TWO-DIMENSIONAL LANDAU-GINZBURG APPROACH TO LINK HOMOLOGY

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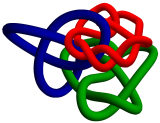
Workshop on quantum fields, knots and integrable systems

Setup

Link sample: 

HOMFLY polynomials $\mathfrak{g} = \mathfrak{su}(N)$ for a link $(\gamma_i:\ S^1 \to \mathcal{M}_3)$, $q = e^{\frac{2\pi i}{\kappa+N}}$

$$P(q, a = q^N | \{\gamma_i, R_i\}) = \int [DA] \left(\prod_i W_{R_i}(\gamma_i) \right) e^{i \frac{\kappa}{4\pi} S_{CS}}$$

During the talk we will be interested mostly in Jones polynomials: $N = 2$.
Khovanov polynomials: bi-graded Poincaré polynomial.

$$\mathscr{K}(q, t) = \sum_{i,j} q^i t^j \dim H^{i,j}$$

It “refines” (categorifies) Jones polynomials $J(q)$:

$$\mathscr{K}(q, -1) = J(q)$$

Our goal: to give a QFT description of link homologies

Statements:

1. Link $L \longrightarrow$ interface in 2d LG theory $\mathfrak{I}(L)$
   Hilbert spaces $\mathcal{H}^{(\mathbf{F},\mathbf{P})}\left[\mathfrak{I}(L)\right]$ are invariants of link L
2. LG link cohomology $\mathrm{LGCoh}(L) := \bigoplus_{(\mathbf{F},\mathbf{P})} \mathcal{H}^{(\mathbf{F},\mathbf{P})}\left[\mathfrak{I}(L)\right]$

   $\mathrm{LGCoh}(L)$ is isomorphic to Khovanov link homology $\mathrm{KHom}(L)$

Plan:

1. Origin of cohomology in QFT
2. Interfaces in LG model
3. Instantons in LG models
4. Tangles as interfaces
5. Explicit examples
6. Obstacles (if time permits)
7. WKB analysis of LG models

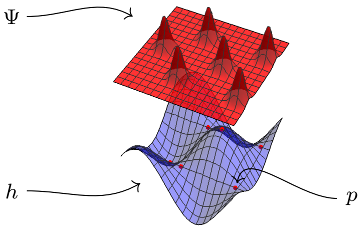
Cohomology origin in QFT. Complex

Supersymmetric Quantum Mechanics (SQM). [Witten '82]
In data: Riemannian Y, real height function $h:\ Y \to \mathbb{R}$

$$L = g_{IJ}\dot{q}^I\dot{q}^J - g^{IJ}\partial_I h \partial_J h + g_{IJ}\bar{\chi}^I\nabla_t\chi^J - g^{IJ}\nabla_I\nabla_J h\, \bar{\chi}^I\chi^J - R_{IJKL}\, \bar{\chi}^I\chi^J\bar{\chi}^K\chi^L$$

Perturbative ground states: Gaussian fluctuations around h-critical points. These states span a Morse-Smale-Witten (MSW) complex as a vector space:

$$\mathcal{E} = \bigoplus_{p:\ dh_p=0} \mathbb{C}|p\rangle$$



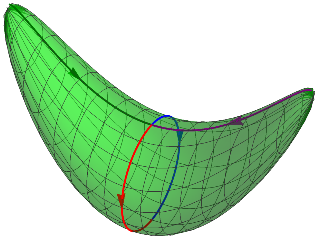
Fermion number $\mathbf{F}$ (homological grading): $\mathbf{F}_p = \frac{1}{2}(n_+(p) - n_-(p))$

Cohomology origin in QFT. Differential

However $\hat{H} = \{\bar{Q}, Q\}$, therefore $Q|\text{Ground State}\rangle = 0$, $Q^2 = 0$:

