Notes on Combinatorics

Claude Auderset

University of Fribourg

1. Ordered choices

Various combinatorial problems can be viewed as involving the choice of k objects from a set of n objects. The number of ways of making such a choice depends on whether or not a given object may be chosen more than once, and whether or not the order of choice is important.

Ordered choices with repetitions

1.1 Theorem. *The number of lists $(a_1, \ldots, a_k)$ of k (non necessarily distinct) elements of a set of n elements is n^k $(k, n \in \mathbb{N})$. This number is to be taken 1 if $k = 0$ (including the case where also $n = 0$).*

Proof. The first element a_1 may be chosen in n ways, the second one a_2 in n ways, and so on, k times. So there are $n \cdot n \cdot n \cdots n = n^k$ possibilities. When $k = 0$, the only list of length 0 is the empty list. □

Such lists are also called *k-tuples*. They can be interpreted as

- ◇ words of length k on an alphabet of n letters,
- ◇ ordered samples of k balls drawn from an urn of n balls *with replacement*.

For instance, there are $26^5 = 11\,881\,376$ five letter words using the Roman alphabet, and $2^8 = 256$ eight bit bytes.

Let A be a set of n elements and $B = \{b_1, \ldots, b_k\}$ a set of k elements. There is a one-to-one correspondence (bijection) between the set A^B of maps $f : B \to A$ and the set A^k of lists $(a_1, \ldots, a_k)$ of k elements of A. To each map $f : B \to A$ corresponds the list $(f(b_1), \ldots, f(b_k)) \in A^k$ and, conversely, a list $(a_1, \ldots, a_k) \in A^k$ defines a map $f \in A^B$ by $f(b_i) = a_i$ $(i = 1, \ldots, k)$. Via this correspondence, the theorem above translates into

1.2 Theorem. *Given finite sets A and B,*

$$\left|A^B\right| = |A|^{|B|},$$

where $|X|$ denotes the cardinality (number of elements) of a finite set X. □

Classical interpretations: there are n^k

- ◇ colourings of k distinct objects with a palette of n colours (each object being painted any one of the n colours),
- ◇ distributions of k distinct objects into n boxes (not considering the order of objects in the boxes).

The *power set* $\mathfrak{P}(X)$ of a set X consists of all subsets of X (including the empty set $\emptyset$). There is a one-to-one correspondence between $\mathfrak{P}(X)$ and $\{0,1\}^X$. To each subset A of X corresponds its *indicator function* (also known as its *characteristic function*)

$$\begin{aligned} \chi_A : X &\longrightarrow \{0,1\} \\ x &\longmapsto \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A. \end{cases} \end{aligned}$$

Conversely, to each function $f : X \to \{0,1\}$ corresponds the subset $A = \{x \in X \mid f(x) = 1\}$. Using the previous result, we get $|\mathfrak{P}(X)| = \left|\{0,1\}^X\right| = 2^{|X|}$.

1.3 THEOREM. *A set of n elements has 2^n subsets ($n \in \mathbb{N}$).* □

Ordered choices without repetitions

1.4 THEOREM. *The number of lists $(a_1, \dots, a_k)$ of k* distinct *elements of a set of n elements is given by the* falling factorial

$$[n]_k = \underbrace{n(n-1)\cdots(n-k+1)}_{k \text{ times}} = \prod_{i=1}^{k}(n-i+1) \qquad (k, n \in \mathbb{N}).$$

This number is taken to be 1 if $k = 0$ (inclusively when $n = 0$ too), and 0 if $k > n$.

PROOF. Suppose that $0 < k \le n$. The first element a_1 may be chosen in n ways, the second one a_2 in only $n-1$ ways since it must be different from a_1, and so on. The i-th choice ($i = 1, \dots, k$) is from the $n-(i-1)$ remaining elements only.

When $k = 0$, there is exactly one list of length k, namely the empty list. When $k > n$, there is no list of k distinct elements among n elements. □

Recall that a map $f : B \to A$ is *injective* when $f(b) = f(b') \Rightarrow b = b'$ or, equivalently, $b \neq b' \Rightarrow f(b) \neq f(b')$ for all $b, b' \in B$. With this terminology,

1.5 THEOREM. *Let A and B be finite sets with $|A| = n$ and $|B| = k$. The number of injective maps from B into A is $[n]_k$.* □

Classical interpretations: there are $[n]_k$

- ◇ ordered samples of k balls drawn from an urn of n balls (without replacement),
- ◇ ordered hands of k cards from a deck of n cards,
- ◇ distributions of k distinct objects into n boxes subject to the *exclusion principle*, i.e., every box containing at most one object.

In the case $|A| = n = k = |B|$, a map $f : B \to A$ is injective if and only if it is surjective, i.e., for every $a \in A$ there is some $b \in B$ such that $f(b) = a$. So, in this case, the three notions 'injective', 'surjective' and 'bijective' (= injective + surjective) coincide. A *permutation* is a bijective map from a set into itself.

1.6 THEOREM. *The number of permutations of a set of cardinality $n \in \mathbb{N}$ is given by the* factorial

$$n! = [n]_n = \underbrace{1 \cdot 2 \cdots n}_{n \text{ times}} = \prod_{i=1}^{n} i.$$

This number is to be taken 1 if $n = 0$. □

For instance, the letters of the word 'STOP' can be arranged in $4! = 24$ ways.

Let us mention without proof the following way of computing the factorial of large numbers.

1.7 STIRLING'S APPROXIMATION

$$n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \qquad (n \in \mathbb{N}, n > 0)$$

with a relative error $< 1/11n$.

1.8 THEOREM. *The number of distributions of k distinct objects into n boxes,* considering the order of the objects in each box, *is given by the* raising factorial

$$[n]^k = \underbrace{n \cdot (n+1) \cdots (n+k-1)}_{k \text{ times}} = \prod_{i=1}^{k}(n+i-1) \qquad (k, n \in \mathbb{N}).$$

This number is taken to be 1 *if* $k = 0$.

PROOF. The first object may be put in one of the boxes in n ways. The second object may be put

- ◇ either in the same box as the first object with 2 possible positions, left or right of the first object,
- ◇ or in one of the $n-1$ other boxes.

So, there are $2 + (n-1) = n+1$ ways of distributing the second object into one of the n boxes.

Let us look in general at how many ways there are of distributing the i-th object into one of the boxes. At that time, $i-1$ objects have already been distributed, say m_j objects into the j-th box ($j = 1, \ldots, n$, $\sum_{j=1}^{n} m_j = i-1$). The i-th object may be put in $m_j + 1$ ways in the j-th box

- ◇ either by placing it just before one of the m_j objects in the box,
- ◇ or by appending it after the m_j objects.

All together, the number of ways of putting the i-th object in one of the n boxes is

$$\sum_{j=1}^{n}(m_j+1) = \left(\sum_{j=1}^{n} m_j\right) + n = (i-1) + n = n + i - 1.$$

The number of distributions of k objects into n boxes is the product of these numbers where i runs from 1 to k. □

2. UNORDERED CHOICES

Unordered choices without repetitions

2.1 THEOREM. *The number of subsets of cardinality k of a set of cardinality n is given by the* binomial coefficient

$$\binom{n}{k} = \frac{[n]_k}{k!} = \frac{n!}{(n-k)!\,k!} \qquad (k, n \in \mathbb{N}).$$

This number is taken to be 1 *if* $k = 0$, *and* 0 *if* $k > n$.

