\ , \qquad \varepsilon_1, \ldots, \varepsilon_n \;\in\; \{1, -1\}$$

and if

$$w \;=\; 1$$

then there exists a "pinch" in w i.e. for some m, $1 \leq m \leq n-1$, either

$$\varepsilon_m \;=\; 1 \;,\; b_m \;\in\; H_{j_m} \qquad \textit{and} \qquad j_{m+1} \;=\; j_m \;,\; \varepsilon_{m+1} \;=\; -1$$

or else

$$\varepsilon_m \;=\; -1 \;,\; b_m \;\in\; K_{j_m} \qquad \textit{and} \qquad j_{m+1} \;=\; j_m \;,\; \varepsilon_{m+1} \;=\; 1 \;.$$

Part (i) of Theorem 3 goes back to Higman, B.H. Neumann and Hanna Neumann in 1949, while (ii) is an observation due to J.L. Britton and is usually referred to as *Britton's Lemma*. It follows immediately from our reconstruction of E as a subgroup of the generalized free product G.

Notice that E contains elements of finite order only if B does. (Prove!)

3. More applications

Theorem 4 (B.H.Neumann) *There exist continuously many non-isomorphic 2-generator groups.*

Proof Let p_i denote the i-th prime and let $C(i)$ denote the cyclic group of order p_i. For each increasing sequence $\big(\sigma(i)\big)_{i=1,2,\ldots}$ of positive integers define

$$A_\sigma \;=\; \bigoplus_{i=1}^{\infty} C\big(\sigma(i)\big) \;,$$

the direct sum of the cyclic groups of order $p_{\sigma(i)}$. Notice that

$$A_\sigma \;\cong\; A_\tau \qquad \text{if and only if} \qquad \sigma \;=\; \tau \;,$$

since $\sigma \neq \tau$ implies A_σ and A_τ do not have the same finite subgroups. Now embed each A_σ in a 2-generator group G_σ, using the method in **1**. Then it follows from the information obtained

in **2** that the elements of finite order are precisely those which are conjugate to elements in A_σ. So

$$G_\sigma \cong G_\tau \qquad if and only if \qquad \sigma = \tau .$$

Now the cardinality of such sequences $\big(\sigma(i)\big)_{i=1,2,\ldots}$ is that of the continuum. This proves Theorem 4. ■

Now recall that the property $\mathcal{M}$ of finitely presented groups is termed a *Markov property* if it is preserved under isomorphism, if there exists a finitely presented group G_1 with $\mathcal{M}$ and a finitely presented group G_2 which cannot be embedded in any finitely presented group with $\mathcal{M}$.

My objective is to give Rabin's proof of the following theorem of Adyan (see the book by R.C. Lyndon and Paul E. Schupp: *Combinatorial Group Theory*, **Ergebnisse der Mathematik und ihrer Grenzgebiete 89**, Springer-Verlag, berlin-Heidelberg- New York (1977)), which I mentioned at the beginning of this course. We have, however, to assume here the existence of a finitely presented group with an insoluble word problem.

Theorem 5 (Adyan) *Let $\mathcal{M}$ be a Markov property. Then there is no algorithm which decides whether or not any finitely presented group has $\mathcal{M}$.*

Proof Let U be a finitely presented group with an insoluble word problem. Put

$$U_0 = U * G_2$$

where G_2 is the finitely presented group which cannot be embedded in any finitely presented group with $\mathcal{M}$. U_0 is finitely presented since U and G_2 are; thus we can find a finite presentation for U_0:

$$U_0 = \langle\, x_1, \ldots, x_m \,;\, r_1, \ldots, r_n \,\rangle .$$

Notice that U_0 also has an insoluble word problem since U does (and U is a subgroup of U_0). We will construct for each $w = w(x_1, \ldots, x_m)$ in U_0 a finitely presented group G_w with the following property:

$$G_w \text{ has } \mathcal{M} \qquad \text{if and only if} \qquad w = 1.$$

This then suffices to prove the theorem since U_0 has an insoluble word problem.

The construction of G_w is carried out in stages.
First we form

$$U_1 \;=\; U_0 * \langle y_0 \rangle$$

the free product of U_0 and the infinite cyclic group on y_0. Notice that if we put

$$y_i \;=\; y_0 x_i \qquad\qquad (i = 1, \ldots, m)$$

then the y_i are all of infinite order and

$$U_1 \;=\; gp(y_0, y_1, \ldots, y_m) \;.$$

Notice also that if $w \neq 1$, then $[w, y_0] \neq 1$; indeed $[w, y_0]$ is of infinite order (it is cyclically reduced and of length at least two). Now form an HNN extension U_2 with base U_1, associated subgroups $gp(y_i),\ gp(y_i^2) \quad (i = 0, \ldots, m)$ and stable letters $t_0, \ldots, t_m$ as follows:

$$U_2 \;=\; \langle\, U_1, t_0, \ldots, t_m \,;\, t_0 y_0 t_0^{-1} = y_0^2, \ldots, t_m y_m t_m^{-1} = y_m^2 \,\rangle \;.$$

Notice that by Britton's Lemma $H = gp(t_0, \ldots, t_m)$ is free on $t_0, \ldots, t_m$ and hence

$$t_i \longmapsto t_i^2 \qquad\qquad (i = 0, \ldots, m)$$

defines an isomorphism from H onto $K = gp(t_0^2, \ldots, t_m^2)$. So we can form another HNN extension U_3:

$$U_3 \;=\; \langle\, U_2, z \,;\, z t_i z^{-1} = t_i^2 \ (i = 0, \ldots, m) \,\rangle \;.$$

Next let V_1 be the free group on r. Form the HNN extension

$$V_2 \;=\; \langle\, V_1, s \,;\, s r s^{-1} = r^2 \,\rangle \;.$$

Again s is of infinite order. So we can form another HNN extension V_3:

$$V_3 \;=\; \langle\, V_2, t \,;\, t s t^{-1} = s^2 \,\rangle \;.$$

Now a major move. Put

$$W \;=\; \{\, U_3 * V_3 \;;\; r = z \,,\, t = [w, y_0] \,\} \;.$$

Observe that if $w \neq 1$, then in U_3, by Britton's Lemma

$$gp(z, [w, y_0]) \quad \text{is free on} \quad z, [w, y_0].$$

Again in V_3

$$gp(r, t) \quad \text{is free on} \quad r, t.$$

So if $w \neq 1$, W is an amalgamated product, where the amalgamated subgroup is a free group of rank two, and therefore containes G_2. This means that

$$W \text{ does not have } \mathcal{M} \quad \text{if} \quad w \neq 1 \text{ !}$$

Let's see what happens if $w = 1$. Tracing our way back through the construction we find

$$[w, y_0] = 1 \implies t = 1 \implies s = 1 \implies r = 1 \implies z = 1 \implies t_0 = \ldots = t_m = 1$$
$$\implies y_0 = \ldots = y_m = 1.$$

In other words

$$W = \{1\} \quad if \quad w = 1 \text{ !}$$

Now put $G_w = W * G_1$.

We have therefore proved that G_w has $\mathcal{M}$ if $w = 1$ ($G_w = G_1$ in this case) and, if $w \neq 1$, $G_w \geq W$ and hence does not have $\mathcal{M}$ i.e. we have proved Adyan's Theorem. ■

It is worth noting that in the case where $\mathcal{M}$ is the property of being of order 1, $G_1 = 1$ and we have concocted a family of finitely presented groups G_w such that $G_w = 1$ if and only if $w = 1$, where again w ranges over the words ofa finitely presented group with an unsolvable word problem! This class of groups is tailor-made for obtaining further negative algorithmic examples.

The following exercise will be of use in the proof of the next theorem.

Exercise 1 *Let $G = A * B \quad (A \neq 1 \neq B)$. Prove the following:*
(i) there exists an element $g \in G$ with infinite cyclic centralizer;
(ii) $\zeta G = 1$;
(iii) G is directly indecomposable [use (i)].

The following theorem is a simple application of the existence of the groups G_w.

Theorem 6 *There is no algorithm which decides whether or not any finitely presented group*

(i) *is isomorphic to its direct square;*

(ii) *has an infinite automorphism group;*

(iii) *is a non-trivial free product;*

(iv) *is centreless;*

(v) *has an infinitely generated subgroup.*

I have concocted these remarks somewhat at random. The listener might want to demonstrate one of her or his favourite problem is algorithmically insoluble as a test of their skill.

To prove (i) for instance, note $G_w * G_w$ is isomorphic to its own direct square if and only if $G_w = 1$ by (iii) of the Exercise. The other parts are left as an exercise to the listener.

Here is one further illustration of this technique. In order to explain let me remind you of some definitions. Suppose G is any group. Then (see the book by P. J. Hilton and U. Stammbach: *A course in Homological Algebra*, **Graduate Texts in Mathematics 4**, Springer-Verlag, New York-Heidelberg-Berlin (1971)) define

$$\mathrm{H}_1(G, \mathbf{Z}) \ = \ G_{ab} \ .$$

$\mathrm{H}_1(G, \mathbf{Z})$ is the first homology group of G with coefficients in the additive group $\mathbf{Z}$ of integers. It is only one of a whole sequence, starting with

$$\mathrm{H}_0(G, \mathbf{Z}) \ = \ \mathbf{Z} \ , \ \mathrm{H}_1(G, \mathbf{Z}) \ , \ \mathrm{H}_2(G, \mathbf{Z}) \ , \ \ldots \ .$$

The one which is of most direct interest in the study of finitely presented groups is $\mathrm{H}_2(G, \mathbf{Z})$ which has a direct group-theoretic definition, like $\mathrm{H}_1(G, \mathbf{Z})$. Indeed let us express G in the form

$$G \ \cong \ F/R \ ,$$

where F is a free group. Then

$$H_2(G, \mathbf{Z}) \ = \ (F' \cap R)/[F, R] \ .$$

This description of $\mathrm{H}_2(G, \mathbf{Z})$ seems to depend on the "presentation" F/R of G. However here is an exercise for those who do not know about these things:

Exercise 2 *(i) Use Tietze transformations to prove that if*

$$G \;\cong\; F/R \;\cong\; E/S$$

where E and F are free, then

$$(F' \cap R)/[F, R] \;\cong\; (E' \cap S)/[E, S] \;.$$

(ii) Prove $\mathrm{H}_2(G, \mathbf{Z}) \;=\; 0$ if G is free.

We have the following simple

Lemma 4 *Suppose G is finitely presented. Then $\mathrm{H}_2(G, \mathbf{Z})$ is finitely generated, indeed if*

$$G \;=\; \langle\, x_1, \ldots, x_m \;;\; r_1, \ldots, r_n \,\rangle \qquad\qquad (m, n < \infty)$$

then $\mathrm{H}_2(G, \mathbf{Z})$ is an abelian group that can be generated by n elements and hence is finitely generated.

Proof We write

$$G \;\cong\; F/R$$

where

$$F \;=\; \langle\, x_1, \ldots, x_m \,\rangle$$

and

$$R \;=\; gp_{_F}(r_1, \ldots, r_n) \;.$$

Then

$$R/[F, R] \;=\; gp\big(\; r_1[F, R] \,, \ldots, \; r_n[F, R] \;\big) \;.$$

Hence $(F' \cap R)/[F, R]$ can be generated by n elements, by the basis theorem for finitely generated abelian groups. ■

Thus to each finitely presented group G we can associate two finitely generated abelian groups, $\mathrm{H}_1(G, \mathbf{Z})$ and $\mathrm{H}_2(G, \mathbf{Z})$. The first of these is computable, by the basis theorem for finitely generated abelian groups. Somewhat surprisingly, the second of them is not effectively calculable. This result is due to Cameron Gordon.

It is easy enough to deduce this fact by making use of the groups G_w. First of all notice that each G_w can be generated by a fixed number of generators and relations:

$$G_w \;=\; \langle\, x_1, \ldots, x_m \;;\; r_1, \ldots, r_n \,\rangle \;.$$

Of course these presentations depend on w. Now form the free product of G_w with the free group on s and t:

$$E_w \;=\; G_w * \langle s, t \rangle \;.$$

Next observe that if $w \neq 1$, then

$$\mathrm{H}_w \;=\; gp\big(\; [w,t] \,,\; s[w,t]s^{-1} \,,\; \ldots \,,\; s^{\alpha}[w,t]s^{-\alpha} \;\big)$$

is free of rank $\alpha + 1$ where we here choose

$$\alpha \;=\; 2m + 4 \;.$$

Let $\overline{E}_w$ be an isomorphic copy of E_w and let $\overline{\mathrm{H}}_w$ be the corresponding copy of H_w in $\overline{E}_w$. Then form the amalgamated product

$$P_w \;=\; \big\{\; E_w * \overline{E}_w \;;\; \mathrm{H}_w = \overline{\mathrm{H}}_w \;\big\} \;.$$

Now there is a sequence, called the Mayer-Vietoris sequence, which links the homology groups of P_w in a long exact sequence, part of which looks like this:

$$\ldots \longrightarrow \mathrm{H}_2(P_w, \mathbf{Z}) \longrightarrow \mathrm{H}_1(\mathrm{H}_w, \mathbf{Z}) \xrightarrow{\gamma} \mathrm{H}_1(E_w, \mathbf{Z}) \oplus \mathrm{H}_1(\overline{E}_w, \mathbf{Z}) \longrightarrow \ldots \;.$$

Now $\mathrm{H}_1(\mathrm{H}_w, \mathbf{Z})$ is free abelian of rank $2m + 5$ and $\mathrm{H}_1(E_w, \mathbf{Z}) \oplus \mathrm{H}_1(\overline{E}_w, \mathbf{Z})$ can be generated by $2m + 4$ elements. Hence $\ker \gamma \neq 0$. This means that $\mathrm{H}_2(P_w, \mathbf{Z}) \neq 0$. This is all predicated on the assumption that $w \neq 1$. If $w = 1$, then $G_w = 1$ and P_w is free. Hence $\mathrm{H}_2(P_w, \mathbf{Z}) = 0$ by the Exercise. Thus we have proved:

Theorem 7 *There is no algorithm whereby one can decide whether any finitely presented group has zero second integral homology group.*

Here is one last application of HNN extensions, providing us with perhaps the simplest of non-hopfian groups.