$$\langle p'|Q|p\rangle = \delta_{F_{p'},F_p+1}\#(\text{Instantons } p \to p')$$

Instantons : gradient flows
$\dot{q}^a = -g^{ab}\partial_b h$

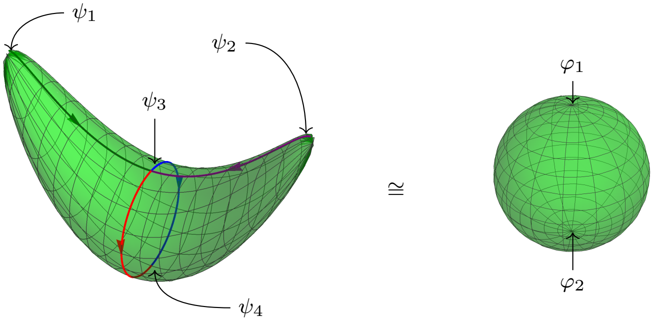


Hilbert space of **non-perturbative** ground states:

$$\mathcal{H} := H^*(\mathcal{E}, Q) = \frac{\text{Ker } Q}{\text{Im } Q}$$

For finite-dimensional compact Y $\mathcal{H} \cong H^*_{dR}(Y)$.

For example...



$$\mathcal{E}(\mathcal{B}) = \mathbb{C}[1]|\psi_1\rangle \oplus \mathbb{C}[1]|\psi_2\rangle \oplus \mathbb{C}[0]|\psi_3\rangle \oplus \mathbb{C}[-1]|\psi_4\rangle$$

$$\langle\psi_3|Q|\psi_1\rangle = [\Rightarrow] = 1,\ \langle\psi_3|Q|\psi_2\rangle = [\Rightarrow] = 1,\ \langle\psi_4|Q|\psi_3\rangle = [\Rightarrow] - [\Rightarrow] = 0$$

$$\mathcal{H}(\mathcal{B}) = H^*(\mathcal{E}(\mathcal{B}), Q) = \mathbb{C}[1]|\psi_1 - \psi_2\rangle \oplus \mathbb{C}[-1]|\psi_4\rangle$$

LG theory. Setup

2d $\mathcal{N} = (2,2)$ Landau-Ginzburg theory: [Gaiotto-Moore-Witten '15]
Kähler manifold X: coordinates ϕ^I, $I = 1, \ldots, N$.
Holomorphic superotential $W: X \to \mathbb{C}$
The potential term: $V = |\nabla W|^2 \Rightarrow$ Critical points of W – vacua $i \in \mathbb{V}$
Any finite energy field configuration approaches vacua at spatial $\pm\infty$:

Central charge of this configuration: $Z_{ij} = W_j - W_i$
Bogomolny-Prasad-Sommerfeld (BPS) bound: $E_{ij} \geq |Z_{ij}|$
This bound is saturated by **solitons**.

LG theory. Solitons

Solitons are solutions to

$$\partial_x \phi^I = \frac{\zeta_{ij}}{2} \overline{\partial_{\phi^I} W},$$

$$\lim_{x \to -\infty} \phi^I(x) = \phi_i^I, \quad \lim_{x \to +\infty} \phi^I(x) = \phi_j^I$$

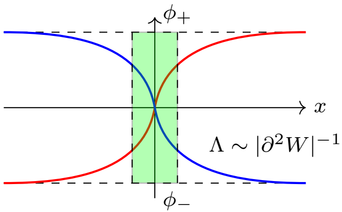
Where phase $\zeta_{ij} = \frac{Z_{ij}}{|Z_{ij}|}$. These field configurations preserve ζ-SUSY $\mathcal{Q}_{\zeta_{ij}}$, $\bar{\mathcal{Q}}_{\zeta_{ij}}$

$$\mathcal{Q}_\zeta := Q_- + \zeta^{-1}\bar{Q}_+, \quad \bar{\mathcal{Q}}_\zeta := \bar{Q}_- + \zeta Q_+, \quad \mathcal{Q}_\zeta^2 = \bar{\mathcal{Q}}_\zeta^2 = 0$$

Example. *Cubic model:* $W = \lambda\left(\frac{\phi^3}{3} - z\phi\right)$

Two vacua: $\phi_\pm = \pm z^{\frac{1}{2}}$, $W_\pm = \mp\frac{2}{3}\lambda z^{\frac{3}{2}}$
Solitons: kinks, quasi-particles, domain walls of width Λ

$$\phi_{-+}(x) = z^{\frac{1}{2}} \tanh |\lambda||z|^{\frac{1}{2}} x, \;\; \phi_{+-}(x) = -z^{\frac{1}{2}} \tanh |\lambda||z|^{\frac{1}{2}} x$$