PROOF. Every subset $\{a_1, \ldots, a_k\}$ of k distinct elements of a set A of cardinality n gives rise to $k!$ lists $(a_{i_1}, \ldots, a_{i_k})$ by permuting the k elements $a_1, \ldots, a_k$. As there are $[n]_k$ lists of k distinct elements of A, the number of subsets of cardinality k of A is $[n]_k/k!$. □

A subset of cardinality k of a set of cardinality n is also called a k-*combination* of n objects. Classical models: there are $\binom{n}{k}$

- ◇ (unordered) samples of k balls drawn from an urn of n balls (without replacement),
- ◇ (unordered) hands of k cards from a deck of n cards,
- ◇ (ordered) lists $(a_1, \ldots, a_k)$ of k elements of an ordered set of cardinality n which are (strictly) *increasing*, i.e., such that $a_1 < a_2 < \cdots < a_k$,
- ◇ distributions of k *identical* objects into n boxes subject to the *exclusion principle*, i.e., with every box containing at most one object,
- ◇ lists of n bits containing k ones.

The symmetry property

$$\binom{n}{k} = \binom{n}{n-k} \qquad (k \le n)$$

is obvious from the formula $\binom{n}{k} = n!/(n-k)!\,k!$. Combinatorial proof: we put the subsets of cardinality k of a set A of cardinality n in one-to-one correspondence with the subsets of cardinality $n-k$ of A by taking their complementary.

2.2 THEOREM (PASCAL'S TRIANGLE). *The binomial coefficients can be computed by the recursive formula*

$$\binom{n}{k} = \begin{cases} 1 & \text{if } k = 0, \\ 0 & \text{if } k \ne 0 \text{ and } n = 0, \\ \binom{n-1}{k} + \binom{n-1}{k-1} & \text{if } k \ne 0 \text{ and } n \ne 0. \end{cases}$$

PROOF. Suppose that k and n are nonzero. Let A_n be a set of cardinality n, a an element of A and put $A_{n-1} = A_n \setminus \{a\}$. The subsets of cardinality k of A_n fall into two disjoint classes according to whether they contain a or not. There are $\binom{n-1}{k}$ subsets of cardinality k of A_n not containing a, since these are the subsets of cardinality k of A_{n-1}. On the other hand, there are $\binom{n-1}{k-1}$ subsets of cardinality k of A_n containing a, since those are obtained from the $\binom{n-1}{k-1}$ subsets of cardinality $k-1$ of A_{n-1} by adding the element a to them. This proves the formula

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1} \qquad (k \ne 0, n \ne 0).$$

If $k = 0$, the only subset of cardinality k is the empty set so that $\binom{n}{k} = 1$. If $k \ne 0$ and $n = 0$, $\binom{n}{k} = 0$ since there are no subsets of cardinality k of $A_n = \emptyset$. □

$n \backslash k$	0	1	2	3	4	5	6	7	8
0	1	0	0	0	0	0	0	0	0
1	1	1	0	0	0	0	0	0	0
2	1	2	1	0	0	0	0	0	0
3	1	3	3	1	0	0	0	0	0
4	1	4	6	4	1	0	0	0	0
5	1	5	10	10	5	1	0	0	0
6	1	6	15	20	15	6	1	0	0
7	1	7	21	35	35	21	7	1	0
8	1	8	28	56	70	56	28	8	1

2.3 Binomial theorem

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k\, b^{n-k}.$$

Proof. If we expand the product

$$(a+b)^n = \underbrace{(a+b)\cdots(a+b)}_{n \text{ times}}$$

term by term, we get a sum of 2^n ordered products $x_1 \cdots x_n$ where the letter $x_i = a$ or b is chosen from the i-th factor of the product $(a+b)^n$ for $i = 1, \ldots, n$. The number of these products containing k a's (hence $n-k$ b's) is $\binom{n}{k}$. So, the coefficient of $a^k\, b^{n-k}$ in the expansion of $(a+b)^n$ is the binomial coefficient $\binom{n}{k}$. □

Unordered choices with repetitions

2.4 Theorem. *The number of k-combinations of n objects* with repetitions *is*

$$\left\langle {n \atop k} \right\rangle = \frac{[n]^k}{k!} = \binom{n+k-1}{k} = \binom{n+k-1}{n-1} \qquad (k, n \in \mathbb{N}).$$

Before proving it, let us mention some classical interpretations of this theorem. There are $\left\langle {n \atop k} \right\rangle$

- ◇ (unordered) samples of k balls drawn from an urn of n balls *with replacement*,
- ◇ lists $(a_1, \ldots, a_k)$ of k elements of an ordered set of cardinality n which are *nondecreasing*, i.e., such that $a_1 \le a_2 < \cdots \le a_k$,
- ◇ distributions of k *identical* objects into n boxes,
- ◇ nonnegative integer solutions $(x_1, \ldots, x_n) \in \mathbb{N}^n$ of the equation

$$x_1 + \cdots + x_n = k.$$

Proof of the theorem. Every distribution of k identical objects into n boxes gives rise to $k!$ distributions of k distinct objects into n boxes with the order of the objects in each box taken into account, by arranging the k distinct objects in the boxes in any one of the $k!$ possible ways. As there are $[n]^k$ distributions of the latter kind, there are $[n]^k/k!$ distributions of the first kind. □

Analogue of Pascal's triangle:

2.5 Theorem. *The number of k-combinations of n objects with repetitions can be computed by the recursive formula*

$$\left\langle {n \atop k} \right\rangle = \begin{cases} 1 & \text{if } k = 0, \\ 0 & \text{if } k \ne 0 \text{ and } n = 0, \\ \left\langle {n-1 \atop k} \right\rangle + \left\langle {n \atop k-1} \right\rangle & \text{if } k \ne 0 \text{ and } n \ne 0. \end{cases}$$

Proof. Suppose that k and n are nonzero. The k-combinations of n objects with repetitions fall into two disjoint classes according to whether they contain the n-th object or not. There are $\left\langle {n-1 \atop k} \right\rangle$ k-combinations of n objects with repetitions not containing the n-th object, since these are the k combinations with repetitions of the $n-1$ first objects. On the other hand, there are $\left\langle {n \atop k-1} \right\rangle$ k-combinations of n objects with repetitions containing the n-th object, since those are obtained from the $\left\langle {n \atop k-1} \right\rangle$ $(k-1)$-combinations of n objects with repetitions by adding the n-th object to them. This proves the formula

$$\left\langle {n \atop k} \right\rangle = \left\langle {n-1 \atop k} \right\rangle + \left\langle {n \atop k-1} \right\rangle \qquad (k \ne 0, n \ne 0).$$

□

$n \backslash k$	0	1	2	3	4	5	6	7	8
0	1	0	0	0	0	0	0	0	0
1	1	1	1	1	1	1	1	1	1
2	1	2	3	4	5	6	7	8	9
3	1	3	6	10	15	21	28	36	45
4	1	4	10	20	35	56	84	120	165
5	1	5	15	35	70	126	210	330	495
6	1	6	21	56	126	252	462	792	1287
7	1	7	28	84	210	462	924	1716	3003
8	1	8	36	120	330	792	1716	3432	6435

Multinomial coefficients

By a *division* of a set X, we mean a list $(X_1, \ldots, X_r)$ of pairwise disjoint subsets of X ($X_i \cap X_j = \emptyset$ if $i \neq j$) whose union $X_1 \cup \cdots \cup X_r$ is the whole set X. This notion is related to, but different from, the notion of *partition* of the set X, which is a set (not a list) of pairwise disjoint, nonempty subsets of X whose union is the whole of X.

2.6 Theorem. *Let X be a set of cardinality n, and $k_1, \ldots, k_r$ nonnegative integers with $k_1 + \cdots k_r = n$. The number of divisions $(X_1, \ldots, X_r)$ of X such that $|X_i| = k_i$ for $i = 1, \ldots r$ is given by the* multinomial coefficient

$$\frac{n!}{k_1! \ldots k_r!},$$

which can also be written as

$$\binom{n}{k_1}\binom{n-k_1}{k_2}\binom{n-(k_1+k_2)}{k_3}\cdots\binom{n-(k_1+\cdots+k_{r-1})}{k_r}.$$

Proof. Every division $(X_1, \ldots, X_r)$ of X with $|X_i| = k_i$ gives rise to $k_1! \ldots k_r!$ lists of n distinct elements of X by putting the elements of $X_1, \ldots, X_r$ one after another in $k_i!$ ways for each X_i. As there are $n!$ lists of n distinct elements of X, there are $n!/k_1! \ldots k_r!$ divisions $(X_1, \ldots, X_r)$ of X with $|X_i| = k_i$.