Theorem 8 *The group*

$$G \;=\; \langle\, a, t\,;\, t^{-1}a^2t = a^3 \,\rangle$$

is non-hopfian.

Proof G is of course an HNN extension with a single stable letter t and an infinite cyclic base $\langle a \rangle$. Consider the map

$$\varphi \;:\; a \longmapsto a^2 \;,\; t \longmapsto t \;.$$

Since

$$t^{-1}(a^2)^2t \;=\; (a^2)^3$$

it follows from W. Dyck's Lemma that φ defines a homomorphism, again denoted φ, of G into G. Observe that

$$G \;=\; gp(\, a^2, t\,) \;.$$

So φ is onto. Now consider

$$g \;=\; (a^{-1}t^{-1}at)^2a^{-1} \;=\; a^{-1}t^{-1}ata^{-1}t^{-1}ata^{-1} \;.$$

There is no pinch in g. So, by Britton's Lemma, $g \neq 1$. But

$$g\varphi \;=\; (a^{-2}t^{-1}a^2t)^2a^{-2} \;=\; a^2a^{-2} \;=\; 1 \;.$$

So $G \cong G/\ker\varphi$ is non-hopfian. ■

This theorem is due to G. Baumslag and D. Solitar (see the references cited in the book by Lyndon and Schupp referenced earlier). Amalgamated products permit the construction of a good many finitely generated groups which are not finitely presented.

Theorem 9 *Let A and B be finitely presented groups. Then*

$$G \;=\; A \underset{H}{*} B$$

is finitely presented if and only if H is finitely generated.

Proof One way is obvious. To prove that G is not finitely presented if H is not finitely generated, let

$$A \;=\; \langle\, X; R \,\rangle \quad , \quad B \;=\; \langle\, Y; S \,\rangle$$

be finite presentations for A and B. Suppose that

$$H \;=\; gp\big(\, h_1(\underset{\sim}{x}), h_2(\underset{\sim}{x}), \dots \big)$$

is an infinite set of generators for H. Now

$$h_i(\underset{\sim}{x}) \;=\; k_i(\underset{\sim}{y}) \qquad (i = 1, 2, \dots) \;.$$

So G can be presented in the form

$$G \;=\; \langle\, X \cup Y \,;\, R \cup S \cup \{\, h_i(\underset{\sim}{x}) k_i(\underset{\sim}{y})^{-1} \;\big|\; i = 1, 2, \dots \} \,\rangle \;.$$

Since $X \cup Y$ is finite, by Neumann's theorem, if G is finitely presented we can present it in the form

$$G \;=\; \langle\, X \cup Y \,;\, R \cup S \cup \{\, h_i(\underset{\sim}{x}) k_i(\underset{\sim}{y})^{-1} \;\big|\; i = 1, \dots, n \} \,\rangle \tag{4}$$

for some positive integer n. Now observe that

$$H \;\neq\; gp\big(\, h_1(\underset{\sim}{x}), \dots, h_n(\underset{\sim}{x}) \big) \;=:\; H_1 \;.$$

So we can choose $h(\underset{\sim}{x}) \in H - H_1$. Thus if $k(\underset{\sim}{y})$ is the corresponding element in B to $h(\underset{\sim}{x})$, then

$$k(\underset{\sim}{y}) \;\notin\; K_1 \;=\; gp\big(\, k_1(\underset{\sim}{y}), \dots, k_n(\underset{\sim}{y}) \big) \;.$$

Let us think of (4) as presenting a group which is supposed to be G but is perhaps more safely denoted by $\widetilde{G}$:

$$\widetilde{G} \;=\; \langle\, X \cup Y \,;\, R \cup S \cup \{\, h_i(\underset{\sim}{x}) k_i(\underset{\sim}{y})^{-1} \;\big|\; i = 1, \dots, n \} \,\rangle \;.$$

Now $\widetilde{G}$ is, by its very presentation, an amalgamated product:

$$\widetilde{G} \;=\; \{\, A * B \,;\, H_1 = K_1 \,\} \;.$$

Now observe that

$$h(\underset{\sim}{x})k(\underset{\sim}{y})^{-1} \ \neq \ 1$$

in $\widetilde{G}$ because it is a strictly alternating product! But $\widetilde{G}$ is supposed to be G given by a finite presentation. Since

$$h(\underset{\sim}{x}) \ = \ k(\underset{\sim}{y})$$

in G, this contradicts this supposition. So G is not finitely presented. ■

Exercise 3 *Suppose $E = \langle\, B, t\,;\, tHt^{-1} = K \,\rangle$ is an HNN extension with finitely presented base B. Prove E is finitely presented if and only if H is finitely generated.*

Examples *(1) If $F \ = \ \langle\, a, b \,\rangle$ then $G \ = \ F \underset{F'}{*} F$ is not finitely presented.*

(2) $A \ = \ \langle\, a, s\,;\, a^s = a^2 \,\rangle \ , \ B \ = \ \langle\, b, t\,;\, b^t = b^2 \,\rangle$ *then* $G \ = \ \{\, A * B\,;\, gp_{_A}(a) = gp_{_B}(b) \,\}$ *is not finitely presented.*

As one last illustration, here is an example of a finitely presented group with a finitely generated subgroup which is not fintely presented.

Example *Let $F_i = \langle a_i; b_i \rangle$ $(i = 1, 2)$ be free of rank two. Consider*

$$D \ = \ F_1 \times F_2$$

the direct product of F_1 and F_2. Then D is clearly finitely presented. We claim that

$$H \ = \ gp\big(\, \alpha = (a_1, a_2)\,,\, \beta = (b_1, 1)\,,\, \gamma = (1, b_2)\,\big)$$

is not finitely presented. To see this let's find a presentation for H:

$$H \ = \ \langle\, \alpha, \beta, \gamma\,;\, [\beta, \gamma^{\alpha^i}] = 1 \ (i \in \mathbf{Z}) \,\rangle \ .$$

Suppose H were finitely presented. Then by Neumann's theorem it has a presentation of the form

$$H \ = \ \langle\, \alpha, \beta, \gamma\,;\, [\beta^{\alpha^j}, \gamma^{\alpha^i}] = 1 \ (-N \leq i, j \leq N) \,\rangle$$

for some positive integer N. We compute now a presentation for

$$K \ = \ gp_{_H}(\beta, \gamma) \ .$$

Notice $H/K = \langle \alpha K \rangle$ *is infinite cyclic. So we have a ready-made Schreier transversal* $S = \{\alpha^i \,\big|\, i \in \mathbf{Z}\}$. *Put*

$$\begin{aligned} \beta_i &= \alpha^i \beta \alpha^{-i} \quad , \quad \gamma_i = \alpha^i \gamma \alpha^{-i} \qquad (i \in \mathbf{Z}); \\ X &= gp(\beta_{-N}, \ldots, \beta_N) \quad , \quad Y = gp(\gamma_{-N}, \ldots, \gamma_N) \ . \end{aligned}$$

Then in H, X *and* Y *are free on the exhibited generators. Indeed*

$$A = gp(X, Y) = X \times Y \ .$$

Similarly

$$B = gp(\, \alpha X \alpha^{-1}, \alpha Y \alpha^{-1} \,) = \alpha X \alpha^{-1} \times \alpha Y \alpha^{-1} \ .$$

Now let's put then these two groups together:

$$gp(A, B) = \{\ A * B\ ;\ \beta_{-N+1} = \beta_{-N+1}, \ldots, \beta_N = \beta_N, \gamma_{-N+1} = \gamma_{-N+1}, \ldots, \gamma_N = \gamma_N\ \} \ .$$

But then $[\beta_{-N}, \gamma_{N+1}] \neq 1$. *This is false in* H. *So* H *is not finitely presented.*

4. Some word, conjugacy and isomorphism problems

Definition 1 *Let* A *be a group given by a finite presentation,* H *a finitely generated subgroup of* A *given by a finite set of generators, each of which comes expressed in terms of the given generators of* A. *Then we say that the occurence problem or extended word problem for* A *relative to* H *is solvable if there exists an algorithm such that for each* $w \in A$ *we can decide whether or not* $w \in H$ *and in this case exhibit* w *as a word in the generators of* H.

Proposition 1 *Suppose*

$$G = \{\ A * B\ ;\ H \overset{\varphi}{=} K\ \}$$

is an amalgamated product in which A *and* B *are given by finite presentations and* H *and* K *are finitely generated subgroups given respectively by means of generators of* A *and* B *and that* φ *is defined by sending each generator of* H *to the correspondingly indexed generator*

of K. If the extended word problem for A relative to H and also that for B relative to K are both solvable, then the word problem for G is solvable provided it is solvable for H relative to its given generators.

The proof rests only on the fact that a strictly alternating product in G is different from 1 and is left to the listener.

There are two classes of groups for which the extended word problem is solvable. In order to explain, we need another definition.

Definition 2 *Let A be a finitely presented group given by an explicit finite presentation, H a finitely generated subgroup of A given explicitly by a finite set of generators. We term H finitely separable from H if for each $w \in A,\ w \notin H$, there exists a normal subgroup N of A of finite index such that*

$$w \ \notin \ NH \ .$$

Lemma 5 *Let A be a finitely presented group, H a finitely generated subgroup of A which is finitely separable from A. Then the extended word problem for A relative to H is solvable provided A has a solvable word problem.*

Proof We simply list the finite quotients of A and also list the elements of H (as products of the given generators of H). Then for any given $w \in A$ we will either find that the image of $w \ \notin$ image of H or else we find $w \in H$, properly expressed. ■

Lemma 6 (Toh) *Let A be a finitely generated nilpotent group, H a finitely generated subgroup of A. Then H is finitely separable from A.*

The proof is by induction of A and is left to the listener.

Lemma 7 *Let A be a finitely generated free group, H a finitely generated subgroup of A.*

Then H is finitely separable from A.

Proof By Marshall Hall's theorem, there exists a subgroup J of A of finite index such that H is a free factor of J. Let $w \in A,\ w \notin J$. Since J is of finite index in A, there exists a normal subgroup N of A contained in J of finite index in A. Then

$$w \ \notin \ NH \ .$$

So we are half-way to proving H finitely separable from A.

Suppose next $w \in J,\ w \notin H$. Now

$$J \ = \ H * L \ .$$

Choose now normal subgroups H_1 of H and L_1 of L both of finite index – H and L are free and hence residually finite – such that if $\overline{J}$ is the canonical image

$$\overline{J} \ = \ H/H_1 \ * \ L/L_1$$

of J, then the image of w, say $\overline{w}$, satisfies

$$\overline{w} \ \notin \ \overline{H} \ = \ H/H_1 \ .$$

Notice $\overline{J}$ is the free product of the two finite groups $\overline{H}$ and $\overline{L} = L/L_1$. Using the Reidemeister-Schreier method it is not hard to see that the kernel K of the homomorphism of $\overline{J}$ onto $\overline{H} \times \overline{L}$ is free. Thus $\overline{J}$ is a finite extension of a free group and hence residually finite. But $\overline{H}$ is a finite subgroup of $\overline{J}$. So we can find a normal subgroup $\overline{S}$ of finite index in $\overline{J}$ such that

$$\overline{w} \ \notin \ \overline{S}\,\overline{H} \ .$$

Pulling this information back into J yields a normal subgroup S of finite index in J such that

$$w \ \notin \ SH$$

Now S is of finite index in J, hence of finite index in A. so the conjugates of S intersect in a subgroup T of S which is of finite index in A and normal in A. But notice

$$w \ \notin \ TH$$

since

$$TH \leq SH \, .$$ ■

So we have proved the

Theorem 10 *The free product of two finitely generated free groups (or two finitely generated nilpotent groups) with a finitely generated subgroup amalgamated has a solvable word problem.*

Corollary 2 (Dehn) *The fundamental groups of closed two-dimensional surfaces have solvable word problem.*

In fact these fundamental groups are even residually finite and so it follows again that they have solvable word problem.

CHAPTER VII Groups acting on trees

1. Basic definitions

The exposition in this chapter is based on the book by Jean-Pierre Serre: *Trees*, **Translated from the French by John Stillwell**, Springer-Verlag, Berlin-Heidelberg-New York (1980). The reader should consult this work for more details, if needed.