LG as SQM

To redefine LG model as SQM we consider the target space of SQM to be a field space of LG model: $Y = \mathrm{Map}(\mathbb{R}_x \to X)$. The height functional reads:

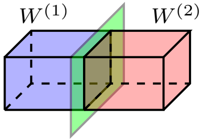
$$h = -\int \left(\sigma - \mathrm{Im}\ \left[\zeta^{-1} W\right] dx\right)$$

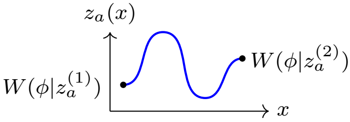
Where σ is a 1-form defined locally as $d\sigma = \frac{\mathrm{i}}{2} g_{I\bar{J}}\, d\phi^I \wedge d\bar{\phi}^{\bar{J}}$.
However there is a boundary term preserving only

$$Q = \mathcal{Q}_\zeta, \quad \bar{Q} = \bar{\mathcal{Q}}_\zeta$$

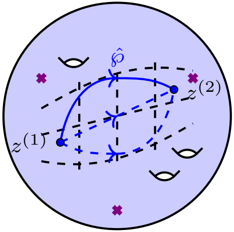
Families of theories. Interfaces.

Interfaces: $W\left(\phi^I(x)\middle|\, z_a(x)\right)$ interpolating between two theories $z_a^{(1)}$ and $z_a^{(2)}$.





Parameter space:



$$h = -\int \left(\sigma - \mathrm{Im}\ \left[\zeta^{-1} W\left(\phi^I(x)\middle|\, z_a(x)\right)\right] dx\right)$$

We expect

$$H^*\left(\mathcal{E}^h, \mathcal{Q}_\zeta\right) \cong H^*\left(\mathcal{E}^{h'}, \mathcal{Q}'_\zeta\right), \text{ for } h \overset{\text{hmtpy}}{\sim} h'$$

1. "Categorified" wall-crossing
2. "Categorified" link invariants

Forced solitons. Hovering solutions

Interface modify soliton equation to a "forced" soliton equation:

$$\delta h = 0: \quad \partial_x \phi^I(x) = -\zeta g^{I\bar{J}} \overline{\partial_J W\left(\phi^I(x) | z_a(x)\right)}$$

We would like to consider adiabatic limits: $\left|\frac{dz}{dx}\right| \ll \Lambda^{-1}$.
Hovering solutions - are simply described by fields which, at each x are critical points for $W(\phi; z(x))$ for the same value of x. In equations: if $\phi_i^I(x)$ is a critical point:

$$dW(\phi^I | z_a(x))\Big|_{\phi_i^I(x)} = 0$$

We can denote this solution by a diagram for vacuum i:

$$i \rule[0.3ex]{4cm}{2pt} i$$

Overall solution is a sum over all vacua:

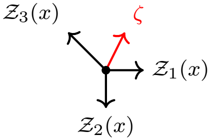
$$\mathcal{E}[\mathfrak{I}_{x_1,x_2}]^{\text{hovering}} = \bigoplus_i \quad \overset{i}{\underset{x_1}{\cdot}} \rule[0.3ex]{4cm}{2pt} \overset{i}{\underset{x_2}{\cdot}}$$

Forced solitons. Binding points

An effective x-dependent central charge for a pair of hovering solutions:

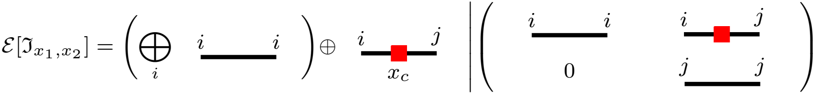
$$\mathcal{Z}_{ij}(x) := W({\phi_j}^I(x)|z_a(x)) - W({\phi_i}^I(x)|z_a(x))$$

Binding points x_c are defined by an equation:



$$\zeta^{-1}\mathcal{Z}_{ij}(x_c) > 0$$

At these points a soliton solution does not break $\mathcal{Q}_\zeta$ and can be bound to the interface:



Composition law: $\mathcal{E}_{x_1,x_n} = \mathcal{E}_{x_1,x_2} \otimes \mathcal{E}_{x_2,x_3} \otimes \ldots \otimes \mathcal{E}_{x_{n-1},x_n}$

$$\begin{pmatrix} i \text{ — } i & i \text{ —}\blacksquare\text{— } j & 0 \\ 0 & j \text{ — } j & 0 \\ 0 & 0 & k \text{ — } k \end{pmatrix} \otimes \begin{pmatrix} i \text{ — } i & 0 & 0 \\ 0 & j \text{ — } j & j \text{ —}\blacksquare\text{— } k \\ 0 & 0 & k \text{ — } k \end{pmatrix} =$$

$$= \begin{pmatrix} i \text{ — } i & i \text{ —}\blacksquare\text{— } j & i \text{ —}\blacksquare\overset{j}{\text{———}}\blacksquare\text{— } k \\ 0 & j \text{ — } j & j \text{ —}\blacksquare\text{— } k \\ 0 & 0 & k \text{ — } k \end{pmatrix}$$

LG model. Instantons.

Instantons are solution to the descend flow equation:

$$\dot{q}^I = -g^{IJ}\partial_J h \Longrightarrow (\partial_x + \mathrm{i}\partial_\tau)\,\phi^I(x,\tau) = -\zeta g^{I\bar{J}}\overline{\partial_J W\left(\phi^I(x,\tau)|z_a(x)\right)}$$

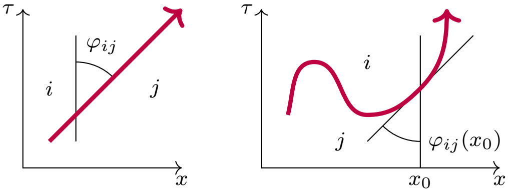
The Euclidean boost acts covariantly on this equation:

$$U_\varphi : \begin{pmatrix} x \\ \tau \end{pmatrix} \mapsto \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} \begin{pmatrix} x \\ \tau \end{pmatrix}, \quad \zeta \mapsto e^{\mathrm{i}\varphi}\zeta$$

This introduces migrating binding points: $e^{-\mathrm{i}\varphi}\zeta^{-1}\mathcal{Z}_{ij}(x) > 0$, $\tan\varphi = \frac{d\tau}{dx}$. Or, re-written,

$$\frac{dx}{d\tau} = -\frac{\mathrm{Im}\,\left[\zeta^{-1}\mathcal{Z}_{ij}(x)\right]}{\mathrm{Re}\,\left[\zeta^{-1}\mathcal{Z}_{ij}(x)\right]}$$

Straight and curved ij-domain walls:

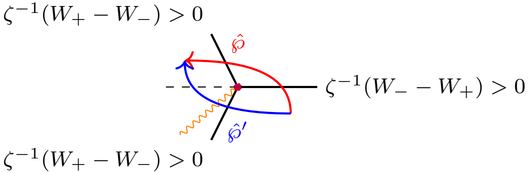


Simple example: Airy (cubic) model

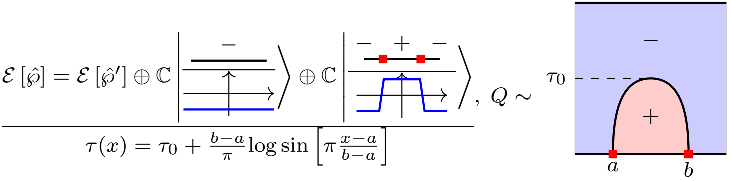
As an illustrative example we will consider Airy model with superpotential

$$W = \frac{\phi^3}{3} - z\phi$$

There are two vacua $\phi_\pm = \pm z^{\frac{1}{2}}$ with values of the superpotential $W_\pm = \mp\frac{2}{3}z^{\frac{3}{2}}$. On the parameter space $z \in \mathbb{C}$ there are three BPS chambers:

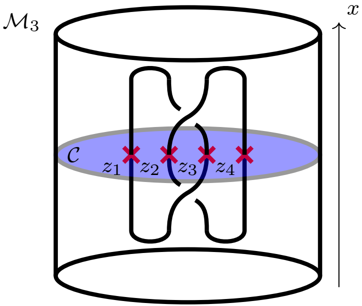


Consider two homotopic paths $\hat{\wp}$ and $\hat{\wp}'$ representing interfaces:

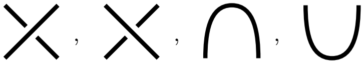


Link as a tangle interface

$$W_{\mathrm{Yang-Yang}} = \sum_{a,I} \log(z_a - \phi^I) - 2\sum_{I<J}\log(\phi^I - \phi^J) + c\sum_I \phi^I$$