We get the latter formula by first choosing one of the $\binom{n}{k_1}$ subsets X_1 of cardinality k_1 of X, then one of the $\binom{n-k_1}{k_2}$ subsets X_2 of cardinality k_2 of $X \setminus X_1$, and so on. □

Classical interpretations: there are $n!/k_1! \ldots k_r!$

- ◇ distributions of n objects into r boxes (taking no account of the order of the objects in the boxes) with k_i objects in the i-th box for $i = 1, \ldots, r$,
- ◇ colourings of n objects with a palette of r colours, using the i-th colour for k_i objects ($i = 1, \ldots, r$),
- ◇ lists of n elements of a set $A = \{a_1, \ldots, a_r\}$ of cardinality r containing k_i times the element a_i for $i = 1, \ldots, r$,
- ◇ words of length n on an alphabet $A = \{a_1, \ldots, a_r\}$ containing k_i times the letter a_i for each $i = 1, \ldots, r$.

For instance, the 11 letters of the word 'MISSISSIPPI' can be arranged in

$$11!/4!\,4!\,2! = 34\,650$$

ways.

2.7 MULTINOMIAL THEOREM

$$(a_1 + \ldots + a_r)^n = \sum_{\substack{(k_1,\ldots,k_r)\in\mathbb{N}^r\\ k_1+\ldots+k_r=n}} \frac{n!}{k_1!\ldots k_r!} a_1^{k_1}\cdots a_r^{k_r}.$$

PROOF. If we expand the product

$$(a_1 + \ldots + a_r)^n = \underbrace{(a_1 + \ldots + a_r)}_{n \text{ times}}$$

term by term, we get a sum of r^n ordered products $x_1 \cdots x_n$ where the letter $x_i \in \{a_1, \ldots, a_r\}$ is chosen from the i-th factor of the product $(a_1 + \ldots + a_r)^n$ for $i = 1, \ldots, n$. The number of these products containing k_i times the letter a_i for $i = 1, \ldots, n$ is $n!/k_1! \ldots k_r!$. So, the coefficient of $a_1^{k_1} \cdots a_r^{k_r}$ in the expansion of $(a_1 + \ldots + a_r)^n$ is the multinomial coefficient $n!/k_1! \ldots k_r!$. □

3. GENERATING FUNCTIONS

The *generating function* of a sequence of numbers $(a_k)_{k\in\mathbb{N}} = (a_0, a_1, \ldots)$ is the power series

$$f(z) = \sum_{k=0}^{\infty} a_k z^k = a_0 + a_1 z + a_2 z^2 + \cdots .$$

Usually, we are not much interested in the convergence of this series, so we need not view $f(z)$ as a genuine function, but merely as a *formal* power series.

For example, the generating function of the constant sequence $(1, 1, \cdots)$ is the *geometric series*

$$\frac{1}{1-z} = \sum_{k=0}^{\infty} z^k = 1 + z + z^2 + \cdots \qquad (|z| < 1).$$

This equality can be verified formally by observing that

$$(1 + z + z^2 + \ldots)(1 - z) = (1 + z + z^2 + \cdots) - (z + z^2 + \cdots) = 1.$$

3.1 THEOREM. *Let $A_1, \ldots, A_n$ ($n \in \mathbb{N}$) be subsets of $\mathbb{N}$. For each $k \in \mathbb{N}$, let a_k be the number of nonnegative integer solutions $(x_1, \ldots, x_n) \in \mathbb{N}^n$ of the equation*

$$x_1 + \ldots + x_n = k$$

subject to the restrictions $x_1 \in A_1, \ldots, x_n \in A_n$. The generating function of the sequence $(a_k)_{k\in\mathbb{N}}$ is

$$\sum_{k=0}^{\infty} a_k z^k = \prod_{i=1}^{n} \left(\sum_{k\in A_i} z^k \right) = \left(\sum_{k\in A_1} z^k \right) \cdots \left(\sum_{k\in A_n} z^k \right) \qquad (|z| < 1).$$

PROOF. If we expand the product

$$\left(\sum_{k_1\in A_1} z^{k_1} \right) \cdots \left(\sum_{k_n\in A_n} z^{k_n} \right)$$

term by term, we get a sum of products

$$z^{k_1} \cdots z^{k_n} = z^{k_1+\cdots+k_n} \quad \text{with } (k_1, \ldots, k_n) \in A_1 \times \cdots \times A_n.$$

The coefficient of z^k ($k \in \mathbb{N}$) in this product is the number a_k of n-tuples

$$(k_1, \ldots, k_n) \in A_1 \times \cdots \times A_n$$

such that $k_1 + \cdots + k_n = k$. □

3.2 EXAMPLE. If $A_1 = A_2 = \cdots = A_n = \mathbb{N}$, a_k is the number of nonnegative integer solutions of the equation $x_1 + \ldots + x_n = k$ without further restrictions. According to the theorem, the generating function of the sequence $(a_k)_{k\in\mathbb{N}}$ is

$$\sum_{k=0}^{\infty} a_k\, z^k = \left(\sum_{k\in\mathbb{N}} z^k\right)^n = \frac{1}{(1-z)^n}.$$

To find the coefficients a_k, we expand their generating function $f(z) = 1/(1-z)^n$ into a power series. The *Taylor formula*

$$f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}\, z^k$$

yields in this case

$$\frac{1}{(1-z)^n} = \sum_{k=0}^{\infty} \frac{[n]^k}{k!} z^k = \sum_{k=0}^{\infty} \left\langle {n \atop k} \right\rangle z^k \qquad (|z| < 1).$$

This expansion can also be obtained by deriving $n-1$ times both sides of the geometric series identity $1/(1-z) = \sum_{k=0}^{\infty} z^k$. For instance,

$$\frac{1}{(1-z)^2} = \left(\frac{1}{1-z}\right)' = (1 + z + z^2 + \cdots)' = 1 + 2z + 3z^2 + \cdots = \sum_{k=0}^{\infty} (k+1) z^k.$$

Anyhow, we find that the number a_k of nonnegative integer solutions of the equation $x_1 + \cdots x_n = k$ is $\left\langle {n \atop k} \right\rangle$. This result was already obtained in the last section by direct counting methods. But the generating function method, relying on algebraic computations, often leads to success whereas purely combinatorial methods fail.

3.3 EXAMPLE. If $A_1 = A_2 = \cdots = A_n = \{0,1\}$, a_k is the number of solutions of the equation $x_1 + \ldots + x_n = k$ with each $x_i = 0$ or 1, i.e., the number of lists of n bits containing k ones. According to the theorem, the generating function of the sequence $(a_k)_{k\in\mathbb{N}}$ is

$$\sum_{k=0}^{\infty} a_k\, z^k = (1+z)^n.$$

We find the expansion

$$(1+z)^n = \sum_{k=0}^{\infty} \binom{n}{k} z^k$$

e.g by the Taylor formula. So $a_k = \binom{n}{k}$ for every $k \in \mathbb{N}$, which was already proved in the last section by direct combinatorial methods.

3.4 EXAMPLE. *What is the number a_k of ways of changing k fr in 1 fr, 2 fr and 5 fr coins?*

It is the number of solutions $(y_1, y_2, y_3) \in \mathbb{N}^3$ of the equation $y_1 + 2y_2 + 5y_3 = k$, where y_1, y_2 and y_3 denote the number of coins of 1 fr, 2 fr and 5 fr respectively. With $x_1 = y_1$, $x_2 = 2y_2$ and $x_3 = 5y_3$, it is also the number of solutions of the equation $x_1 + x_2 + x_3 = 0$ subject to the restrictions $x_1 \in A_1$, $x_2 \in A_2$ and $x_3 \in A_3$, where $A_1 = \mathbb{N}$, A_2 consists of all nonnegative even numbers, and A_3 of all nonnegative multiples of 5.