Definition 1 *A graph X is a pair of sets, $V = V(X) \neq \emptyset$ and $E = E(X)$, termed the vertices and edges of X, equipped with three maps*

$$o \ : \ E \longrightarrow V \ , \qquad t \ : \ E \longrightarrow V \ , \qquad \overline{\ } \ : \ E \longrightarrow E$$

satisfying the following conditions: if $e \in E$, then
(i) $e \neq \overline{e}$ and $\overline{\overline{e}} = e$ (i.e. the map $\overline{\ }$ is of order two and is fixed point free);
(ii) $o(e) \ = \ t(\overline{e})$.

We term $t(e)$ the terminus of e, $o(e)$ its origin and $\overline{e}$ the inverse of e. Sometimes we refer to $o(e)$ and $t(e)$ as the extremities of e. It is possible for $o(e) = t(e)$ and in this case e is termed a loop. Two distinct vertices are termed adjacent if they are the extremities of some edge.

Graphs are often represented by diagrams in the plane, the vertices by points and the edges by line segments joining its extremities. We usually affix to such line segments an arrow, whose direction emanates from the origin of the edge and terminates in its origin. We customarily omit one of $e,\ \overline{e}$. Usually diagrams are drawn in such a way that the graphs can be reconstructed from them without ambiguity.

We take for granted the usual notions involving morphisms of graphs, with the automorphism group $\mathrm{Aut}\, X$ of the graph X consisting of the invertible morphisms $X \longrightarrow X$ using composition as the binary operation.

Definition 2 *A group G acts on a graph X if it comes equipped with a homomorphism*

$$\varphi \ : \ G \longrightarrow \mathrm{Aut}\, X \ .$$

We often denote the image of $v \in V(X)$ under the action of $g \in G$ by gv etc.

Examples *(1) Let G be a group, S a set of generators of G. We define the Cayley graph $X = X(G,S)$ of G relative to S by*
(i) $V(X) \ = \ G$;
(ii) $E(X)$ is the disjoint union of the sets $G \times S$ and $S \times G$;
(iii) $o(g,s) \ = \ g \ ; \quad t(g,s) \ = \ gs \ ; \quad \overline{(g,s)} \ = \ (s,g)$.

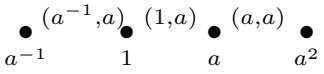
Notice that G acts on $X(G,S)$ by left multiplication:

(2) $G = \{1\}$. Then $X\big(G,\{1\}\big)$ is a loop:

Notice that despite the notation, we have the inverse edge $\overline{(1,1)}$ *which is different from* $(1,1)$.

(3) $G = \langle a; a^n = 1 \rangle \quad (n \in \{1, 2, \ldots\})$, $\quad S = \{a\}$. *Then* $X(G,S)$ *is what is often referred to as a circuit of length* n*:*

(4) $G = \langle a \rangle$, $\quad S = \{a\}$. *Then* $X(G,S)$ *can be drawn as follows:*



(5)

(6) A point: •

(7) A segment: • •

(8) A path of length n: P_n $\underset{0}{\bullet}$ $\underset{1}{\bullet}$ $\underset{2}{\bullet}$ $\underset{n-1}{\bullet}$ $\underset{n}{\bullet}$ $(n \in \{0, 1, 2, \ldots\})$.

Definition 3 *Let X be a graph. A morphism $f : P_n \longrightarrow X$ is again termed a path (of length n). We term $f(0)$ and $f(n)$ the extremities of f in X and say f goes from $f(0)$ to $f(n)$. The path f is called a closed path if $f(0) = f(n)$.*

Definition 4 *A graph is said to be connected if any two vertices are the extremities of at least one path. The maximal connected subgraphs (under inclusion) are called the connected components of a given graph.*

A path $f : P_n \longrightarrow X$ can be identified as a succession of edges

$$e_1, \ldots, e_n$$

where

$$e_i \;=\; f(\underset{i-1}{\bullet} \quad \underset{i}{\bullet}) \qquad (i = 1, \ldots, n) \ .$$

A consecutive pair of edges $e_i,\ e_{i+1}$ is termed a backtracking if $e_{i+1} = \overline{e_i}$. A path f with extremities P and Q is termed a geodesic if it is of minimal length i.e. any path with the same extremities has at least as great a length.

Definition 5 *A graph X is termed a tree if it is connected and every closed path in X of positive length contains a backtracking.*

Examples of trees

(1)

(2) $\bullet \quad \bullet \quad \bullet \quad \bullet$ *an infinite path.*

A rich source of trees comes from graphs of free groups. Here is a simple lemma which clarifies the kinds of Cayley graphs one can get.

Lemma 1 *Let G be a group, S a set of generators of G, $X = X(G,S)$ the Cayley graph of G relative to S. Then the following hold:*
(i) X is connected;
(ii) X contains a loop if and only if $1 \in S$;
(iii) G acts on X without inversion i.e. $ge \neq \overline{e}$ $(e \in E(X),\ g \in G)$.

The proof of Lemma 1 is straightforward. The one feature I want to emphasize is that of an inversion, which plays a crucial role in constructing quotients by group actions. I will return to this in a moment. But first let me look at another example.

Example *Let G be free on x,y, let $S = \{x, y\}$ and let $X = X(G,S)$. Then we claim X is a tree. To begin with, of course, by Lemma 1, X is connected and contains no loops and G acts on X without inversion. Here is an attempt to draw X in the plane:*

It is not hard to see X is a tree, for suppose we have a closed path in X of length $n > 0$, beginning and ending at w. Then it can be viewed as a succession of edges

$$e_1, \ldots, e_n$$

where

$$o(e_1) \;=\; w \;, \quad t(e_n) \;=\; w \;.$$

Now notice

$$t(e_1) \;=\; wz_1 \;=\; o(e_2) \;, \quad t(e_2) \;=\; wz_1z_2 \;=\; o(e_3) \;, \;\ldots\;, \quad t(e_n) \;=\; wz_1 \ldots z_n \;.$$

Here each $z_i \in \{x, y, x^{-1}, y^{-1}\}$. Since $t(e_n) = w$,

$$z_1 \ldots z_n \;=\; 1 \;.$$

So $z_1 \ldots z_n$ is not a reduced product i.e. for some i,

$$z_i z_{i+1} \;=\; 1 \;.$$

This means that the given closed path in X contains a backtracking, as required.

The relevance of groups acting without inversion on a graph is clarified next by Lemma 2. But first we need some additional notions.

Definition 6 *An orientation of a graph X is a decomposition*

$$E \;=\; E_+ \cup E_-$$

of $E = E(X)$ into two disjoint sets E_+ and E_- such that

$$\overline{E_+} \;=\; E_- \quad , \quad \overline{E_-} \;=\; E_+ \;.$$

Every graph has such an orientation since the map $\overline{}$ is of order two and is fixed point free. Thus every orbit has two elements and we can take for E_+ any set of representatives of these

orbits and for E_- the complementary set. Sometimes we then refer to the edges in E_+ as positive edges, those in E_- as negative ones. A graph X with a prescribed orientation is usually referred to as an oriented graph. Most of our diagrams represent oriented graphs.

Definition 7 *A morphism of graphs $f : X \longrightarrow X'$ is termed orientation preserving if there exists orientations of X and X' preserved by f i.e. if there exists an orientation*

$$E(X) \;=\; E_+(X) \cup E_-(X)$$

of X and an orientation

$$E(X') \;=\; E_+(X') \cup E_-(X')$$

of X' such that

$$f\big(E_+(X)\big) \;\subseteq\; E_+(X')\;, f\big(E_-(X)\big) \;\subseteq\; E_-(X')\;.$$

Lemma 2 *Suppose G acts on a graph X. Then G acts without inversion if and only if G is orientation preserving i.e. if there exists an orientation $E = E_+ \cup E_-$ of X preserved by G.*

Proof Decompose E into disjoint G-orbits:

$$E \;=\; \dot{\bigcup}\langle e\rangle$$

where

$$\langle e\rangle \;=\; Ge \qquad \textit{and} \qquad \langle v\rangle \;=\; Gv \qquad (v \in V(X))\;.$$

If G acts without inversion, $\overline{}$ acts as an involution on the orbits $\langle e\rangle$ and we can therefore partition E into $E = E_+ \cup E_-$ in such a way that

$$E_+ \;=\; \bigcup\langle e\rangle \qquad , \qquad E_- \;=\; \bigcup\langle f\rangle\;.$$

The converse is an immediate consequence of this definition. ■

Definition 8 *Let G act without inversion on X. Define the quotient graph $G\backslash X$ as follows. First*

$$o\langle e\rangle \;=\; \langle o(e)\rangle \quad , \quad t\langle e\rangle \;=\; \langle t(e)\rangle \quad , \quad \overline{\langle e\rangle} \;=\; \langle \overline{e}\rangle \;.$$

The point here is that $\overline{}$ makes sense since G acts without inversion on X.

The map

$$f \;:\; X \longrightarrow G\backslash X$$

defined by

$$f \;:\; e \longmapsto \langle e\rangle \quad , \quad v \longmapsto \langle v\rangle$$

is a morphism of graphs.

Example *Let G be free of rank two on x and y, $S = \{x, y, x^{-1}, y^{-1}\}$ and $X = X(G, S)$ as before. So X is a tree. We now compute $G\backslash X$.*

So $G\backslash X$ is a two-leaved rose, reflecting the fact that G is free of rank two. We shall obtain a general stucture theorem for groups acting on a tree. The clue to the structure of these groups will come from an examination of the corresponding quotient graphs, from which we will be able to reconstruct G itself. In order to try to motivate what follows let me digress for a few minutes and take you through a quick trip into covering space theory.

2. Covering space theory

In this quick trip though covering space theory, I will take for granted many elementary notions of topology, such as a topological space, arc-wise connectedness and the fundamental group $\pi_1(X, *)$ of a space X based at a point $*$. All spaces will be Hausdorff, i.e., distinct points have disjoint neighborhoods, and locally arc-wise connected, i.e., if V is an open set containing a point x, there exists an open subset U contained in V and containing x such that any pair of points in U have a path in U joining them.

Definition 9 *Let X and $\widetilde{X}$ be two arc-wise connected, locally arc-wise connected spaces, $p : \widetilde{X} \longrightarrow X$ a continuous map. We term $\left(\widetilde{X}, p\right)$ a covering space of X if*
(i) p is onto;
(ii) each $x \in X$ has an open neighborhood U such that $p^{-1}(U)$ is a disjoint union of open sets homeomorphic via p to U. (Such sets U are usually called elementary neighbourhoods.)

Examples *(1)* $X = S^1 = \left\{ z \in \mathbf{C} \;\middle|\; |z| = 1 \right\}$,
$\widetilde{X} = \mathbf{R}^1$,
$p : \widetilde{X} \longrightarrow X$
$r \longmapsto e^{2\pi i r}$.

(2) $X = \mathbf{P}^2_{\mathbf{R}}$ the real projective plane. Points of X are lines in $\mathbf{R}^3$ through 0. Open sets in X are open "cones" of lines:

$\widetilde{X} = S^2 = \{ (x, y, z) \in \mathbf{R}^3 \mid x^2 + y^2 + z^2 = 1 \}$,
$p : \widetilde{X} \longrightarrow X$ *maps a point $(x, y, z) \in \widetilde{X}$ to the line through 0 passing through $(x, y, z).Noticethatp^{-1}(line) =$ two points, i.e., "p is a so-called 2-sheeted covering".*

(3) $X = T^2 = S^1 \times S^1$ torus,

$$\begin{aligned}\widetilde{X} &= \mathbf{R}^2 \\ p &: \widetilde{X} \longrightarrow X \\ (r_1, r_2) &\longmapsto \left(e^{2\pi i r_1}, e^{2\pi i r_2}\right)\end{aligned}$$

A covering space of a space X is termed a universal covering space if it is simply connected, i.e. if its fundamental group is of order 1. Such universal covering spaces are, in a sense, unique and so are usually referred to as *the* universal covering of the space X. (See the book by William S. Massey: **Algebraic Topology: An Introduction**, published by Harcourt, Brace & World, Inc. (1968), New York, Chicago, San Francisco, Atlanta.) **Definition 10** *Let $\widetilde{X}$ be an arc-wise connected, locally arc-wise connected space. Then a group G is said to act properly discontinuously on $\widetilde{X}$ if it comes equipped with a homomorphism*

$$\varphi \ : \ G \longrightarrow \operatorname{Aut} \widetilde{X} \quad (= \textit{the group of homeomorphisms of } \widetilde{X})$$

such that every point $\widetilde{x} \in \widetilde{X}$ is contained in a so-called proper open neighbourhood V such that

$$V \cap gV \ = \ \emptyset \qquad (g \in G, g \neq 1) \ .$$

Let $X = G\backslash\widetilde{X}$ denote the quotient space of $\widetilde{X}$. Points of X are the orbits $G\widetilde{x}$ of points $\widetilde{x} \in \widetilde{X}$. The topology of X is obtained by taking as a basis for X the sets $U = p(V)$ where V is a proper open set in $\widetilde{X}$. Then the canonical projection $p : \widetilde{X} \longrightarrow X$ makes $\left(\widetilde{X}, p\right)$ a covering space of X.