We consider a link as a tangle embedded in $M_3 = \mathbb{C} \times \mathbb{R}_x$. x is evolution direction. A link can be encoded in link strand trajectories $z_a(x)$.
Elementary interfaces:

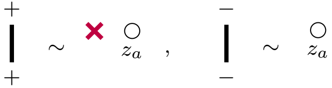


Vacua

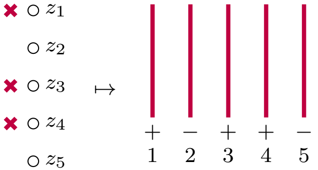
Vacuum equations ≈ Bethe ansatz equations:

$$\partial_{\phi^I} W = 0 : \quad \sum_a \frac{1}{\phi^I - z_a} - 2 \sum_{J \neq I} \frac{1}{\phi^I - \phi^J} + c = 0, \quad \forall I$$

Critical points are $\phi^I = z_{a(I)} - \frac{1}{c} + O\left(\frac{1}{c^2}\right)$.

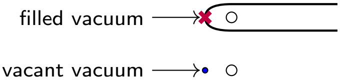


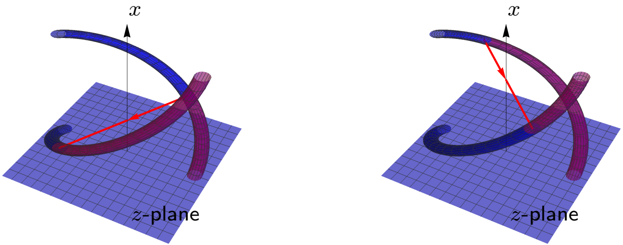
For example,



Brading interfaces and grading I

$$\mathrm{Im}\left[\zeta^{-1}\partial_\phi W(\phi(s))\dot{\phi}(s)\right] = 0$$

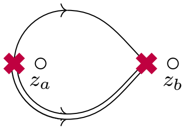




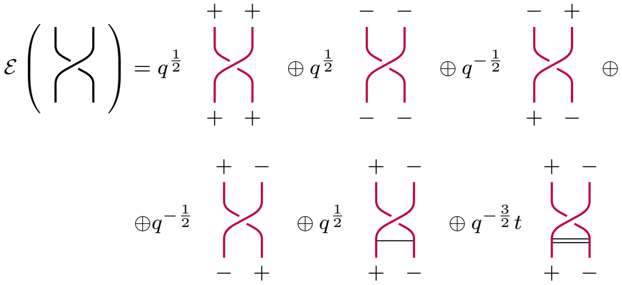
■: "+" vacuum, ■: "−" vacuum

Braiding interfaces and grading II

Two emerging soliton solutions: [G.-Moore '16]



All the states weighted with $q^{\mathbf{P}} t^{\mathbf{F}}$.
$\mathbf{P} = \frac{1}{\pi \mathrm{i}} \Delta W$, $\mathbf{F}$ – fermion number (η-invariant)



Cap/cup interfaces and unknot

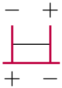
The Euler characteristic reduce to the R-matrix for $SU_q(2)$, $\square \otimes \square$:

$$\chi\left[\mathcal{E}(\mathcal{R})\right] = q^{\frac{H \otimes H}{2}} \left(1 + \left(q - q^{-1}\right) E \otimes F\right)$$

Similarly we have:

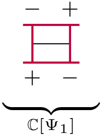
$$\mathcal{E}\left(\bigcap\right) = q^{\frac{1}{2}}$$