According to the theorem, the generating function of the sequence $(a_k)_{k\in\mathbb{N}}$ is

$$\begin{aligned} f(z) = \sum_{k=0}^{\infty} a_k\, z^k &= \left(\sum_{k=0}^{\infty} z^k\right)\left(\sum_{k=0}^{\infty} z^{2k}\right)\left(\sum_{k=0}^{\infty} z^{5k}\right) \\ &= \frac{1}{1-z}\cdot\frac{1}{1-z^2}\cdot\frac{1}{1-z^5} \\ &= \frac{1}{(1-z)^3(1+z)(1+z+z^2+z^3+z^4)}. \end{aligned}$$

To expand $f(z)$ into a power series, we first decompose it into partial fractions

$$f(z) = \frac{13}{40(1-z)} + \frac{1}{4(1-z)^2} + \frac{1}{10(1-z)^3} + \frac{1}{8(1+z)} + \frac{1+z+2z^2+z^3}{5(1+z+z^2+z^3+z^4)}$$

either by hand or with the help of a symbolic computation program.

According to the formula $1/(1-z)^n = \sum_{k=0}^{\infty} \frac{[n]^k}{k!}\, z^k$ or by termwise derivation of the geometric series, we get

$$\frac{1}{1-z} = \sum_{k=0}^{\infty} z^k, \qquad \frac{1}{(1-z)^2} = \sum_{k=0}^{\infty}(k+1)z^k,$$

$$\frac{1}{(1-z)^3} = \sum_{k=0}^{\infty} \frac{(k+1)(k+2)}{2}\, z^k, \qquad \frac{1}{1+z} = \sum_{k=0}^{\infty}(-1)^k z^k.$$

We expand the last partial fraction of $f(z)$ in the following way

$$\begin{aligned} \frac{1+z+2z^2+z^3}{1+z+z^2+z^3+z^4} &= \frac{(1+z+2z^2+z^3)(1-z)}{(1+z+z^2+z^3+z^4)(1-z)} = \frac{1+z^2-z^3-z^4}{1-z^5} \\ &= (1+z^2-z^3-z^4)(1+z^5+z^{10}+\cdots) \\ &= (1+z^2-z^3-z^4) + (z^5+z^7-z^8-z^9) + \cdots \\ &= \sum_{k=0}^{\infty} \gamma_k z^k, \end{aligned}$$

where

$$\gamma_k = \begin{cases} 1 & \text{if } k \bmod 5 = 0 \text{ or } 2, \\ 0 & \text{if } k \bmod 5 = 1, \\ -1 & \text{if } k \bmod 5 = 3 \text{ or } 4. \end{cases}$$

The coefficients of the expansion $f(z) = \sum_{k=0}^{\infty} a_k\, z^k$ are thus given by

$$\begin{aligned} a_k &= \frac{13}{40} + \frac{k+1}{4} + \frac{(k+1)(k+2)}{20} + \frac{(-1)^k}{8} + \frac{\gamma_k}{5} \\ &= \frac{k^2+8k}{20} + \frac{27+5(-1)^k+8\gamma_k}{40}. \end{aligned}$$

We can put this result into a nicer form by using one of the three ways of rounding a real number x:

- ⋄ $\lfloor x \rfloor$ is the largest integer $\leq x$ (rounding down, floor),
- ⋄ $[x]$ is the closest integer to x, rounding to even when x is halfway between two integers,
- ⋄ $\lceil x \rceil$ is the least integer $\geq x$ (rounding up, ceiling).

As

$$a_k = \frac{k(k+8)}{20} + 1 + \frac{5(-1)^k + 8\gamma_k - 13}{40} \qquad \text{with} \quad -1 < \frac{5(-1)^k + 8\gamma_k - 13}{40} \le 0,$$
$$a_k = \frac{(k+4)^2}{20} + \frac{5(-1)^k + 8\gamma_k - 5}{40} \qquad \text{with} \quad -\frac{1}{2} < \frac{5(-1)^k + 8\gamma_k - 5}{40} < \frac{1}{2},$$
$$a_k = \frac{(k+1)(k+7)}{20} + \frac{13 + 5(-1)^k + 8\gamma_k}{40} \qquad \text{with} \quad 0 \le \frac{13 + 5(-1)^k + 8\gamma_k}{40} < 1,$$

the number of ways of changing k fr in 1 fr, 2 fr and 5 fr coins is

$$a_k = \left\lfloor \frac{k(k+8)}{20} + 1 \right\rfloor = \left[\frac{(k+4)^2}{20} \right] = \left\lceil \frac{(k+1)(k+7)}{20} \right\rceil .$$

k	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
a_k	1	1	2	2	3	4	5	6	7	8	10	11	13	14	16	18	20	22	24

4. PARTITIONS OF INTEGERS

A *partition* of an integer $k \in \mathbb{N}$ is a decomposition of k into a sum $k = x_1 + \cdots + x_n$ ($n \in \mathbb{N}$) of positive integers $x_1, \ldots, x_n$, called *parts* of the partition, *not considering their order*. We usually sort them in decreasing order $x_1 \ge x_2 \ge \cdots \ge x_n \ge 1$.

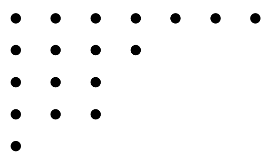
We denote by $p(k)$, resp. $P_n(k)$, resp. $p_n(k)$, the number of partitions of $k \in \mathbb{N}$ into an arbitrary number of parts, resp. into exactly n parts, resp. into at most n parts. For instance, there are $p(5) = 7$ partitions

$$5 = 4+1 = 3+2 = 3+1+1 = 2+2+1 = 2+1+1+1 = 1+1+1+1+1.$$

of the number 5. Obviously,

$$p_n(k) = \sum_{i=1}^{n} P_i(k), \qquad P_n(k) = p_n(k) - p_{n-1}(k), \quad \text{and} \quad p(k) = p_k(k) = \sum_{i=1}^{k} P_i(k).$$

We can display a partition $k = x_1 + \ldots + x_n$ ($x_1 \ge x_2 \ge \cdots \ge x_n$) of the number k into n parts by its *dot diagram*, or *Ferrer's diagram*, which is made up of k dots arranged in n horizontal rows of $x_1, \ldots, x_n$ dots respectively. For example, the dot diagram



represents the partition $7+4+3+3+1$ of 18.

4.1 THEOREM. *If $n \le k$, then $P_n(k) = p_n(k-n)$.*

PROOF. If we delete the first column of a diagram of k dots with n rows, we get a diagram of $n - k$ dots with at most n rows. Conversely, if we put a column of n dots in front of a diagram of $k - n$ dots with at most n rows, we get a diagram of k dots with n rows. This defines a one-to-one correspondence between the partitions of k into n parts and the partitions of $k - n$ into at most n parts. □

If $0 < n \le k$, then $p_n(k) = p_{n-1}(k) + p_n(k-n)$ since $p_n(k) - p_{n-1}(k) = P_n(k) = p_n(k-n)$, so

4.2 THEOREM. *The number of partitions of k into at most n parts can be computed by the recursive formula*

$$p_n(k) = \begin{cases} 1 & \text{if } k = 0, \\ 0 & \text{if } k \neq 0 \text{ and } n = 0, \\ p_{n-1}(k) + p_n(k-n) & \text{if } 0 < n \le k, \\ p_k(k) & \text{if } n > k. \quad \square \end{cases}$$