Theorem 1 *Let $\widetilde{X}$ be simply connected, arc-wise connected and locally arc-wise connected.*

Suppose that G acts properly discontinuously on $\widetilde{X}$. Then

$$G \;\cong\; \pi_1\left(G\backslash\widetilde{X}, *\right) \;.$$

In other words if G acts properly discontinuously on a space $\widetilde{X}$ with the right properties then we can recapture G from the fundamental group of the quotient space $G\backslash\widetilde{X}$ of $\widetilde{X}$. This is, roughly speaking, the plan that we will follow here. More precisely, suppose that a group G acts without inversion on a tree X. We form the quotient graph $G\backslash\widetilde{X}$, keeping track of the stabilisers of some of the vertices and edges in X under the action of G. This information is codified in terms of a so-called *graph of groups*, a subject to which we will turn next. The group G is then recaptured using this information.

3. Graphs of groups

The following definition turns out to be an important tool in studying groups acting without inversion on a tree.

Definition 11 *A pair $(\mathcal{G}, Y)$ satisfying the following conditions is termed a graph of groups:*
(1) Y is a connected graph;
(2) $\mathcal{G}$ is a mapping from $V(Y) \cup E(Y)$ into the class of all groups;
(3) the image of $P \in V(Y)$ under $\mathcal{G}$ is usually denoted by $G_{_P}$ and is termed the vertex group at P or simply a vertex group;
(4) the image of $y \in E(Y)$ under $\mathcal{G}$ is usually denoted by G_y and is termed the edge group at y or simply an edge group;
(5) $G_y = G_{\overline{y}}$ for every $y \in E(Y)$;
(6) each edge group G_y comes equipped with a monomorphism

$$G_y \longrightarrow G_{t(y)} \qquad \textit{denoted by} \qquad a \longmapsto a^y \quad (a \in G_y) \;.$$

We can now amplify a little the comment made above. To this end, suppose that G acts without inversion on a tree X. We then associate with this action a graph $(\mathcal{G}, Y)$ of groups. The graph Y is the quotient graph $G\backslash X$. The vertex groups and edge groups of the graph $(\mathcal{G}, Y)$ of groups are stabilisers of a carefully selected set of edges and vertices of X. Now given any graph $(\mathcal{G}, Y)$ of groups we associate to it a group which is analogous to the fundamental group of a topological space, termed its fundamental group and denoted by $\pi_1(\mathcal{G}, Y)$. This group $\pi_1(\mathcal{G}, Y)$ is constructed from its vertex groups and edge groups by using amalgamated products and HNN extensions. The point here is that if we go back to our given group G acting without inversion on a tree X and construct the corresponding graph of groups $(\mathcal{G}, Y)$, then it turns out that

$$G \cong \pi_1(\mathcal{G}, Y).$$

This then yields the desired structure theorem for groups acting without inversion on a tree.

Finally, the theory of groups acting on trees is completed by proving that every such fundamental group $\pi_1(\mathcal{G}, Y)$ of a graph of groups $(\mathcal{G}, Y)$ acts also on a suitably defined tree. This theorem contains, in particular, our earlier observations that free groups, amalgamated products, and HNN extensions act on trees. Now if a group acts without inversion on a tree, so does every one of its subgroups. This means, e.g., that a subgroup of an amalgamated product or of an HNN extension is the fundamental group of a graph of groups. Consequently we have obtained subgroup theorems for amalgamated products and HNN extensions. These subgroup theorems were first obtained by A. Karrass and D. Solitar: *The subgroups of a free product of two groups with an amalgamated subgroup*, **Transactions of the American Math. Soc. vol. 150**, pp. 227-250 (1970).

We give some important examples of graphs of groups and describe the associated fundamental groups.

Examples *(1) A loop of groups.*

Y : $(\mathcal{G}, Y)$:

So a loop of groups consists of a group G_P, a second group G_y and two monomorphisms of G_y into G_P:

$$\begin{array}{ccc} G_y \longrightarrow G_P & , & G_y \longrightarrow G_P \\ a \longmapsto a^y & & a \longmapsto a^{\overline{y}} \ . \end{array}$$

Here the fundamental group $\pi_1(\mathcal{G}, Y)$ turns out to be an HNN extension with one stable letter, base group G_P and associated subgroups the two images of G_y.

(2) A segment of groups.

$$Y \ : \ \underset{P}{\bullet} \ {}^{y} \ \underset{Q}{\bullet} \ ; \ \underset{G_P}{\bullet} \ {}^{G_y} \ \underset{G_Q}{\bullet}$$

Here $\pi_1(\mathcal{G}, Y) = G = \{G_P * G_Q; G_y\}$.

As we have already noted such an amalgamated product $G = A \star_U B$ acts on a tree T. This tree is easy enough to describe. We define first $V(T)$ to be the disjoint union of the set $\{gA \mid g \in G\}$ of left cosets of A in G and the set $\{gB \mid g \in G\}$ of left cosets of B in G. $E_+(T)$ is then defined to be the set $\{gU \mid g \in G\}$ of left cosets of U in G, with $o(gU) = gA$ and $t(gU) = gB$. G acts on this graph by left multiplication. In order to prove that T is a tree, we need to prove first that T is connected. The point here is that if $f = a_1 b_1 \ldots a_m b_m$ and $g = \alpha_1 \beta_1 \ldots \alpha_n \beta_n$ where $a_i, \alpha_j \in A$ and $b_i, \beta_j \in B$ then we have the following path in T from fA to gB.

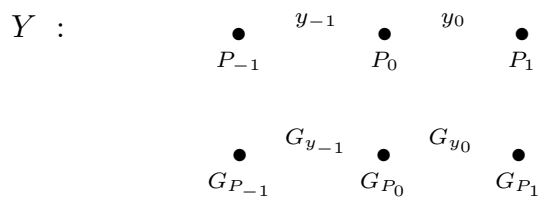
A similar argument shows that there is a path connecting any pair of distinct points in T. In order to verify that T is a tree, we need to prove that there are no closed paths in T of length $n > 0$ which do not contain bactrackings. Since T is connected, it suffices to show that there is no closed path in T of length $n > 0$ based at A. Such a closed path then takes the following form, where $a_i \in A$ and $b_i \in B$:

It follows that $a_1 b_1 \dots a_n b_n \in A$ and then that a pair of consecutive elements, say a_j, b_j belong to the same factor. Assume, for example, that $a_j \in B$. Then

which is the desired backtracking.

(3) A tree of groups.

Here, as in (2), $\pi_1(\mathcal{G}, Y)$ turns out to be a generalized free product. We shall only consider the special case of an infinite path Y, where the graph $(\mathcal{G}, Y)$ and the fundamental group are given below.



Then $\pi_1(\mathcal{G}, Y)$ is described as being generated by the G_{P_i} with the G_{y_i} identified by the indicated isomorphisms.

4. Trees

Lemma 3 *Let X be a graph. Then every tree in X is contained in a maximal tree.*

Proof Since an inductive limit of trees is again a tree, Lemma 3 follows immediately from Zorn's Lemma. ■

Lemma 4 *Let T be a maximal tree in a connected graph X. Then $V(T) = V(X)$.*

Proof Let $v \in V(X),\ v \notin V(T)$. Let $w \in V(T)$. Then there is a path in X joining v to w. We may assume v and w are adjacent. Let e be an edge whose extremities are v and w. Since $v \notin V(T),\ e \notin E(T)$.

Now adjoin to T the edges e and $\overline{e}$ and the vertex v. This then defines a graph $Y \subseteq X$, say. Notice that Y is connected since T is and every closed path in Y must contain a backtracking. ■

Example

The squiggles outline a maximal tree. Notice that there are usually plenty such maximal trees.

Now suppose G acts without inversion on a connected graph $\widetilde{X}$. Then we can form the quotient graph $G\backslash\widetilde{X} = X$, say. Notice that the map

$$p \ : \ \widetilde{X} \longrightarrow X$$

defined by

$$p(v) \ = \ \langle v \rangle \qquad , \qquad p(e) \ = \ \langle e \rangle$$

where $v \in V(\widetilde{X}),\ e \in E(\widetilde{X}),\ \langle v \rangle = Gv,\ \langle e \rangle = Ge$, is a morphism of graphs. Moreover since $\widetilde{X}$ is connected, so is X. Let T be a maximal tree in X. We say T lifts to a tree $\widetilde{T}$ in $\widetilde{X}$ if $p \,\big|\, \widetilde{T}$ is an isomorphism between $\widetilde{T}$ and T.

Lemma 5 *T lifts to a tree $\widetilde{T}$ in $\widetilde{X}$.*

Proof Let $\widetilde{T}_1$ be a tree in $\widetilde{X}$ which is maximal subject to p mapping $\widetilde{T}_1$ injectively into T. We want to prove that

$$p\left(\widetilde{T}_1\right) \ = \ T \ .$$

Suppose the contrary. Then there is a vertex $\langle v \rangle \in V(T)$ such that $\langle v \rangle \notin V(p(\widetilde{T}_1))$. Let $\langle w \rangle \in V(p\left(\widetilde{T}_1\right))$. Then there is a path in T from $\langle w \rangle$ to $\langle v \rangle$. After replacing $\langle w \rangle$ and $\langle v \rangle$ by different vertices if necessary, we may assume $\langle w \rangle$ and $\langle v \rangle$ are adjacent vertices in T. Let $\langle e \rangle$ be the edge of T with

$$o\big(\langle e \rangle\big) \ = \ \langle w \rangle \qquad , \qquad t\big(\langle e \rangle\big) \ = \ \langle v \rangle \ .$$

Notice that

$$\langle e \rangle \ = \ Ge \quad , \quad \langle w \rangle \ = \ Gw \quad , \quad \langle v \rangle \ = \ Gv \ .$$

In particular since $\langle w \rangle \in p(\widetilde{T}_1)$ we find that

$$gw \ \in \ \widetilde{T}_1 \qquad for\ some \quad g \in G \ .$$

Consider now the edge

$$ge \ \in \ \widetilde{X} \ .$$

Then $ge \notin \widetilde{T}_1$ because

$$p(ge) \;=\; \langle e\rangle \;.$$

Now adjoin to $\widetilde{T}_1$ the edges $ge,\ g\,\overline{e}$ and the vertex $t(ge)$. Since

$$p\,t(ge) \;=\; t\langle e\rangle \;=\; \langle v\rangle,$$

$t(ge) \notin \widetilde{T}_1$. It follows then, as in the proof of Lemma 4, that $\widetilde{T}_2 = \widetilde{T}_1 \cup \{ge, g\,\overline{e}, t(ge)\}$ is a tree and p is injective on $\widetilde{T}_2$. So $p\left(\widetilde{T}_1\right) = T$ after all. ■

Definition 12 *We sometimes term a lift $\widetilde{T}$ of a maximal tree T in $G\backslash\widetilde{X}$ a tree of representatives (modulo G).*

Notice that every vertex of $\widetilde{X}$ is in one of the subtrees $g\widetilde{T}$ ($g \in G$), where $\widetilde{T}$ is a tree of representatives modulo G. For if $v \in V(\widetilde{X})$, then $\langle v\rangle \in V(X)$. So $\langle v\rangle \in V(T)$. Hence

$$\langle v\rangle \;=\; Gv$$

has a unique preimage gv in $\widetilde{T}$ where $g \in G$. Therefore

$$v \;\in\; V(g^{-1}\widetilde{T}) \;.$$

If $\langle e\rangle \in E(X)$ and $py = \langle e\rangle$ then we sometimes term y a lift of $\langle e\rangle$. Notice that

$$py \;=\; Gy \;=\; Ge$$

and hence

$$y \;=\; ge$$

for some $g \in G$. So the lifts of $\langle e\rangle$ are simply the elements in the G-orbit of e. Notice also that $o(\langle e\rangle) \in V(T)$. Now $o(\langle e\rangle) = Go(e)$ i.e. the lifts of $o(\langle e\rangle)$ comprise the orbit $Go(e)$. Since $o(\langle e\rangle)$ lifts to a vertex of $\widetilde{T}$ it follows that $\langle e\rangle$ lifts to an edge of $\widetilde{X}$ whose origin is in $\widetilde{T}$. We shall have need of such lifts later.

5. The fundamental group of a graph of groups

Let $(\mathcal{G}, Y)$ be a graph of groups. We shall define the fundamental group $\pi_1(\mathcal{G}, Y)$ of $(\mathcal{G}, Y)$ in two ways.