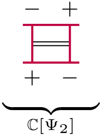

$$\oplus\, q^{-\frac{1}{2}} t$$


$$\mathcal{E}\left(\bigcup\right) = q^{\frac{1}{2}} t^{-1}$$


$$\oplus\, q^{-\frac{1}{2}}$$

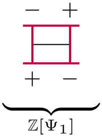

The unknot is the easiest calculation. Using our rules we get the following MSW complex:

$$\mathcal{E}\left(\bigcirc\right) = q t^{-1}$$


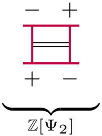
$$\oplus\, q^{-1} t$$


Unknot (continuation)

$$\mathcal{E}\left(\bigcirc\right) = qt^{-1}$$



$$\oplus\, q^{-1}t$$



The q-grading splits this complex in two one-dimensional subcomplexes

$$\mathcal{E}_1 = 0 \stackrel{\mathcal{Q}_\zeta}{\to} \mathbb{Z}[\Psi_1] \stackrel{\mathcal{Q}_\zeta}{\to} 0, \quad \mathcal{E}_{-1} = 0 \stackrel{\mathcal{Q}_\zeta}{\to} \mathbb{Z}[\Psi_2] \stackrel{\mathcal{Q}_\zeta}{\to} 0,$$

and in each of them the differential acts trivially. The fermion numbers are

$$\mathbf{F}(\Psi_1) = -1, \quad \mathbf{F}(\Psi_2) = +1.$$

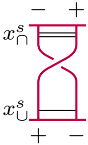
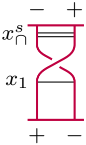
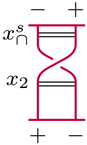
Accordingly the link homology is rank 2 and may be denoted

$$H^{*,*}(\mathrm{Unknot}) = \left(qt^{-1}\right)\mathbb{C} \oplus \left(q^{-1}t\right)\mathbb{C}$$

We will generally summarize this kind of data by specifying the Poincaré polynomial:

$$\mathcal{P}(q,t|\mathrm{Unknot}) = \frac{q}{t} + \frac{t}{q}.$$

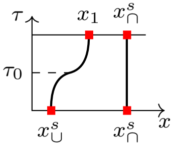
Example: twisted unknot

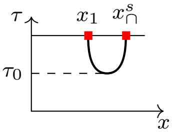
$\mathcal{E}\Big($  $\Big) = q^{-\frac{1}{2}} \Big($  $\oplus$  $\oplus\, t$  $\Big) \oplus q^{-\frac{5}{2}} t^2$ 

We denote generators by $\mathbb{C}[\Psi_\alpha]$, $\alpha = 1, \ldots, 4$, reading from left to right.

$$\mathcal{E}_{-\frac{1}{2}} = (0 \to (\mathbb{C}[\Psi_1] \oplus \mathbb{C}[\Psi_2]) \to \mathbb{C}[\Psi_3] \to 0)\,, \quad \mathcal{E}_{-\frac{5}{2}} = (0 \to \mathbb{C}[\Psi_4] \to 0)$$

We will draw explicit curved webs with a single time-translation modulus:

$\langle \Psi_3 | \mathcal{Q}_\zeta | \Psi_1 \rangle = 1 \sim$ 

, $\langle \Psi_3 | \mathcal{Q}_\zeta | \Psi_2 \rangle = 1 \sim$ 

Thus, in the basis Ψ_α, the differential $\mathcal{Q}_\zeta$ takes the form

$$\mathcal{Q}_\zeta = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad H^0(\mathcal{E}_{-\frac{1}{2}}, \mathcal{Q}_\zeta) = \mathbb{C}[\Psi_1 - \Psi_2]$$

Twisted Unknot (continuation) and Figure-eight Knot

$$H^{*,*}\Bigg($$

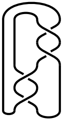
$$\Bigg) = \left(q^{-\frac{1}{2}}t^{-1}\right)\mathbb{C} \oplus \left(q^{-\frac{5}{2}}t^{2}\right)\mathbb{C}$$

Thus the Poincaré polynomial is

$$\mathcal{P}\Bigg(q, t\Big|$$

$$\Bigg) = q^{-\frac{3}{2}}t\left(\frac{q}{t} + \frac{t}{q}\right)$$

For figure-eight knot (4_1)

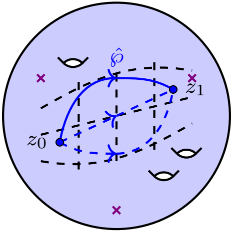


calculation is more involved, however the result is in agreement with Khovanov's Poincaré 4_1 polynomial $\mathrm{Kh}(q,t)$:

$$\mathcal{P}(q, t|\mathbf{4_1}) = q^{-5}t^3 + q^{-1}(1+t) + q(1+t^{-1}) + q^5 t^{-3}$$