$k \setminus n$	0	1	2	3	4	5	6	7	8	9	10	11	12	13
0	**1**	1	1	1	1	1	1	1	1	1	1	1	1	1
1	0	**1**	1	1	1	1	1	1	1	1	1	1	1	1
2	0	1	**2**	2	2	2	2	2	2	2	2	2	2	2
3	0	1	2	**3**	3	3	3	3	3	3	3	3	3	3
4	0	1	3	4	**5**	5	5	5	5	5	5	5	5	5
5	0	1	3	5	6	**7**	7	7	7	7	7	7	7	7
6	0	1	4	7	9	10	**11**	11	11	11	11	11	11	11
7	0	1	4	8	11	13	14	**15**	15	15	15	15	15	15
8	0	1	5	10	15	18	20	21	**22**	22	22	22	22	22
9	0	1	5	12	18	23	26	28	29	**30**	30	30	30	30
10	0	1	6	14	23	30	35	38	40	41	**42**	42	42	42
11	0	1	6	16	27	37	44	49	52	54	55	**56**	56	56
12	0	1	7	19	34	47	58	65	70	73	75	76	**77**	77
13	0	1	7	21	39	57	71	82	89	94	97	99	100	**101**

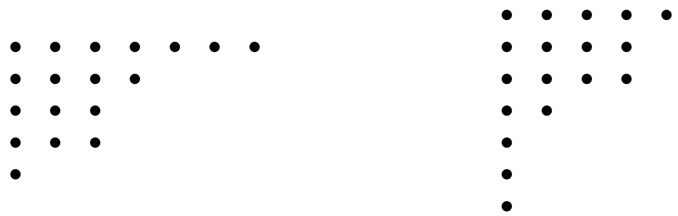
4.3 THEOREM. *The number of partitions of k into n parts can be computed by the recursive formula*

$$P_n(k) = \begin{cases} 1 & \text{if } n = 0 \text{ and } k = 0, \\ 0 & \text{if } n = 0 \text{ and } k \neq 0, \\ P_{n-1}(k-1) + P_n(k-n) & \text{if } 0 < n \le k, \\ 0 & \text{if } n > k. \end{cases}$$

$k \setminus n$	0	1	2	3	4	5	6	7	8	9	10	11	12	13
0	1	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	1	0	0	0	0	0	0	0	0	0	0	0	0
2	0	1	1	0	0	0	0	0	0	0	0	0	0	0
3	0	1	1	1	0	0	0	0	0	0	0	0	0	0
4	0	1	2	1	1	0	0	0	0	0	0	0	0	0
5	0	1	2	2	1	1	0	0	0	0	0	0	0	0
6	0	1	3	3	2	1	1	0	0	0	0	0	0	0
7	0	1	3	4	3	2	1	1	0	0	0	0	0	0
8	0	1	4	5	5	3	2	1	1	0	0	0	0	0
9	0	1	4	7	6	5	3	2	1	1	0	0	0	0
10	0	1	5	8	9	7	5	3	2	1	1	0	0	0
11	0	1	5	10	11	10	7	5	3	2	1	1	0	0
12	0	1	6	12	15	13	11	7	5	3	2	1	1	0
13	0	1	6	14	18	18	14	11	7	5	3	2	1	1

PROOF. Suppose that $0 < n \leq k$. The $P_n(k)$ partitions of k into n parts fall into two disjoint classes according to whether they contain the part 1 or not. There are $P_{n-1}(k-1)$ partitions of k into n parts containing the part 1, since they are obtained from the $P_{n-1}(k-1)$ partitions of $k-1$ into $n-1$ parts by adding the part 1 to them. On the other hand, there are $P_n(k-n)$ partitions of k into n parts not containing the part 1, since their Ferrer diagrams are obtained by putting a column of n dots in front of the Ferrer diagram of one of the $P_n(k-n)$ partitions of $k-n$ into n parts. Thus $P_n(k) = P_{n-1}(k-1) + P_n(k-n)$. □

If we transpose the rows and columns of the dot diagram of a partition of k, we get the dot diagram of another partition of k, called the *transposed*, or *dual*, partition. For instance, the transposed partition of $7+4+3+3+1$ is $5+4+4+2+1+1+1$



4.4 THEOREM. *The number $P_n(k)$ of partitions of k into exactly n parts is equal to the number of partitions of k into parts whose maximum is n.*

The number $p_n(k)$ of partitions of k into at most n parts is equal to the number of partitions of k into parts that are all $\leq n$.

PROOF. In both statements, the two indicated kinds of partitions are in one-to-one correspondence by transposition. □

4.5 THEOREM. *The equation $y_1 + 2y_2 + \cdots + ny_n = k$ has $p_n(k)$ nonnegative integer solutions $(y_1, \ldots y_n) \in \mathbb{N}^n$.*

PROOF. We put the partitions of k into parts that are all $\leq n$ in one-to-one correspondence with the solutions $(y_1, \ldots y_n) \in \mathbb{N}^n$ of the equation $y_1 + 2y_2 + \cdots + ny_n = k$ by defining y_i as the number of occurences of i in such a partition $(i = 1, \ldots, n)$. □

According to the theorem of the previous section,

4.6 THEOREM. *For any $n \in \mathbb{N}$, the generating function of the sequence $(p_n(k))_{k\in\mathbb{N}}$ is*

$$\sum_{k=0}^{\infty} p_n(k) z^k = \frac{1}{(1-z)(1-z^2)\cdots(1-z^n)} \qquad (|z| < 1).$$

Similarily, the generating function of the sequence $(p(k))_{k\in\mathbb{N}}$ is given by the infinite product

$$\sum_{k=0}^{\infty} p(k) z^k = \prod_{i=1}^{\infty} \frac{1}{1-z^i} = \frac{1}{(1-z)(1-z^2)(1-z^3)\cdots} \qquad (|z| < 1).$$

□

For example, we find the explicit formulas

$$p_2(k) = \left\lfloor \frac{k}{2} \right\rfloor + 1 \quad \text{and} \quad p_3(k) = \left[\frac{(k+3)^2}{12} \right]$$

by expanding the generating functions of $(p_2(k))_{k\in\mathbb{N}}$ and $(p_3(k))_{k\in\mathbb{N}}$.

5. LINEAR RECURSIVE EQUATIONS

Consider a sequence of numbers $(a_n)_{n\in\mathbb{N}}$ recursively defined by

$$a_n = \begin{cases} \alpha_n & \text{if } n < r, \\ \gamma_1 a_{n-1} + \gamma_2 a_{n-2} + \cdots + \gamma_r a_{n-r} + \varphi(n) & \text{if } n \geq r, \end{cases}$$

where $\alpha_0, \alpha_1, \ldots, \alpha_{r-1}$ and $\gamma_1, \ldots, \gamma_r$ are given numbers ($\gamma_r \neq 0$), and φ is a function. The equations $a_n = \alpha_n$ for $n < r$ are called *initial conditions*, and $a_n = \gamma_1 a_{n-1} + \gamma_2 a_{n-2} + \cdots + \gamma_r a_{n-r} + \varphi(n)$ a *linear recursive equation*, or *linear recurrence relation, of order r with constant coefficients.* In the special case $\varphi(n) = 0$, the linear recursive equation is called *homogeneous.*

This section presents methods to solve such recursive equations, i.e., to find an explicit formula for a_n.