The first way involves a maximal tree T in Y. Suppose $y \in E(T)$. then we have two monomorphisms

of the edge group G_y into the vertex groups of y. We then define $G_{_T}$ to be the group generated by the vertex groups $G_{_P}$ $(P \in V(T))$ with the two images of the edge group G_y in the adjacent vertex groups $G_{o(y)}$ and $G_{t(y)}$ identified according to the prescription

$$a^y \;=\; a^{\overline{y}} \qquad (a \in G_y),$$

where here y ranges over all the edges in T. In other words $G_{_T}$ is simply obtained from the vertex groups by repeatedly forming amalgamated products where the graph $(\mathcal{G}, Y)$ of groups determines the subgroups to be amalgamated. The group $G_{_T}$ can be described in more precise terms as follows. We choose first a vertex $v_0 \in V(T)$ and define

$$L_0(T) = \{v_0\}.$$

We term $L_0(T)$ the set of vertices at level 0. We then define $L_{n+1}(T)$, the set of vertices of T at level $n+1$, inductively to consist of those vertices of T which are the terminuses of edges in T whose origins lin in $L_n(T)$. It is worth noting that since T is a tree none of the vertices in $L_n(T)$ are the extremities of an edge in T. Since T is connected

$$V(T) = \bigcup_{n=0}^{\infty} L_n(T).$$

We now define G_T as follows. First we dfine

$$G(0) = G_{v_0}.$$

We then define $G(n+1)$ inductively, assuming that $G(n)$ has already been defined in such a way that it is generated by all the vertex groups at levels at most n. For each $v \in L_{n+1}(T)$ there is a unique edge $y \in E(T)$ such that $o(y) \in L_n(T)$ with $t(y) = v$. We define $G(v)$ to be the generalised free product of $G(n)$ and G_v with G_y amalgamated according to the monomorphisms of G_y into the vertex groups at its extremities given by the the graph of groups $(\mathcal{G}, Y)$. $G(n+1)$ is then defined to be the generalised free product of all of these groups $G(v)$ with $G(n)$ amalgamated. Finally we define

$$G_T = \bigcup_{n=0}^{\infty} G_n.$$

It follows that G_T contains an isomorphic copy of each of the vertex groups G_v $(v \in V(T)$ and that

$$gp(G_{o(y)}, G_{t(y)}) = G_{o(y)} *_{G_y} G_{t(y)}$$

for every edge $y \in E(T)$. It is clear on inspecting the obvious presentation for G_T that this description of G_T does not depend on the choice of v_0. The fundamental group

$$\pi_1(\mathcal{G}, Y, T) \ ,$$

of the graph of groups $(\mathcal{G}, Y)$ at T is then defined to be an HNN extension with possibly infinitely many stable letters. The base group is $G_{_T}$. The choice of the stable letters depends on an orientation, say

$$E(Y) \ = \ E_+ \cup E_- \ .$$

of Y. Then for each edge $y \in E_+,\ y \notin E(T)$ we choose a stable letter t_y and define

$$\pi_1(\mathcal{G}, Y, T) \ = \ \langle \, G_{_T} \ \{t_y \,|\, y \in E_+ - E(T)\} \ | \ t_y a^y t_y^{-1} = a^{\overline{y}} \, (a \in G_y, y \in E_+ - E(T)) \, \rangle \ .$$

Examples

(1) Y : $(\mathcal{G}, Y)$:

$$T \ = \ \{P\} \quad , \quad G_{_T} \ = \ G_{_P} \quad , \quad E_+ \ = \ \{y\}$$

$$\pi_1(\mathcal{G}, Y, T) \;=\; \langle\, G_t\,,\, t_y \;;\; t_y\, a^y\, t_y^{-1} = a^{\overline{y}} \,\rangle$$

So the fundamental group of a loop of groups is simply an HNN extension with a single stable letter.

(2) Y : $\underset{P}{\bullet}$ ^{y} $\underset{Q}{\bullet}$

$(\mathcal{G}, Y)$:

Here T=Y. So

$$\pi_1(\mathcal{G}, Y, T) \;=\; G_{_T} \;=\; \{\, G_{_P} * G_{_Q} \,;\, a^y = a^{\overline{y}}\ (a \in G_y)\,\}\ .$$

Suppose now that $(\mathcal{G}, Y)$ is a graph of groups and that T is a maximal tree in Y. Furthermore, suppose that S is a subtree of T. Then the restriction of $\mathcal{G}$ to S gives rise to a graph of groups, which we denote by $(\mathcal{G}, S)$ and in turn to a corresponding group $G_{_S}$. Let $G(S)$ be the subgroup of $G_{_T}$ generated by its subgroups $G(P)$, where P ranges over all of the vertices of S. Then there is a canonical homomorphism of $G_{_S}$ onto $G(S)$. Our next objective is to prove that this homomorphism is a monomorphism. This will be accomplished by using a process which can be termed 'contraction of subtrees to points 'More precisely, let Y be a graph and let Z be a subgraph of Y. Suppose that Z is the union of a family of disjoint trees Z_i, where here i ranges over an index set I. We now form a new graph denoted Y/Z, the quotient graph of Y by Z. The vertices of Y/Z are denoted by $[v]$, where v is a vertex of Y, and are defined as follows:

$$[v] = v \;if\; v \notin V(Z);\quad [v] = V(Z_i),\; if\; v \in Z_i \;for\; some\; i \in I.$$

The edges of Y/Z are denoted by $[e]$, where e is an edge in Y which is not an edge of Z, and are defined simply by

$$[e] = e.$$

The map $\overline{\ }$ is simply $\overline{\ }$ in Y restricted to the edges of Y which are not in Z and

$$o([e]) = [o(e)], \quad t([e]) = [t(e)].$$

It is easy to deduce from the definition of Y/Z that Y/Z is a tree if and only if Y is a tree.

The following lemma, where we make use of the notation introduced above, holds.

Lemma 6 *Let $(\mathcal{G}, Y)$ be a graph of groups, let T be a maximal tree in Y and let S be a subtree of T. Then the canonical homomorphism of G_S into G_T is a monomorphism.*

We contract the subtree S to a point and turn Y/S into a graph of groups by taking the vertex group at the 'vertex S 'to be the group G_S, the vertex groups at the other vertices to be those given by the graph $(\mathcal{G}, Y)$. Now if $[y]$ is an edge with origin S, then $o(y) \in V(S)$ and we define the monomorphism from $G_{[y]}$ into G_S to be the monomorphism $a \mapsto a^{\overline{y}}$ followed by the inclusion of $G_{o(y)}$ into G_S. The other monomorphisms on the edges which have neither origin nor terminus in T are then those that are provided by the graph of groups $(\mathcal{G}, Y)$. Now T/S is a maximal tree in Y/S and so the vertex group G_S embeds into $G_{T/S}$. If we now write down the natural presentation of $G_{T/S}$ using the natural presentation for G_S, then the resultant presentation is simply a presentation for G_T. But the route whereby this presentation was obtained reveals that G_S maps injectively into G_T with image $G(S)$. this completes the proof of Lemma 6. ■

6. The fundamental group of a graph of groups (continued)

The definition of $\pi_1(\mathcal{G}, Y)$ given above seems to depend on the choice of T. There is a somewhat different approach which reveals that this dependence is illusory. This is our next objective.

Suppose then that $(\mathcal{G}, Y)$ is a graph of groups. We define now a group $F(\mathcal{G}, Y)$ whose definition is dictated by the graph of groups $(\mathcal{G}, Y)$. To this end let us choose an orientation $E(Y) = E_+(Y) \cup E_-(Y)$ of Y. Then $F(\mathcal{G}, Y)$ is an HNN extension of the free product of all

of the vertex groups $G_{_P}$:

$$F(\mathcal{G},Y) \;=\; \langle\, (\underset{P\in V(Y)}{\ast} G_{_P}) \cup E(Y)\,;\; ya^{y}y^{-1} = a^{\overline{y}}\ (a\in G_y,\ y\in E_+(Y))\,\rangle\ .$$

It follows that we can present $F(\mathcal{G},Y)$ as follows:

$$F(\mathcal{G},Y) \;=\; \langle\, (\underset{P\in V(Y)}{\ast} G_{_P}) \cup E(Y)\,;\; \overline{y}=y^{-1}\,,\, ya^{y}y^{-1} = a^{\overline{y}}\ (a\in G_y,\ y\in E(Y))\,\rangle\ .$$

It is easy to obtain a presentation for

$$\pi_1(\mathcal{G},Y,T)\ ,$$

where T is a maximal tree in Y, from $F(\mathcal{G},Y)$. Indeed define

$$Q(\mathcal{G},Y,T) \;=\; \langle\, F(\mathcal{G},Y) \;\big|\; y=1 \ \ if\ \ y\in E(T)\,\rangle\ .$$

In other words we add to $F(\mathcal{G},Y)$ the relations $y=1$ for every edge y of T. So if we denote the image of y in $Q(\mathcal{G},Y,T)$ by t_y $(y\in E(T))$ then in $Q(\mathcal{G},Y,T)$

$$a^{y} \;=\; t_y a^{y} t_y^{-1} \;=\; a^{\overline{y}}\ (a\in G_y).$$

Since $t_y=1$ if $y\in E(T)$, it follows that

$$a^{y} \;=\; a^{\overline{y}}\ (a\in G_y, y\in E(T)).$$

It follows from these remarks that the image of the subgroup $\ast_{P\in V(Y)}G_{_P}$ of the group $F(\mathcal{G},Y)$ in $Q(\mathcal{G},Y,T)$ is (isomorphic to) the group $G_{_T}$ and the t_y, with y ranging over the positive edges not in $E(T)$ (in some orientation of Y), are the stable letters in the HNN extension of $G_{_T}$ that defines $\pi_1(\mathcal{G},Y,T)$. In short

$$Q(\mathcal{G},Y,T) \;\cong\; \pi_1(\mathcal{G},Y,T)\ .$$

We are actually more interested in a *subgroup* of $F(\mathcal{G},T)$, which we will compare with $Q(\mathcal{G},Y,T)$. To this end let P_0 be a chosen vertex in Y and let c be a path in Y with origin P_0. As usual we think of c as a sequence of edges

$$y_1,\ldots,y_n\ .$$

Put $P_i = t(y_i)$ $(i = 1, \ldots, n)$. Notice that $o(y_{i+1}) = t(y_i)$ $(i = 1, \ldots, n-1)$.

Definition 13 *A word of type c in $F(\mathcal{G}, Y)$ is a pair (c, μ) where*

$$\mu \ = \ (r_0, \ldots, r_n)$$

is a sequence of elements

$$r_i \ \in \ G_{P_i} \ (i = 0, \ldots, n)$$

Definition 14 *Let (c, μ) be a word of type c. Then we define*

$$|c, \mu| \ = \ r_0 y_1 r_1 y_2 \ldots y_n r_n \qquad \big(\in F(\mathcal{G}, Y)\big)$$

and say that $|c, \mu|$ is the element or word in $F(\mathcal{G}, Y)$ associated with (c, μ). Notice that when $n = 0$, $|c, \mu| = r_0$.

Definition 15 *Let*

$$\pi_1(\mathcal{G}, Y, P_0) \ = \ \{\, |c, \mu| \ \big| \ c \text{ is a closed path in } Y \text{ beginning (and ending) at } P_0 \,\} \ .$$

It is clear that $\pi_1(\mathcal{G}, Y, P_0)$ is a subgroup of $F(\mathcal{G}, Y)$.

Proposition 1 $\pi_1(\mathcal{G}, Y, P_0) \ \cong \ \pi_1(\mathcal{G}, Y, T)$.

Proof It suffices, by the remarks above, to prove that

$$\pi_1(\mathcal{G}, Y, P_0) \ \cong \ Q(\mathcal{G}, Y, T) \ .$$

We first concoct a homomorphism

$$f \ : \ Q(\mathcal{G}, Y, T) \longrightarrow \pi_1(\mathcal{G}, Y, P_0) \ .$$

In order to do so, suppose $P \in V(Y)$. Then there is a unique geodesic $c_{_P}$ in T joining P_0 to P:

$$c_{_P} \;=\; y_1, \ldots, y_n \;.$$

Put

$$\gamma_{_P} \;=\; y_1 \ldots y_n \;\in\; F(\mathcal{G}, Y) \;.$$

Then for each $x \in G_{_P}$ define

$$x' \;=\; \gamma_{_P}\, x\, \gamma_{_P}^{-1}$$

and for each edge $y \in E(Y)$ define

$$y' \;=\; \gamma_{o(y)} y \gamma_{t(y)}^{-1} \;.$$

Notice that $x', y' \in \pi_1(\mathcal{G}, Y, P_0)$.