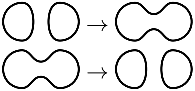
$$\mathrm{Kh}(q, t|L) = \mathcal{P}(qt, t|L)$$

Khovanov vs LG



Claim: There is a distinguished path $\hat{\mathcal{P}}_{\mathrm{Kh}}$ such that LG link homology complex is isomorphic to Khovanov homology complex:

$$\begin{gathered}\tilde{\mathcal{E}}_{\mathrm{Kh}}\left[\times\right] = \tilde{\mathcal{E}}_{\mathrm{Kh}}\left[)\ (\right] \oplus qt\,\tilde{\mathcal{E}}_{\mathrm{Kh}}\left[\asymp\right] \\ \tilde{\mathcal{E}}_{\mathrm{Kh}}\left[\bigcirc\right] = q\,\mathbb{C} \oplus q^{-1}\,\mathbb{C}\end{gathered},$$

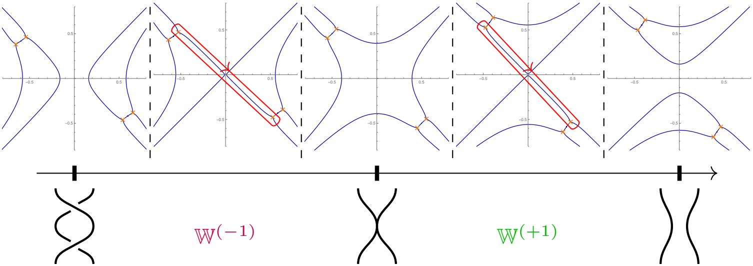


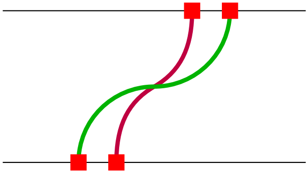
In terms of Poincaré polynomials we have:

$$\mathrm{Kh}(q,t|L) = \mathcal{P}(qt,t|L)$$

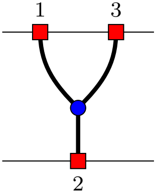
WKB analysis of LG instantons

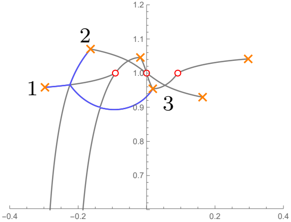
Null WKB webs: $\oint_{\gamma} \lambda = 0$ [G. '17]





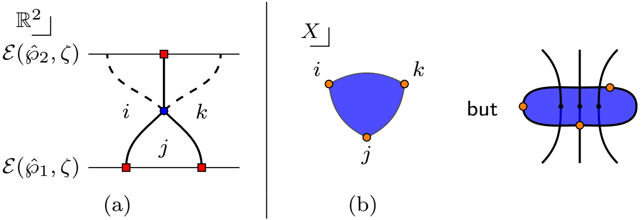
WKB analysis of LG instantons: soliton scattering

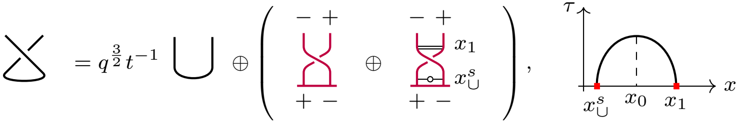


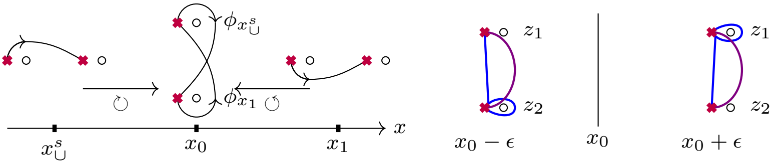


Scattering
$1 + 2 \to 3$

Do not go deep into IR!







■ – Yang-Yang, ■ – Monopole moduli space

Conclusion

- Supersymmetric quantum mechanics provides a simple physical construction of cohomology
- Hilbert spaces of grond states of interfaces in 2d Landau-Ginzburg model on the moduli space of monopoles are naturally bi-graded and are link invariants
- Homotopic interfaces give quasi-isomorphic families of complexes
- Landau-Ginzurg link homology is equivalent to Khovanov link homology
- One can use asymptotic analysis to count critical field configurations and instantons in LG models
- One can easily describe invariants for higher rank groups and representations

Thank you for your attention!