5.1 EXAMPLE of a first order recursive equation

$$a_n = \begin{cases} 1 & \text{if } n = 0, \\ 2a_{n-1} + 3n + 4 & \text{if } n \geq 1. \end{cases}$$

n	0	1	2	3	4	5	6	7	8	9	10	11	12
a_n	1	9	28	69	154	327	676	1377	2782	5595	11224	22485	45010

Generating function method. We translate this recursive equation into an ordinary equation for the generating function $A(z) = \sum_{n=0}^{\infty} a_n z^n$ according to the following dictionary.

sequence			generating function
0	...	$n \geq 1$	
$a_0 = 1$	...	a_n	$A(z)$
0	...	a_{n-1}	$zA(z)$
1	...	1	$1/(1-z)$
1	...	$n+1$	$1/(1-z)^2$
1	...	0	1
1	...	$2a_{n-1} + 3n + 4$	$2zA(z) + 3/(1-z)^2 + 1/(1-z) - 3$

We find the equation

$$A(z) = 2zA(z) + \frac{3}{(1-z)^2} + \frac{1}{1-z} - 3,$$

with the solution

$$A(z) = \frac{3}{(1-2z)(1-z)^2} + \frac{1}{(1-2z)(1-z)} - \frac{3}{1-2z}$$

or, decomposed into partial fractions,

$$= \frac{11}{1-2z} - \frac{7}{1-z} - \frac{3}{(1-z)^2}.$$

Expanding $A(z)$ gives

$$a_n = 11 \cdot 2^n - 3n - 10.$$

Ansatz method. We first solve the associated homogeneous recursive equation

$$a_n = 2a_{n-1} \qquad (n \geq 1).$$

It is obvious in this case that $a_n = a_0 2^n$. In a general, systematic approach, we try a solution of the form $a_n = \lambda^n$. By substitution into $a_{n+1} = 2a_n$, we get $\lambda^{n+1} = 2\lambda^n$, hence $\lambda = 2$. So, $a_n = 2^n$ is a solution, and the general solution of the homogenous linear recursive equation is a multiple $a_n = \alpha\, 2^n$ of this solution.

Then, we look for a particular solution of the inhomogeneous linear equation $a_n = 2a_{n-1} + 3n + 4$ $(n \geq 1)$. We guess a solution of the form $a_n = \beta n + \gamma$ with some constants β and γ. Substitution yields

$$(\beta + 3)n + \gamma - 2\beta + 4 = 0 \qquad (n \geq 1).$$

Our guess works with $\beta = -3$ and $\gamma = -10$.

The general solution of the inhomogeneous linear recursive equation is the sum

$$a_n = \alpha\, 2^n - 3n - 10$$

of the general solution of the associated homogeneous equation and of our particular solution of the inhomogenous equation. Taking into account the initial value $a_0 = 1$, we find $\alpha = 11$.

5.2 Example of a second order recursive equation

$$a_n = \begin{cases} 2 & \text{if } n = 0, \\ 1 & \text{if } n = 1, \\ a_{n-1} + 6a_{n-2} + 2^n & \text{if } n \geq 2. \end{cases}$$

n	0	1	2	3	4	5	6	7	8	9	10	11
a_n	2	1	17	31	149	367	1325	3655	11861	34303	106493	314359

Generating function method. In terms of $A(z) = \sum_{n=0}^{\infty} a_n z^n$, the recursive equation becomes

$$A(z) = zA(z) + 6z^2 A(z) + \frac{1}{1-2z} + 1 - 3z.$$

sequence				generating function
0	1	...	$n \geq 2$	
$a_0 = 2$	$a_1 = 1$	...	a_n	$A(z)$
0	$a_0 = 2$	...	a_{n-1}	$zA(z)$
0	0	...	a_{n-2}	$z^2 A(z)$
1	2	...	2^n	$1/(1-2z)$
1	0	...	0	1
0	1	...	0	z
2	1	...	$a_{n-1} + 6a_{n-2} + 1$	$zA(z) + 6z^2 A(z) + 1/(1-2z)$ $+1 - 3z$

Solution

$$A(z) = \frac{1}{(1-3z)(1+2z)} \left(\frac{1}{1-2z} + 1 - 3z \right).$$

Decomposition into partial fractions

$$A(z) = \frac{9}{5} \cdot \frac{1}{1-3z} + \frac{6}{5} \cdot \frac{1}{1+2z} - \frac{1}{1-2z}.$$

Expansion

$$a_n = \frac{9}{5}\, 3^n + \frac{6}{5}\, (-2)^n - 2^n.$$

Ansatz method. We first solve the associated homogeneous recursive equation $a_n = a_{n-1} + 6a_{n-2}$ $(n \geq 2)$ or, equivalently, $a_{n+2} - a_{n+1} - 6a_n = 0$ $(n \in \mathbb{N})$. With the Anzatz $a_n = \lambda^n$ we find $\lambda^{n+2} - \lambda^{n+1} - 6\lambda^n$. The *characteristic equation* $\lambda^2 - \lambda - 6 = 0$ has two roots $\lambda_1 = 3$ and $\lambda_2 = -2$. The general solution of the homogeneous equation is a linear combination

$$a_n = \alpha\, 3^n + \beta\,(-2)^n$$

of the solutions $(\lambda_1^n)_{n\in\mathbb{N}}$ and $(\lambda_2^n)_{n\in\mathbb{N}}$.

Then we guess a particular solution $a_n = \gamma\, 2^n$ of the inhomogeneous equation $a_n = a_{n-1} + 6a_{n-2} + 2^n$ $(n \geq 2)$. By substitution, we get $\gamma\, 2^n = \gamma\, 2^{n-1} - 6\gamma\, 2^{n-2} + 2^n$ $(n \geq 2)$, or $\gamma\, 2^{n+2} = \gamma\, 2^{n+1} + 6\gamma\, 2^n + 2^{n+2}$ $(n \in \mathbb{N})$, or $4\gamma\, 2^n = 2\gamma\, 2^n + 6\gamma\, 2^n + 4\cdot 2^n$, so that $\gamma = -1$.

The general solution of the inhomogeneous equation is the sum

$$a_n = \alpha\, 3^n + \beta\,(-2)^n - 2^n$$

of the general solution of the associated homogeneous equation and the particular solution of the inhomogenous equation. The initial conditions yield a system of two linear equations for α and β

$$\begin{aligned} a_0 &= \alpha + \beta - 1 = 2, \\ a_1 &= 3\alpha - 2\beta - 2 = 1, \end{aligned}$$

hence $\alpha = 9/5$ and $\beta = 6/5$.

Fibonacci numbers

The Fibonacci numbers $(F_n)_{n\in\mathbb{N}}$ are recursively defined by

$$F_n = \begin{cases} 1 & \text{if } n = 0 \text{ or } 1, \\ F_{n-1} + F_{n-2} & \text{if } n \geq 2. \end{cases}$$

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
F_n	1	1	2	3	5	8	13	21	34	55	89	144	233	377	610	987

The generating function $F(z) = \sum_{n=0}^{\infty} F_n z^n$ satisfies the equation

$$F(z) = zF(z) + z^2F(z) + 1.$$

With

$$\alpha = \frac{\sqrt{5}+1}{2} \approx 1.618 \quad (\textit{golden ratio}) \quad \text{and} \quad \beta = \alpha - 1 = \frac{\sqrt{5}-1}{2} \approx 0.618,$$

$$F(z) = \frac{1}{1-z-z^2} = \frac{1}{(1-\alpha z)(1+\beta z)} = \frac{1}{\alpha+\beta}\left(\frac{\alpha}{1-\alpha z} + \frac{\beta}{1+\beta z}\right),$$

hence

$$F_n = \frac{\alpha^{n+1} + (-1)^n\beta^{n+1}}{\sqrt{5}}.$$

As $\beta^{n+1}/\sqrt{5} < 1/2$,

5.3 THEOREM. *The Fibonacci numbers are given by the formula*

$$F_n = \left[\frac{\alpha^{n+1}}{\sqrt{5}}\right] \qquad (n \in \mathbb{N}),$$

where α denotes the golden ratio $(1+\sqrt{5})/2$.