Now if $y \in E(T)$, then $y' = 1$. Furthermore, if $y \in E(Y)$,

$$\overline{y}' y' \;=\; \gamma_{o(\overline{y})} \overline{y} \gamma_{t(\overline{y})}^{-1} \cdot \gamma_{o(y)} y \gamma_{t(y)}^{-1} \;=\; 1 \;.$$

And if $a \in G_y$,

$$\begin{aligned} y'(a^y)' y'^{-1} \;&=\; \gamma_{o(y)} y \gamma_{t(y)}^{-1} \cdot \gamma_{t(y)} a^y \gamma_{t(y)}^{-1} \cdot \gamma_{t(y)} y^{-1} \gamma_{o(y)}^{-1} \\ &=\; \gamma_{o(y)} y a^y y^{-1} \gamma_{o(y)}^{-1} \\ &=\; \gamma_{o(y)} a^{\overline{y}} \gamma_{o(y)}^{-1} \\ &=\; \left(a^{\overline{y}}\right)' \;. \end{aligned}$$

We now map a set of generators of $\pi_1(\mathcal{G}, Y, T)$ to $\pi_1(\mathcal{G}, Y, P_0)$ as follows

$$\begin{aligned} x &\longmapsto x' \qquad (x \in G_{_P},\, P \in V(Y)) \\ t_y &\longmapsto y' \qquad \left(y \in E_+ - E(T)\right) \end{aligned} \;.$$

Notice that the images under this mapping satisfy a set of defining relations of $\pi_1(\mathcal{G}, Y, T)$. So, by von Dyck's Lemma, the given mapping defines a homomorphism

$$h \;:\; \pi_1(\mathcal{G}, Y, T) \longrightarrow \pi_1(\mathcal{G}, Y, P_0) \;.$$

Consequently using the canonical isomorphism between $\pi_1(\mathcal{G}, Y, T)$ and $Q(\mathcal{G}, Y, T)$ we have defined a homomorphism

$$f \;:\; Q(\mathcal{G}, Y, T) \longrightarrow \pi_1(\mathcal{G}, Y, P_0) \;.$$

Let now p be the canonical projection of $F(\mathcal{G}, Y)$ onto $Q(\mathcal{G}, Y, T)$ and let i be the restriction of p to $\pi_1(\mathcal{G}, Y, P_0)$:

$$i \;=\; p \,\big|\, \pi_1(\mathcal{G}, Y, P_0) \;.$$

Observe that if $y \in E(T)$, then

$$i(y) \;=\; 1 \;.$$

Hence

$$i(\gamma_{_P}) \;=\; 1 \qquad for\ all \qquad P \in V(Y) \;.$$

Consequently

$$i(x') \;=\; x \;(x \in G_P, P \in V(Y)),\; i(y') \;=\; y \;(y \in E_+ - E(T)).$$

Thus

$$i \circ f \;=\; 1 \;. \tag{1}$$

In order to complete the proof of the proposition it suffices to prove that

$$f \circ i \;=\; 1 \;. \tag{2}$$

Suppose then that c is a closed path with origin P_0, (c, μ) a word of type c and that

$$|c, \mu| \;=\; r_0 y_1 r_1 \ldots y_n r_n \;.$$

Now

$$r'_i \;=\; \gamma_{_{P_i}} r_i \gamma_{_{P_i}}^{-1} \,(i = 0, \ldots, n),\; y'_i \;=\; \gamma_{_{P_{i-1}}} y_i \gamma_{_{P_i}}^{-1} \,(i = 1, \ldots, n).$$

Therefore

$$r'_0 y'_1 r'_1 \ldots y'_n r'_n \;=\; \gamma_{_{P_0}} (r_0 y_1 r_1 \ldots y_n r_n) \gamma_{_{P_0}}^{-1} \;.$$

Since $\gamma_{_{P_0}} = 1$ by definition,

$$r'_0 y'_1 r'_1 y'_n r'_n \;= r_0 y_1 r_1 \ldots y_n r_n$$

i.e. (2) holds. This completes the proof. ■

Corollary 1 *The groups $\pi_1(\mathcal{G}, Y, T)$, $\pi_1(\mathcal{G}, Y, P_0)$ are all isomorphic, hence independent of either the choice of T or P_0.*

Proof Let $P_0,\ \overline{P}_0$ be arbitrarily chosen and fix T. Then

$$\pi_1(\mathcal{G}, Y, P_0) \ \cong\ \pi_1(\mathcal{G}, Y, T) \ \cong\ \pi(\mathcal{G}, Y, \overline{P}_0)\ .$$

(In fact $\pi_1(\mathcal{G}, Y, P_0)$ and $\pi_1(\mathcal{G}, Y, \overline{P}_0)$ are conjugate in $F(\mathcal{G}, Y)$). But if $\overline{T}$ is now any other maximal tree in Y we again find that

$$\pi_1(\mathcal{G}, Y, P_0) \ \cong\ \pi_1(\mathcal{G}, Y, \overline{T})\ .$$

This completes the proof. ■

One more remark before we move on to determine some graphs of groups.

Suppose $(\mathcal{G}, Y)$ is a graph of groups, $y \in E(Y)$. Then we denote the image of G_y in $G_{t(y)}$ under the map

$$a \longmapsto a^y \qquad (a \in G_y)$$

by G_y^y.

Definition 16 *Let (c, μ) be a word of type c where c is a closed path in Y beginning at P_0 with edges $y_1, \ldots, y_n$. Suppose that*

$$\mu \ =\ (r_0, \ldots, r_n)\ .$$

Then we term (c, μ) a reduced word if
(i) $r_0 \neq 1 \quad if \quad n = 0$;
(ii) $r_i \notin G_{y_i}^{y_i} \quad whenever \quad y_{i+1} = \overline{y}_i \quad (i = 1, \ldots, n-1)$.

Now $F(\mathcal{G}, Y)$ is an HNN extension of the free product of the groups G_P. It follows immediately from Britton's Lemma that

Proposition 2 *If (c, μ) is a reduced word, then $|c, \mu| \neq 1$.*

7. Group actions and graphs of groups

Let G be a group acting without inversion on a connected graph $\widetilde{Y}$. The key to understanding the structure of a group acting without inversion on a tree is the fact that we can associate to this action of G on $\widetilde{Y}$, a graph $(\mathcal{G}, Y)$ of groups, where $Y = G\backslash\widetilde{Y}$. To this end, let T be a maximal tree in Y, $\widetilde{T}$ a lift of T in $\widetilde{Y}$. Let

$$p \ : \ \widetilde{Y} \longrightarrow Y$$

be the canonical projection. Then $p^{-1} \,\big|\, T$ is an isomorphism from T to $\widetilde{T}$. Our objective is to extend this isomorphism to a map

$$q \ : \ Y \longrightarrow \widetilde{Y}.$$

it is important to note that q need not be a morphism of graphs, even though $p^{-1} \,\big|\, T$ is such a morphism.

Now q is already defined on T, hence on all the vertices of Y. We need to define q on the edges y of Y. If $y \in E(T)$, then qy has already been defined. We are left with the edges $y \notin E(T)$. Choose an orientation $E(Y) = E_+(Y) \cup E_-(Y)$ of Y. We will define q on the positive edges y not in T and extend q to the negative edges by defining $q\,\overline{y} = \overline{(qy)}$. So let y be a positive edge not in T. Then there exists an edge $\widetilde{y} \in E(\widetilde{Y})$ which is a lift of y such that

$$o\,(\widetilde{y}) \ \in \ V(\widetilde{T}) \ .$$

Define

$$qy \ = \ \widetilde{y} \ .$$

This completes the definition of q. Notice that

$$q\,\overline{y} = \overline{(qy)} \ \ for \ all \ \ y \in E(Y).$$

Let us now put

$$e(y) = \begin{cases} 0 & \text{if } y \in E_+; \\ 1 & \text{if } y \notin E_+. \end{cases}$$

Notice that

$$p\,t\,(\widetilde{y}) \ = \ t(y) \ .$$

So $t\,(\widetilde{y})$ and $q\,t(y)$ are both lifts of $t(y)$. Hence there exists an element $g_y \in G$ such that

$$t\,(\widetilde{y}) \;=\; g_y\; q\,t(y)\;.$$

Furthermore, we define

$$g_{\overline{y}} \;=\; g_y^{-1}\; if\; y \in E_-,\; g_y = 1\; if\; y \in E(T).$$

It follows that

$$g_{\overline{y}} \;=\; g_y^{-1}\; (y \in E(Y))\; and\; g_y = 1\; if\; y \in E(T).$$

It follows that for each $y \in E(Y)$

$$o(qy) = g_y^{-e(y)} q o(y)$$
$$t(qy) = g_y^{1-e(y)} q t(y).$$

We are now in a position to define the graph $(\mathcal{G}, Y)$ of groups. The edge and vertex groups of $(\mathcal{G}, Y)$ are the stabilizers of the images of vertices and edges of Y under q. In detail, then, we define the vertex and edge groups of $(\mathcal{G}, Y)$ as follows:

$$\mathcal{G}(P) \;=\; G_{_{qP}} \;=\; \{\, g \in G \;\big|\; g(qP) = qP \,\} \;\stackrel{\text{def}}{=}\; G_{_P}\big(P \in V(Y)\big);$$

$$\mathcal{G}(y) \;=\; G_{qy} \;=\; \{\, g \in G \;\big|\; g(qy) = qy \,\} \;\stackrel{\text{def}}{=}\; G_y\big(y \in E(Y)\big)\;.$$

We now define the edge monomorphisms

$$a \longmapsto a^y \qquad (a \in G_y, y \in E(Y))$$

by putting

$$a^y \;=\; g_y^{e(y)-1} a g_y^{1-e(y)}\; (y \in E(Y)).$$

The verification that the above prescription does define a graph of groups is straightforward.

We have two things to check. First we have

$$G_y = G_{qy} = G_{\overline{qy}} = G_{q\overline{y}} = G_{\overline{y}}.$$

Second we have to make sure that the mapping $a \longmapsto a^y$ is a monomorphism from G_y into $G_{t(y)}$, i.e. from G_{qy} into $G_{qt(y)}$. Now $qt(y) = g^{e(y)-1}t(qy)$ and $G_{qy} \leq G_{t(qy)}$. Consequently $g_y^{e(y)-1} a g_y^{1-e(y)} \in G_{qt(y)}$. It follows immediately that the mapping $a \longmapsto a^y$ is indeed a monomorphism from G_y into $G_{t(y)}$, as desired.

Since T plays a role here in the definition of $(\mathcal{G}, Y)$ we sometimes denote $(\mathcal{G}, Y)$ by $(\mathcal{G}, Y, T)$ and refer to $(\mathcal{G}, Y, T)$ as the graph of groups $(\mathcal{G}, Y)$ at T.

We are now in a position to formulate the

Theorem 2 *Let G be a group acting without inversion on a tree $\widetilde{Y}$. Then*

$$\pi_1(\mathcal{G}, Y, T) \;\cong\; G$$

where $Y = G\backslash\widetilde{T}$ and $(\mathcal{G}, Y, T)$ is the graph of groups associated to the action of G on Y at the maximal tree T of Y.

The connection between $\pi_1(\mathcal{G}, Y, T)$ and G is not hard to fathom in view of the very definition of $(\mathcal{G}, Y)$. To begin with we want to concoct a homomorphism from $\pi_1(\mathcal{G}, Y, T)$ onto G. Now $\pi_1(\mathcal{G}, Y, T)$ is made up from the vertex groups of Y together with a whole bunch of stable letters t_y, one for each positive edge $y \notin E(T)$ and each vertex group is simply a subgroup of G, namely the stabilizer G_{qP} of a vertex qP in $\widetilde{T}$. Indeed there is a canonical map from $\pi_1(\mathcal{G}, Y, T)$ to G which is the identity on the G_{qP} and maps t_y to g_y. This map defines a homomorphism

$$\Theta \;:\; \pi_1(\mathcal{G}, Y, T) \longrightarrow G$$

because we have arranged that the relations defining $\pi_1(\mathcal{G}, Y, T)$ mimic those that hold between the various subgroups of G. Our first step then in the proof of Theorem 2 is to prove Θ is onto. This is taken care of by the following

Lemma 7 *Let G be a group acting without inversion on a connected graph X, U a tree of representatives of X modulo G and Z a subgraph of X such that*
(i) if $z \in E(Z)$ then either $o(z) \in V(U)$ or $t(z) \in V(U)$;
(ii) $GZ = X$;

(iii) for each $z \in E(Z)$ with $o(z) \in V(U)$ let $g_z \in G$ be such that $g_z^{-1}\, t(z) \in V(U)$.
Then

$$G \;=\; gp\left(\left\{\, G_{_P} \;\middle|\; P \in V(U) \,\right\} \cup \left\{\, g_z \;\middle|\; z \in E(Z)\,,\, o(z) \in V(U) \,\right\}\right) \tag{3}$$

where as usual

$$G_{_P} \;=\; \{\, g \in G \;\big|\; gP = P \,\}\ .$$

To see how Lemma 7 implies that Θ is onto, we take $X = \widetilde{Y},\ U = \widetilde{T},\ Y = G\backslash\widetilde{Y},\ Z = qY$ and the g_z to be the g_y.

Proof of Lemma 7. Let us denote the right-hand-side of (3) by H. Our objective then is to prove $H = G$.

Let $p : X \longrightarrow G\backslash X$ be the canonical projection. By definition pU is a maximal tree in $G\backslash X$ and hence contains all of the vertices of $G\backslash X$. So, as we have already noted previously,

$$V(X) \;=\; V(GU)\ .$$

Suppose that $V(HU) = V(GU)$. Let $g \in G$, $P \in V(U)$. Then $gP \in V(HU)$. So there exists $Q \in V(U),\ h \in H$ such that $gP = hQ$. Since p is injective on U, this implies $Q = P$. Hence

$$h^{-1}g \in G_{_P}\ .$$

So by the very definition of $H,\ h^{-1}g \in H$ and hence so does $h\cdot h^{-1}g = g$. It suffices, therefore, for the proof of Lemma 7, to prove that $V(HU) = V(X)$. Suppose the contrary. This means that there is a vertex Q of X which is not in any of the subtrees hU $(h \in H)$. Let Q be chosen to be as "close" to one of these subtrees as possible, $Q \notin V(HU)$. Since X is connected, we can assume that there is an edge $y \in E(X)$ such that

$$P \;=\; o(y) \;\in\; V(HU) \quad , \quad t(y) \;=\; Q\ .$$

On replacing y by hy, where h is a suitably chosen element of H, we can assume that $P \in V(U)$. Now

$$GZ \;=\; X\ .$$

So there exists $f \in G$ such that

$$z \;=\; fy \;\in\; E(Z)\ .$$

We then have two possibilities.