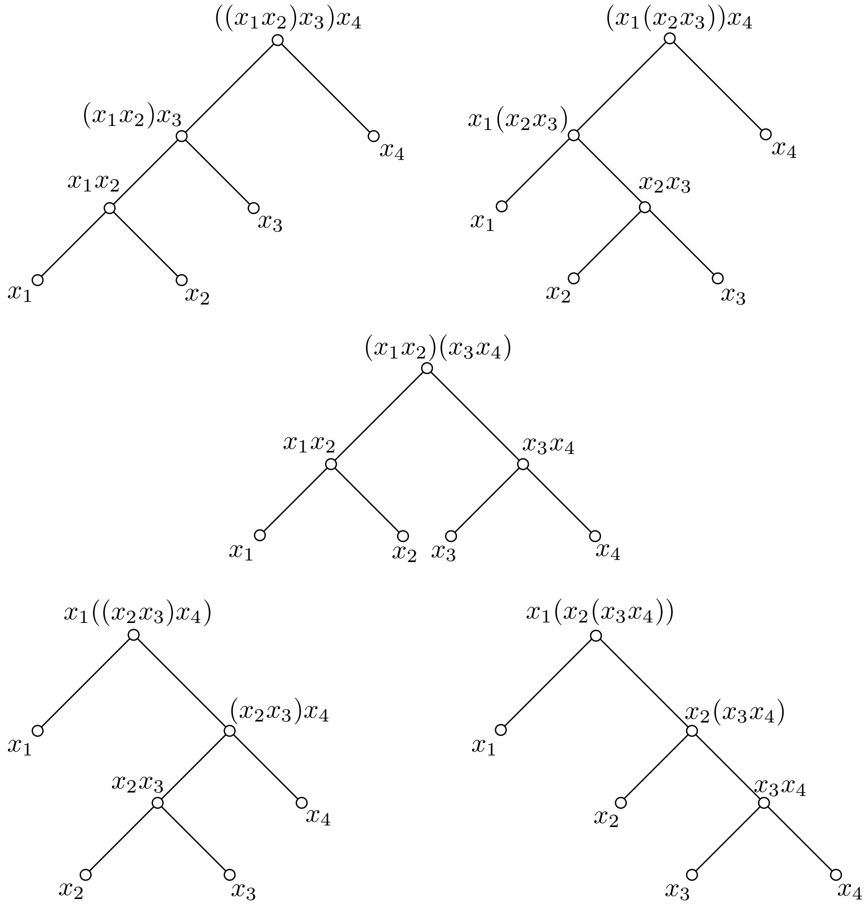
6. Catalan numbers

Bracketings, binary trees and triangulations

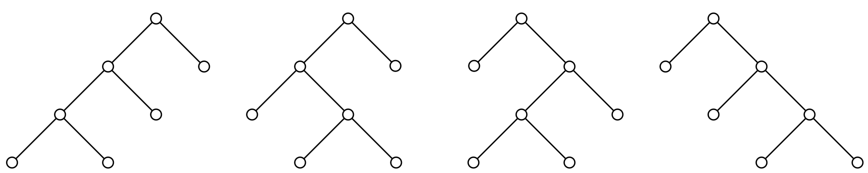
Consider a product of n numbers $x_1, \ldots, x_n$ in this order. The *Catalan number* c_n is the number of ways of putting brackets in this product, each way corresponding to a computation of the product $x_1 \cdots x_n$ by successive multiplications of precisely two numbers. Associativity is used, *but not commutativity.* For instance, the fourth Catalan number is $c_4 = 5$

$$((x_1x_2)x_3)x_4, \quad (x_1(x_2x_3))x_4, \quad (x_1x_2)(x_3x_4), \quad x_1((x_2x_3)x_4), \quad x_1(x_2(x_3x_4)).$$

Bracketings can be represented in tree form



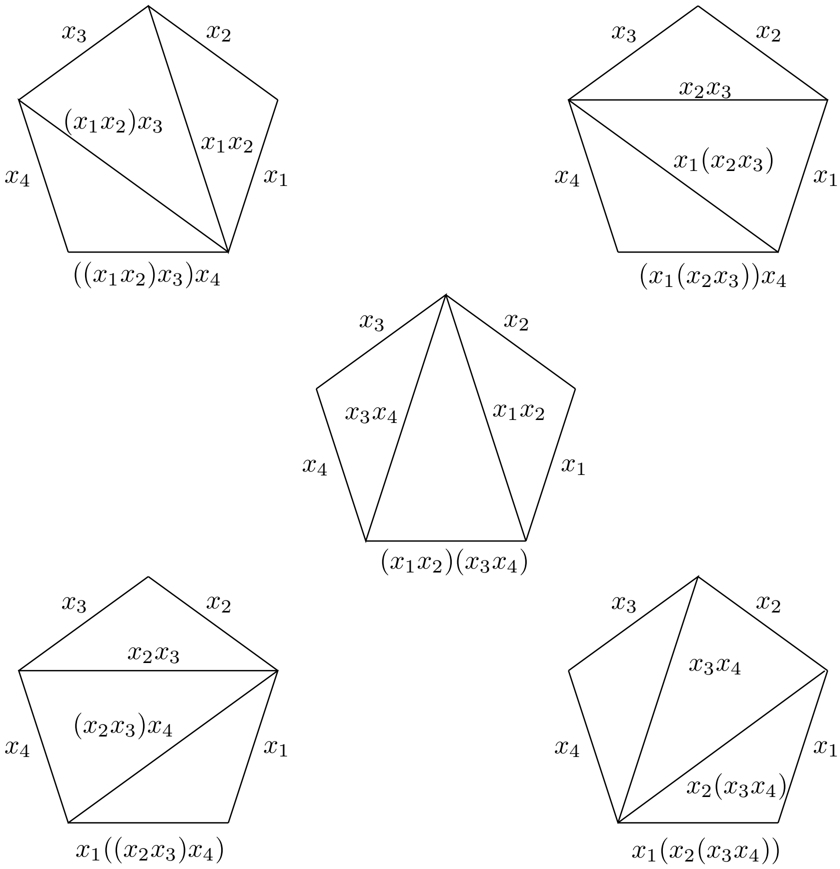
By a *planar tree*, we mean an unlabeled rooted tree, *considering the order of the sons of each node.* For example, the following planar trees, although representing the same unlabeled rooted tree, are distinct.



By a *binary tree*, more precisely a *binary planar tree*, we mean here a planar tree whose nodes have either no sons (*leaves*) or two sons (*inner nodes*). A binary tree with n leaves has $n-1$ inner nodes, $2n-1$ nodes and $2n-2$ edges. As there is a one-to-one correspondence between bracketings of a product of n letters and binary trees with n leaves,

6.1 THEOREM. *The number of binary trees with n leaves is the Catalan number c_n.* □

The following illustration explains a one-to-one correspondence between bracketings and triangulations of convex polygons.



6.2 THEOREM. *The number of triangulations of a convex polygon with n vertices ($n \geq 3$) is the Catalan number c_{n-1}.* □

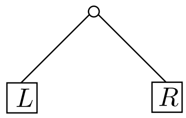
Recursive equation, generating function and explicit formula

6.3 THEOREM. *The Catalan numbers can be computed by the recursive formula*

$$c_n = \begin{cases} 0 & \text{if } n = 0, \\ 1 & \text{if } n = 1, \\ \sum_{k=1}^{n-1} c_k c_{n-k} & \text{if } n \geq 2. \end{cases}$$

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13
c_n	0	1	1	2	5	14	42	132	429	1430	4862	16796	58786	208012

Proof. A binary tree with $n \geq 2$ leaves has the form



where L is a binary tree with say k leaves, and R a binary tree with $n-k$ leaves $(1 \leq k \leq n-1)$. The number of binary trees which split up into two binary trees, L with k leaves, and R with $n-k$ leaves, is the product $c_k c_{n-k}$. All together, the number c_n of binary trees with n leaves is the sum of these products, where k runs from 1 to $n-1$. □

Let us translate the recursive equation of the Catalan numbers into an algebraic equation for their generating function.

$$\begin{aligned}
C(z) &= \sum_{n=0}^{\infty} c_n z^n = z + \sum_{n=2}^{\infty} c_n z^n && (c_0 = 0, c_1 = 1)\\
&= z + \sum_{n=2}^{\infty}\left(\sum_{k=0}^{n} c_k c_{n-k}\right) z^n && \Big(c_n = \sum_{k=1}^{n-1} c_k c_{n-k} = \sum_{k=0}^{n} c_k c_{n-k} \text{ since } c_0 = 0\Big)\\
&= z + \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} c_k c_{n-k}\right) z^n && \Big(\sum_{k=0}^{n} c_k c_{n-k} = 0 \text{ for } n = 0 \text{ and } n = 1\Big)\\
&= z + \sum_{k=0}^{\infty}\sum_{h=0}^{\infty} c_k c_h z^{k+h} && (h = n-k)\\
&= z + \left(\sum_{k=0}^{\infty} c_k z^k\right)\left(\sum_{h=0}^{\infty} c_h z^h\right) && \text{(distributivity)}\\
&= z + C(z)^2,
\end{aligned}$$

hence

$$C(z)^2 - C(z) - z = 0.$$

6.4 Theorem. *The generating function of the Catalan numbers is*

$$C(z) = \sum_{n=0}^{\infty} c_n z^n = \frac{1-\sqrt{1-4z}}{2} \qquad (|z| < 1/4).$$

Proof. In the solution $C(z) = (1 \pm \sqrt{1-4z})/2$ of the second degree equation, the $\pm$ sign must always be the same. It can not depend on the values of z since the generating function $C(z)$ is analytic. Only the minus sign yields the correct value $c_0 = C(0) = 0$. □

We expand $C(z)$ by means of Newton's series (an instance of Taylor's series)

$$(1+z)^\alpha = \sum_{n=0}^{\infty} \binom{\alpha}{n} z^n, \qquad (|z| < 1)$$

which holds for any real number α with the (generalized) binomial coefficients

$$\binom{\alpha}{n} = \frac{[\alpha]_n}{n!} = \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!} \qquad (\alpha \in \mathbb{R}, n \in \mathbb{N}).$$

In our case,

$$\begin{aligned}
\sqrt{1-4z} = (1-4z)^{1/2} &= \sum_{n=0}^{\infty}\binom{1/2}{n}(-4z)^n \\
&= 1 + \sum_{n=1}^{\infty}\frac{\frac{1}{2}(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})\cdots(-\frac{2n-3}{2})}{n!}(-4)^n z^n \\
&= 1 - \sum_{n=1}^{\infty}\frac{1\cdot 3\cdot 5\cdots(2n-3)}{n!\,2^n}\,4^n z^n \\
&= 1 - 2\sum_{n=1}^{\infty}\frac{1\cdot 3\cdot 5\cdots(2n-3)\,2^{n-1}}{n!}\,z^n \\
&= 1 - 2\sum_{n=1}^{\infty}\frac{1\cdot 3\cdot 5\cdots(2n-3)\,2^{n-1}(n-1)!}{n!\,(n-1)!}\,z^n \\
&= 1 - 2\sum_{n=1}^{\infty}\frac{1\cdot 3\cdot 5\cdots(2n-3)\cdot 2\cdot 4\cdot 6\cdots(2n-2)}{n\cdot(n-1)!\,(n-1)!}\,z^n \\
&= 1 - 2\sum_{n=1}^{\infty}\frac{1}{n}\cdot\frac{(2n-2)!}{(n-1)!\,(n-1)!}\,z^n \\
&= 1 - 2\sum_{n=1}^{\infty}\frac{1}{n}\binom{2n-2}{n-1}\,z^n,
\end{aligned}$$

hence

$$C(z) = \sum_{n=0}^{\infty} c_n z^n = \frac{1-\sqrt{1-4z}}{2} = \sum_{n=1}^{\infty}\frac{1}{n}\binom{2n-2}{n-1}\,z^n.$$

6.5 Theorem. *The Catalan numbers are given by the explicit formula*

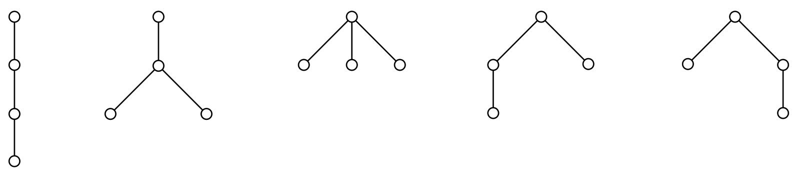
$$c_n = \frac{1}{n}\binom{2n-2}{n-1} \qquad (n \geq 1).$$

□

Planar trees

6.6 Theorem. *The number of planar trees with n nodes is the Catalan number c_n.*

For instance, there are $c_4 = 5$ planar trees of order 4.



To prove the theorem, we define a one-to-one correspondence between planar trees of order n and binary trees with n leaves.

Planar trees $\xrightarrow{b}$ *binary trees.* We associate a binary tree $b(P)$ with n leaves to a planar tree P with n nodes in the following recursive way.

If P has one node only, then $b(P)$ must be the same tree, with one leaf only. Otherwise, we decompose P into two planar trees,

- ◇ the tree F of the eldest (leftmost) son of the root with his descendance,
- ◇ and the tree R of the younger sons, obtained by deleting F from P.

We define the left part of the binary tree $p(B)$ as $b(F)$, and the right part as $b(R)$.

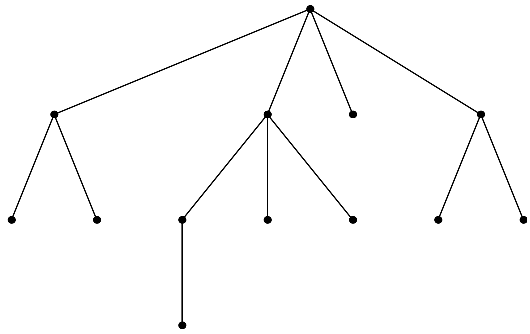
Binary trees $\xrightarrow{p}$ *planar trees.* We associate a planar tree $p(B)$ with n nodes to a binary tree B with n leaves in the following recursive way.

If B has a single leaf, then $p(B)$ is the same tree, with a single node. Otherwise, let L and R be the left and right parts of B. We define the planar tree $p(B)$ in such a way that

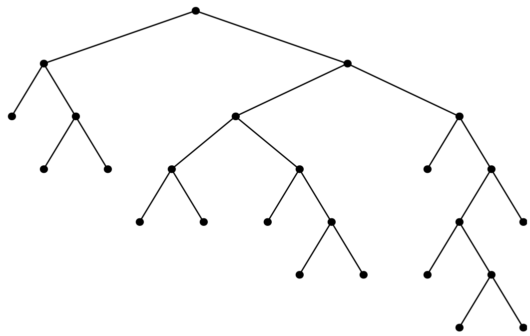
- ◇ the tree of the eldest son of the root of $p(B)$ is $p(L)$,
- ◇ and the rest of the tree $p(B)$, i.e. the tree of the younger sons, is $p(L)$.

Example

Planar tree



Binary tree



A planar tree can be recursively described as the list of the sons of its root. For example, the planar tree above is the list

$$\Big(\big((),()\big),\big(((()),(),()),()\big),(),\big((),()\big)\Big).$$

If we denote the Lisp constructor

$$\langle a,(x_1,\ldots,x_n)\rangle=(a,x_1,\ldots,x_n) \qquad (n\in\mathbb{N}),$$

by $\langle\ ,\ \rangle$, and the empty list by a dot, this list can also be written as

$$\Big\langle\big\langle\cdot,\langle\cdot,\cdot\rangle\big\rangle,\big\langle\langle\langle\cdot,\cdot\rangle,\langle\cdot,\langle\cdot,\cdot\rangle\rangle\rangle,\langle\cdot,\langle\langle\cdot,\langle\cdot,\cdot\rangle\rangle,\cdot\rangle\rangle\big\rangle\Big\rangle.$$

This expression describes the binary tree above in a recursive way, as an ordered pair $\langle L,R\rangle$ of its left and right parts.

In fact, planar trees and binary trees are implemented the same way, with pointers called 'First' and 'Rest' (or 'Next') in the case of planar trees, and 'Left' and 'Right' in the case of binary trees.