(1) $o(z) \in V(U)$. Notice that $o(z) = fP$ and $P \in V(U)$. Again using the injectivity of p we deduce that $fP = P$ i.e. $f \in G_{_P} \leq H$. By (iii) of the hypothesis,

$$\left(g_z^{-1} f\right) Q \;\in\; V(U)\ ,$$

where $g_z \in H$. Hence $Q \in V(HU)$ after all.

(2) $t(z) \in V(U)$. There exists again g_z such that

$$g_z^{-1}\, o(z) \;\in\; V(U)\ .$$

Now $o(z) = fP$. So $g_z^{-1}\, fP \in V(U)$. Since $P \in V(U)$ this implies as before that $g_z^{-1}\, f \in G_{_P} \leq H$. This means $f \in H$. But

$$t(z) \;=\; fQ \;\in\; V(U)\ .$$

Therefore

$$Q \;\in\; V(HU)$$

once again.

We have proved the surjectivity of Θ. In order to complete the proof of Theorem 2 we need to find a 'model 'of X. This is the objective of our next section.

8. Universal covers

Let $(\mathcal{G}, Y)$ be a graph of groups, T a maximal tree in Y and

$$G \;=\; \pi_1(\mathcal{G}, Y, T)\ .$$

Suppose that $\widetilde{Y}$ is a tree, that G acts on $\widetilde{Y}$, that

$$G\backslash\widetilde{Y} \xrightarrow{\sim} Y$$

and that the graph of groups that we can associate to this action of G on $\widetilde{Y}$ is (isomorphic in the obvious sense) to $\mathcal{G}(Y,T)$. This means that we can concoct a map, as before, $q : Y \longrightarrow \widetilde{Y}$ which is an injective morphism on T. Now $GV(qT) = V\Big(\widetilde{Y}\Big)$ and $G_{_{qP}} = G_{_P}$; this implies that $V\Big(\widetilde{Y}\Big)$ can be identified with the disjoint union of all of the left cosets $G/G_{_P}$ $\big(P \in V(Y)\big)$. Furthermore $GqE(Y) = E\Big(\widetilde{Y}\Big)$ and $G_{qy} = G_y$. So $E\Big(\widetilde{Y}\Big)$ can be identified with the disjoint union of the set of all the left cosets G/G_y $\big(y \in E(Y)\big)$.

The rest of the graph of groups can also be reconstructed in a similar way, as we have already detailed before.

Our objective here , given a graph $(\mathcal{G}, Y)$ of groups, T a maximal tree in Y and $G = \pi_1(\mathcal{G}, Y, T)$, the fundamental group of this graph of groups, is to reverse the process described above. That is, to construct a graph $\widetilde{Y} = \widetilde{Y}(\mathcal{G}, Y, T)$ and an action of G on $\widetilde{Y}$ so that the graph of groups associated to this action is (isomorphic to) $(\mathcal{G}, Y)$.

To this end, let $E(Y) = E_+(Y) \cup E_-(Y)$ be an orientation of Y. For each $y \in E_+(Y)$, define

$$G(y) = G_{\overline{y}}^{\overline{y}} \; (\leq G_{t(\overline{y})} = G_{o(y)}).$$

We now *define* $\widetilde{Y}$ as follows :

(i) $V\Big(\widetilde{Y}\Big) = \bigcup\limits_{P \in V(Y)}^{\cdot} G/G_{_P} \qquad \big(G/G_{_P} = \{\, gG_{_P} \mid g \in G \,\}\big)$.

(ii) $E_+\Big(\widetilde{Y}\Big) = \bigcup\limits_{y \in E_+(Y)}^{\cdot} G/G_y \qquad \big(G/G(y) = \{\, gG_y \mid g \in G \,\}\big)$.

(Notice that $E_-(Y)$ is then automatically taken to be the set of 'inverse 'edges of the edges in $E_+(\widetilde{Y})$.)

(iii) If $y \in E_+(Y)$ then

$$o\,(gG(y)) \;=\; gG_{o(y)} \quad , \quad t\,(gG(y)) \;=\; g\,t_y\,G_{o(y)}\ .$$

We need to verify that these definitions are unambiguous, i.e. do not depend on the choice of the representatives of the cosets $gG(y)$. November 18 $\widetilde{Y}$ is a graph, by definition. We term $\widetilde{Y}$ the *universal covering* of Y relative to the graph of groups $(\mathcal{G}, Y)$ at T.

Notice that G acts without inversion on $\widetilde{Y}$ in the obvious way, by left multiplication. Notice also that

$$V\Big(G\backslash\widetilde{Y}\Big) \;=\; \big\{\, \langle G_{_P}\rangle \;\big|\; P \in V(Y)\,\big\}$$

$$E_+\Big(G\backslash\widetilde{Y}\Big) \;=\; \big\{\, \langle G_y\rangle \;\big|\; y \in E_+(Y)\,\big\}$$

Thus the maps

$$\langle G_{_P}\rangle \longmapsto P \quad , \quad \langle G_y\rangle \longrightarrow y$$

define a morphism of graphs – indeed an isomorphism

$$G\backslash\widetilde{Y} \xrightarrow{\sim} Y\ !$$

The point of this construction is contained in the following

Theorem 3 *$\widetilde{Y}$ is a tree.*

There are two steps in the proof. The first is that $\widetilde{Y}$ is connected. This follows from the fact that G is generated by the $G_{_P}$ together with the t_y. The second step in the proof is to show that every closed path in $\widetilde{Y}$ of length $n > 0$ contains a backtracking. But every such closed path corresponds to a word w in the elements of the $G_{_P}$ and the t_y with $w = 1$. It follows from Britton's Lemma and the normal form theorem for amalgamated products that w has a special form which implies that the given path does indeed have the desired backtracking. Alternatively we can deduce that $\widetilde{Y}$ has no circuits from Proposition 2.

Examples *(1) $(\mathcal{G}, Y)$ a loop of groups.*

$$G \;=\; \pi_1(\mathcal{G}, Y, T) \;=\; \langle\, G_{_P}\,,\, t_y \;;\; t_y\, a^y\, t_y^{-1} = a^{\overline{y}} \,\rangle\ .$$

$$V\Big(\widetilde{Y}\Big) \;=\; G/G_{_P} \;=\; \big\{\, gG_{_P} \;\big|\; g \in G \,\big\}$$

$$E_+\Big(\widetilde{Y}\Big) \;=\; G/G_y \;=\; \big\{\, gG_y \;\big|\; g \in G \,\big\}\ .$$

Thus the positive edges of $\widetilde{Y}$ *all have the form*

$$\underset{gG_{_P}}{\bullet} \overset{gG_y}{\qquad} \underset{g\,t_y\,G_{_P}}{\bullet}\ .$$

It is instructive to try to see why $\widetilde{Y}$ *is connected. The thing to notice is that if* $g \in G$*, then*

$$g \;=\; g_0\, t_y^{\varepsilon_1}\, g_1\, \ldots\, t_y^{\varepsilon_n}\, g_n$$

where

$$g_i \;\in\; G_{_P} \quad (i = 0, \ldots, n)\ ,\ \varepsilon_i = \pm 1 \quad (i = 1, \ldots, n)\ .$$

One wants to find a path e.g. from

$$G_{_P} \qquad \textit{to} \qquad gG_{_P}\ .$$

Let's look at the case $n = 1,\ \varepsilon_1 = 1$*, which is nice and easy but instructive: we have the edge*

$$\begin{array}{ccc} \underset{G_{_P}}{\bullet} & & \underset{g_0\,t_y\,g_1\,G_{_P}}{\bullet} \\ \| & & \| \\ g_0 G_{_P} & & g_0\,t_y\,G_{_P} \end{array}$$

(2) $(\mathcal{G}, Y)$ *a segment of groups.*

We have already described the universal covering $\widetilde{Y}$ *of this segment of groups at the maximal tree* T *in* Y *which here is the graph* Y *itself. We remind the reader that if*

$$Y \ : \ \underset{P}{\bullet} \quad {}^{y} \quad \underset{Q}{\bullet}$$

then

$$V\left(\widetilde{Y}\right) \ = \ G/G_{_P} \bigcup\!\!\!\cdot\, G/G_{_Q}$$

$$E_{+}\left(\widetilde{Y}\right) \ = \ G/G_y$$

and the positive edges of $\widetilde{Y}$ *all take the form*

$$Y \ : \ \underset{gG_{_P}}{\bullet} \quad {}^{gG_y} \quad \underset{gG_{_Q}}{\bullet}$$

9. The proof of Theorem 2

The easiest way to prove Theorem 2 is to make use of the existence of a universal covering. Suppose then that G acts without inversion on a tree X and let $Y = G\backslash X$. Form the corresponding graph of groups $(\mathcal{G}, Y)$ associated to this action of G on X relative to the choice of a maximal tree T in Y. Now form

$$H \ = \ \pi_1(\mathcal{G}, Y, T) \ .$$

As we noted before there is an epimorphism

$$\Theta \ : \ H \longrightarrow G \ .$$

Let $\widetilde{Y}$ be the universal covering of Y corresponding to this graph of groups $(\mathcal{G}, Y)$ relative to T. Now the definition of $(\mathcal{G}, Y)$ involves the use of a map

$$q \ : \ Y \longrightarrow X \ .$$

As we have already observed before

$$V(X) \ = \ \left\{ \ g\, q(P) \ \middle| \ g \in G \, , \, P \in V(Y) \ \right\} \ .$$

Define a map φ on $V\left(\widetilde{Y}\right)$ by

$$\varphi \ : \ hG_{_P} \longmapsto \Theta(h)\, q(P) \qquad (h \in H) \ .$$

(Notice that since we are using H to denote $\pi_1(\mathcal{G}, Y, T)$ the $G_{_P}$ are subgroups of H now.) It is clear that φ is surjective. Similarly notice that

$$E_+(X) \ = \ \left\{ \ g\, q(y) \ \middle| \ g \in G \, , \, y \in E_+(Y) \ \right\} \ .$$

Define a map, again denoted φ, on $E_+\left(\widetilde{Y}\right)$ by

$$\varphi \ : \ hG_y \longmapsto \Theta(h) q(y) \ .$$

Again φ is surjective. It is not hard to check that φ is a morphism of graphs. Our objective is to show that φ is an isomorphism. There are two main steps. The first is to show that

$$\Theta \,|\, gp\left(G_{_P} \,\middle|\, P \in V(Y)\right) \longrightarrow G$$

is an isomorphism. In this instance, working inside G now, if we set

$$J \ = \ gp\left(G_{_P} \,\middle|\, P \in V(Y)\right)$$

and if we take

$$W = \text{the connected graph generated by } qT$$

then J acts on W and

$$J\backslash W \ \cong \ T \ .$$

This allows us to prove that $J \cong G_{_T}$ i.e. that

$$\Theta \,|\, gp\left(G_{_P} \,\middle|\, P \in V(Y)\right) \longrightarrow J$$

is an isomorphism.

The second step, which is more graph-theoretical, involves the use of the fact that X is a tree and leads to the conclusion that φ is an isomorphism of graphs and that $G \ \cong \ H$.

10. Some consequences of Theorems 2 and 3

Let $G = \pi_1(\mathcal{G}, Y)$ be the fundamental group of a graph of groups $(\mathcal{G}, Y)$. Then G acts on a tree $\widetilde{Y}$. So if $H \leq G,\ H$ acts also on $\widetilde{Y}$. Hence

$$H \ = \ \pi_1(\mathcal{H}, X)$$

is again a fundamental group of a graph of groups. This simple observation constitutes, in essence, a rather remarkable subgroup theorem, as one sees from the remarks that follow.

We need a definition.

Definition 17 *A group G is said to act freely on a tree if it acts without inversion and only the identity element leaves either a vertex or an edge invariant.*

Then we have the

Theorem 4 *Let G act freely on a tree. Then G is free.*

Proof As usual

$$G \ \cong \ \pi_1(\mathcal{G}, Y, T) \ .$$

Now

$$G \ = \ \langle \, G_{_T} \, , \, t_y \ (y \in E_+(Y) - E(T)) \ ; \ t_y \, a^y \, t_y^{-1} = a^{\overline{y}} \ (a \in G_y, y \in E_+(Y) - E(T)) \, \rangle$$

is therefore an HNN extension with base $G_{_T}$. But $G_{_T}$ is generated by the stabilizers of the vertices of a lift of T. Since G acts freely, these stabilizers are all trivial. Hence $G_{_T} = 1$ and so the a^y are also all 1. Hence

$$G \ = \ \langle \, t_y \ (y \in E_+(Y) - E(T)) \, \rangle$$

is free on the t_y ! ■

Now we have already noted that the Cayley graph of a free group, relative to a free set of generators, is a tree. And the free group then acts freely on this tree. Hence so does every one of its subgroups. It follows then from Theorem 4 that subgroups of free groups are free.

For our next examples we need some extra information.

Lemma 8 *Let G be a group acting on a set X. Then*

$$G_{gx} \;=\; gG_xg^{-1} \quad (g \in G, x \in X) \;.$$

Here G_z denotes the stabilizer of $z \in X$.

The proof is easy and is left to the reader.

Lemma 9 *Let G be a group, $A \leq G$ and let*

$$X \;=\; \left\{\, gA \;\middle|\; g \in G \,\right\}$$

be the set of all left cosets of A in G. Let H now be a second subgroup of G. Then if we let H act on X by left multiplication then

$$H_{_{gA}} \;=\; gAg^{-1} \cap H \;.$$

Again the proof is straightforward and is left to the listener.

We come now to our second illustration.

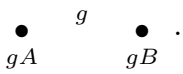
Theorem 5 (A.G. Kurosh) *Let*

$$G \;=\; A * B$$

be the free product of its subgroups A and B and let $H \leq G$. Then H is a tree product of conjugates of subgroups of A and B and a free group.

Proof We note first that G acts on a tree X. Recall

(1) $V(X) \;=\; \{\, gA \;|\; g \in G \,\} \cup \{\, gB \;|\; g \in G \,\}$.
(2) $E_+(X) \;=\; \{\, g \;|\; g \in G \,\}$ (the set of all elements of G since it is only the trivial subgroup that is amalgamated). So a typical positive edge takes the form



Now H acts also on X. So

$$H \;=\; \pi_1(\mathcal{H}, Y, T)$$

where

$$Y \;=\; H\backslash X$$

and T is a maximal tree in Y. So

$$H \;=\; \langle\, H_{_T}\,,\, t_y \;(y \in E_+(Y) - E(T))\,;\; t_y\, a^y\, t_y^{-1} = a^{\overline{y}} \;(a \in H_y\,,\, y \in E_+(Y) - E(T))\,\rangle\ .$$

These edge groups H_y are easy to determine. They are simply all of the form

$$H_y \;=\; H_{qy} \;=\; H_g \;=\; \{\, h \;|\; hg = g \,\} \;=\; 1\ .$$

So the edge groups are all trivial.

Next we need to know the vertex groups $H_{_P}\ \ (P \in V(Y))$. But as before we find either

$$H_{_P} \;=\; H_{_{qP}} \;=\; H_{_{gA}} \;=\; gAg^{-1} \cap H$$

or

$$H_{_P} \;=\; H_{_{qP}} \;=\; H_{_{gB}} \;=\; gBg^{-1} \cap H\ .$$

so $H_{_T}$ is a free product of conjugates of subgroups of A and B (since the edge groups are all trivial) and hence H is a free product of the free group on the t_y and $H_{_T}$, as claimed.

Similarly we can deduce

Theorem 6 (Hanna Neumann, A. Karrass and D. Solitar) *Suppose*

$$G \;=\; A \underset{C}{*} B$$

is an amalgamated product. then every subgroup H of G is an HNN extension of a tree product (i.e. an H_T, T a maximal tree in a graph Y) in which the vertex groups are conjugates of subgroups of either A or B and the edge groups are conjugates of subgroups of C. The associated subgroups involved in the HNN extension are also conjugates of subgroups of C.

We need only recall that G acts on a tree X whose edges are all of the form

$$\underset{gA}{\bullet} \overset{gC}{\qquad} \underset{gB}{\bullet} .$$

One can then deduce the following

Theorem 7 (Karrass and Solitar) *Let*

$$G \;=\; A \underset{C}{*} B$$

where A and B are free and C is cyclic. Then every finitely generated subgroup of G is finitely presented.

11. The tree of SL_2

Our next objective is to show that if F is a field with a discrete valuation then there is a tree X upon which $\mathrm{SL}_2(\mathrm{F})$ acts. Let me begin by recalling some definitions and facts.

Definition 18 *Let F be a commutative field, $\mathrm{F}^* = \mathrm{F} - \{0\}$ viewed as a multiplicative group. Then a surjective map*

$$v \;:\; \mathrm{F} \longrightarrow \mathbf{Z} \cup \{\infty\}$$

is called a discrete valuation if

(i) $v(0) = \infty$ where $a + \infty = \infty = \infty + a$ for every $a \in \mathrm{F}$;

(ii) $v : \mathrm{F}^* \longrightarrow \mathbf{Z}$ *is a surjective homomorphism from the multiplicative group* F^* *to the additive group* $\mathbf{Z}$ *i.e.*

$$v(xy) \;=\; v(x) + v(y) \qquad (x, y \in \mathrm{F}^*) \;;$$

(iii) $v(x+y) \;\geq\; \min\{\, v(x)\,,\, v(y)\,\}$ *[where* $a+\infty = \infty = \infty + a,\ \infty \geq a$ *for every* $a \in F$*]*

.

We form

$$\mathcal{O} \;=\; \left\{\; x \in \mathrm{F} \;\middle|\; v(x) \geq 0 \;\right\}$$

the *valuation ring* of F. Of course $\mathcal{O}$ is a subring of F. It has the additional property that if $a \in \mathrm{F},\ a \neq 0$ then either $a \in \mathcal{O}$ or $a^{-1} \in \mathcal{O}$.

We gather together some properties of $\mathcal{O}$.

Lemma 10 *(i) The non-units of* $\mathcal{O}$ *form a maximal ideal* $\mathcal{M}$*.*
(ii) $\mathcal{O}$ *is a principal ideal domain.*

Proof (i) Notice first that if $a \in \mathcal{O}$ and $v(a) = 0$, then $v(a^{-1}) = -v(a) = 0$. So $a^{-1} \in \mathcal{O}$. Thus the set $\mathcal{M}$ of non-units of $\mathcal{O}$ consists of the elements $a \in \mathcal{O}$ with $v(a) > 0$. This is clearly an ideal by the properties of v. Moreover $\mathcal{M}$ is maximal since the elements outside $\mathcal{M}$ are invertible i.e. $\mathcal{O}/\mathcal{M}$ is a field.

(ii) Let $\mathcal{I} \neq \mathcal{O}$ be a non-zero ideal of $\mathcal{O}$ and let $\pi \in \mathcal{O}$ be chosen so that

$$v(\pi) \;=\; 1 \;.$$

(π is sometimes called a *uniformizer.*) Now if

$$m \;=\; \min\left\{\, v(a) \;\middle|\; a \in \mathcal{I} \right\}$$

we claim that $\mathcal{I}$ is the ideal (π^m) generated by π^m. To see this first note that if $a \in \mathcal{I},\ v(a) = m$ then

$$v(a\pi^{-m}) \;=\; 0 \;.$$

Hence $a\pi^{-m} \in \mathcal{O}$ and is invertible in $\mathcal{O}$; thus

$$\pi^m \;=\; au \;\in\; \mathcal{I} \qquad (u \text{ a unit in } \mathcal{O})\ .$$

So

$$a \;\in\; (\pi^m)\ .$$

That the other elements of $\mathcal{I}$ are also contained in (π^m) requires the use of the Euclidean algarithm and uses the minimality of m.

Incidentally it follows from this argument that π generates $\mathcal{M}$.

Now suppose that V is a 2-dimensional left vector space over F. We can also think of V as a left $\mathcal{O}$-module.

Definition 19 *An $\mathcal{O}$-lattice L of V is any $\mathcal{O}$-submodule of the form*

$$\mathrm{L} \;=\; \mathcal{O}x \;+\; \mathcal{O}y \qquad (xy \in \mathrm{V})$$

where x and y are linearly independent over F.

Now the group F^* acts on the set of all such $\mathcal{O}$-lattices L of V by left multiplication:

$$a\mathrm{L} \;=\; \mathcal{O}ax \;+\; \mathcal{O}ay\ .$$

The orbit of such a lattice L under this action is called its *class* and will be denoted $\mathrm{cl}(\mathrm{L})$. Two $\mathcal{O}$-lattices are termed *equivalent* if they lie in the same class. We denote by X the set of all these classes of $\mathcal{O}$-lattices. Our claim is that X can be thought of as a tree. This will follow in due course.

Now suppose L and L' are $\mathcal{O}$-lattices in V. Then $(\mathrm{L}+\mathrm{L}')/\mathrm{L}'$ is a finitely generated torsion $\mathcal{O}$-module. Now a submodule of a free module over a principal ideal domain is again free. This leads to the conclusion that a finitely generated module over a principal ideal domain is a direct sum of cyclic modules. It follows that

$$((\mathrm{L}+\mathrm{L}')/\mathrm{L}' \;\cong\;)\ \mathrm{L}'/(\mathrm{L}\cap\mathrm{L}' \;\cong\; \mathcal{O}/\pi^c\mathcal{O} \,\oplus\, \mathcal{O}/\pi^d\mathcal{O} \qquad (c,d \geq 0)\ .$$

It follows that

$$\pi^e \mathrm{L}' \ \leq \ \mathrm{L} \quad \text{for some} \quad e \geq 0 \ .$$

Now again the "basis theorem" for submodules of finitely generated free modules over principal ideal domains allows us to choose a basis x, y for L such that

$$\left\{ \pi^f x \, , \, \pi^g y \right\} \ \text{ is a basis for } \ \pi^e \mathrm{L}' \ .$$

Hence

$$\mathrm{L}' \ \text{ has a basis } \left\{ \pi^i x \, , \, \pi^j y \right\} \ .$$

We claim that

$$|i - j| \ = \ |c - d|$$

depends only on $\mathrm{cl}(\mathrm{L})$ and $\mathrm{cl}(\mathrm{L}')$. To see this consider instead the $\mathcal{O}$-lattices $a\mathrm{L}$ and $b\mathrm{L}'$ $(a, b \in \mathrm{F}^*)$. Then $a\mathrm{L}$ has a basis $\{ax, ay\}$ and $b\mathrm{L}'$ a basis $\{b\pi^i x, b\pi^j y\}$. Notice now that if $v(ba^{-1}) = n$, then $ba^{-1} = u\pi^n$ (u a unit in $\mathcal{O}$). So

$$b \ = \ au\pi^n \ .$$

Hence $b\mathrm{L}'$ has an $\mathcal{O}$-basis $\left\{ \pi^{i+n}(ax) \, , \, \pi^{j+n}(ay) \right\}$ and once again

$$|(i + n) - (j + n)| \ = \ |c - d|$$

as claimed.

If we now denote $\mathrm{cl}(\mathrm{L})$ by Λ and $\mathrm{cl}(\mathrm{L}')$ by Λ' then we *define*

$$\mathrm{d}(\Lambda, \Lambda') \ = \ |c - d|$$

and term $\mathrm{d}(\Lambda, \Lambda')$ the *distance between* Λ *and* Λ'.

We now term Λ and Λ' *adjacent* if

$$\mathrm{d}(\Lambda, \Lambda') \ = \ 1 \ .$$

This allows us to turn X, the set of all such equivalence classes Λ of $\mathcal{O}$-lattices in V, into a graph. Here

$$V(X) \ = \ \left\{ \ \Lambda \ \middle| \ \Lambda = \mathrm{cl}(\mathrm{L}), \ \mathrm{L} \ an \ \mathcal{O} - lattice \ in \ \mathrm{V} \ \right\} \ .$$

We then define

$$E(X) \;=\; \left\{\; (\Lambda,\Lambda') \;\middle|\; \Lambda,\Lambda' \; adjacent \;\right\} \;.$$

For each such edge (Λ,Λ') we define

$$o\big((\Lambda,\Lambda')\big) \;=\; \Lambda \quad , \quad t\big((\Lambda,\Lambda')\big) \;=\; \Lambda' \;.$$

Finally we define

$$\overline{(\Lambda,\Lambda')} \;=\; (\Lambda,\Lambda') \;.$$

What we have left then is to prove the

Theorem 8 *X is a tree.*

The proof is not hard, once one figures out what has to be proved. It can be found on page 70 of Serre's book cited at the beginning of this chapter

Now $\mathrm{GL(V)}$ acts on X in the obvious way and so $\mathrm{SL(V)}$ does as well. Unfortunately $\mathrm{GL(V)}$ acts on X with inversion, but $\mathrm{SL(V)}$ acts without inversion. So our structure theorems apply to $\mathrm{SL(V)}$. Indeed one can e.g. deduce

Theorem 9 (Ihara) $\mathrm{SL}_2(\mathrm{F})$ *is an amalgamated product:*

$$\mathrm{SL}_2(\mathrm{F}) \;=\; \mathrm{SL}_2(\mathcal{O}) \underset{\Gamma}{*} \mathrm{SL}_2(\mathcal{O})$$

where here Γ is the subgroup of $\mathrm{SL}_2(\mathcal{O})$ consisting of the matrices

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad c \equiv 0 \,(\bmod \pi) \;.$$

Now suppose that G is a group and that there exist representations of G in $\mathrm{SL(V)}$. Such representations provide us with an action of G on the tree X of $\mathrm{SL(V)}$ that we discussed above and as a consequence yields a description of G as the fundamental group of a graph of groups:

$$G \;\cong\; \pi_1(\mathcal{G},Y) \;.$$

Unravelling stabilizers of edges and vertices is the next task if this description is to be useful. In the event that one is able to find the right kind of representations of G, the task of geometric representation theory, then this approach does turn out to be useful. This point of view was introduced by Culler and Shalen in their fundamental paper on the fundamental groups of three manifolds. Similar techniques apply also to groups given by generators and defining relations and yield, in particular, a proof of Theorem 8 of Chapter V.