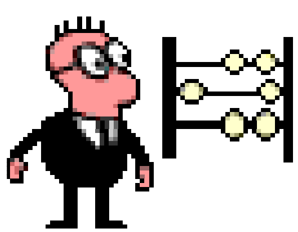
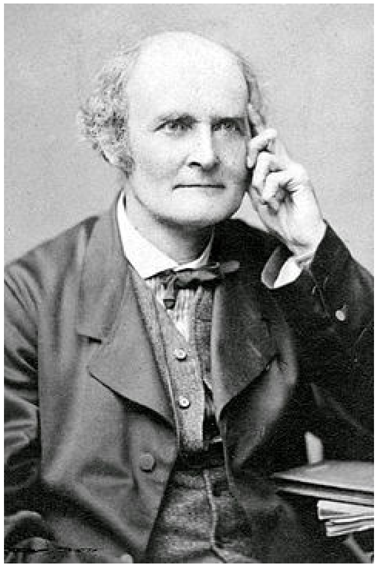
Max Planck Institute for Mathematics
California Institute of Technology

The sound of Knots and 3-manifolds

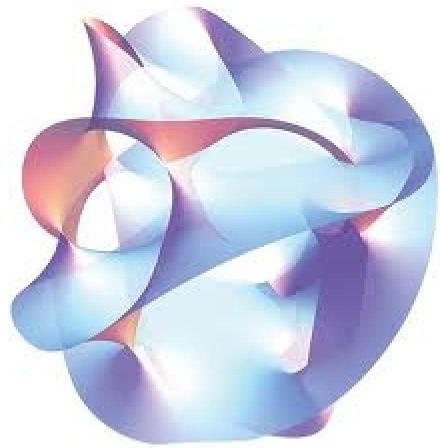
Sergei Gukov

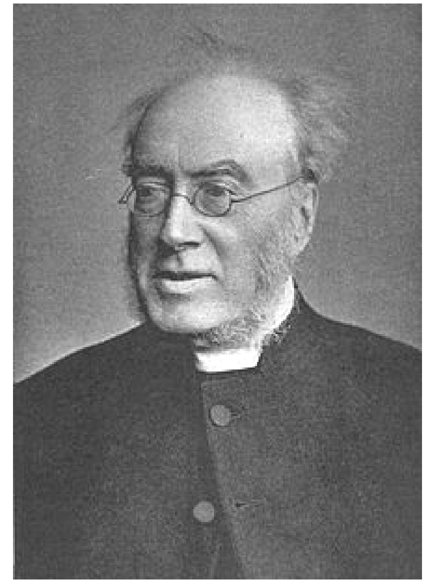


27 lines on a general cubic surface



Arthur Cayley (1821-1895)

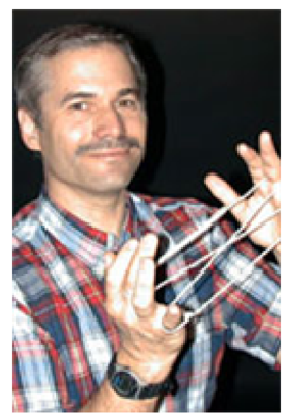




George Salmon (1819-1904)

2,875 lines (degree-1) on a quintic 3-fold

609,250 conics (degree-2) on a quintic 3-fold



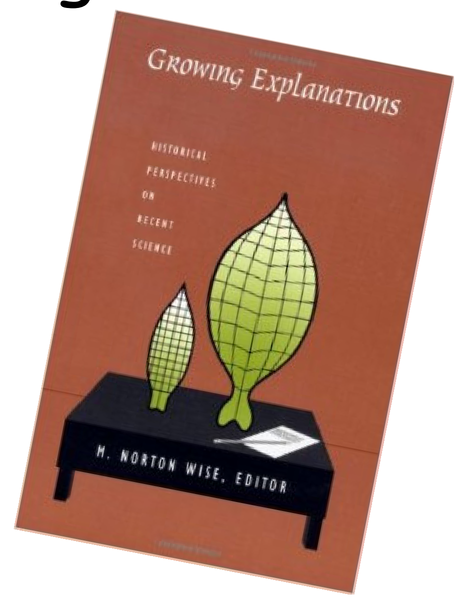
Sheldon Katz, 1986

317,206,375 degree-3 curves on a quintic 3-fold

"Suddenly, in late 1990, Candelas and his collaborators Xenia de la Ossa, Paul Green, and Linda Parkes (COGP) saw a way to use mirror symmetry to barge into the geometers' garden.

...

Candelas, who liked calculating things, thought maybe it was in fact tractable using some algebra and a home computer."

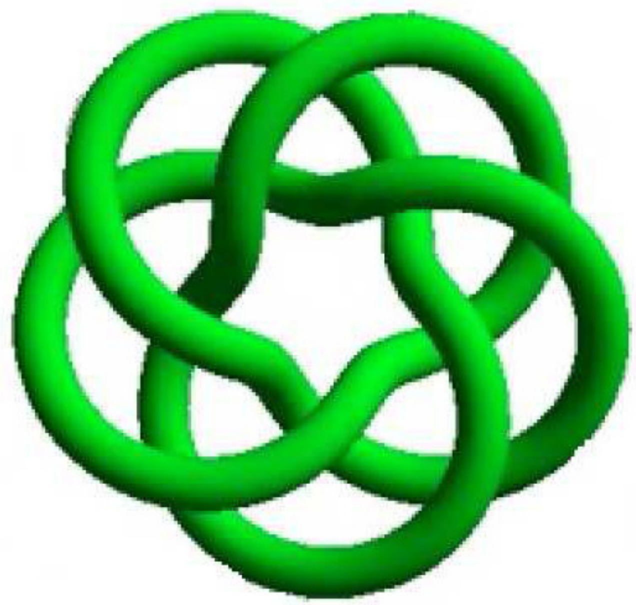


2,682,549,425 degree-3 curves on a quintic 3-fold

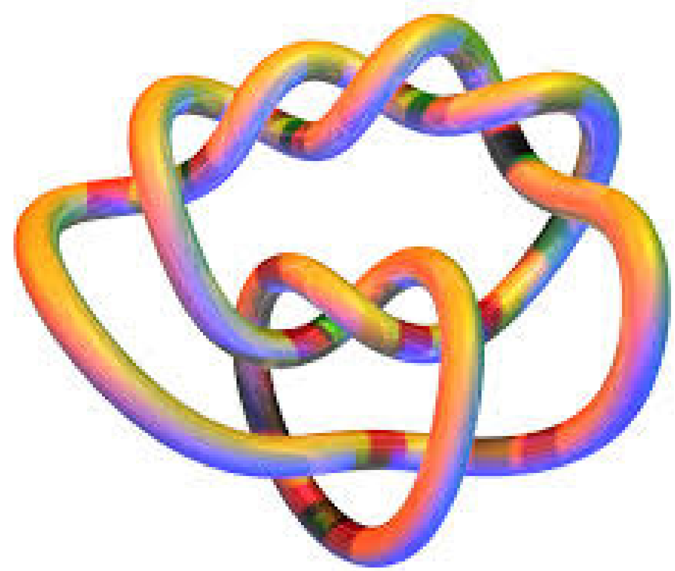
Geir Ellingssrud and Stein Arild Stromme, May 1991

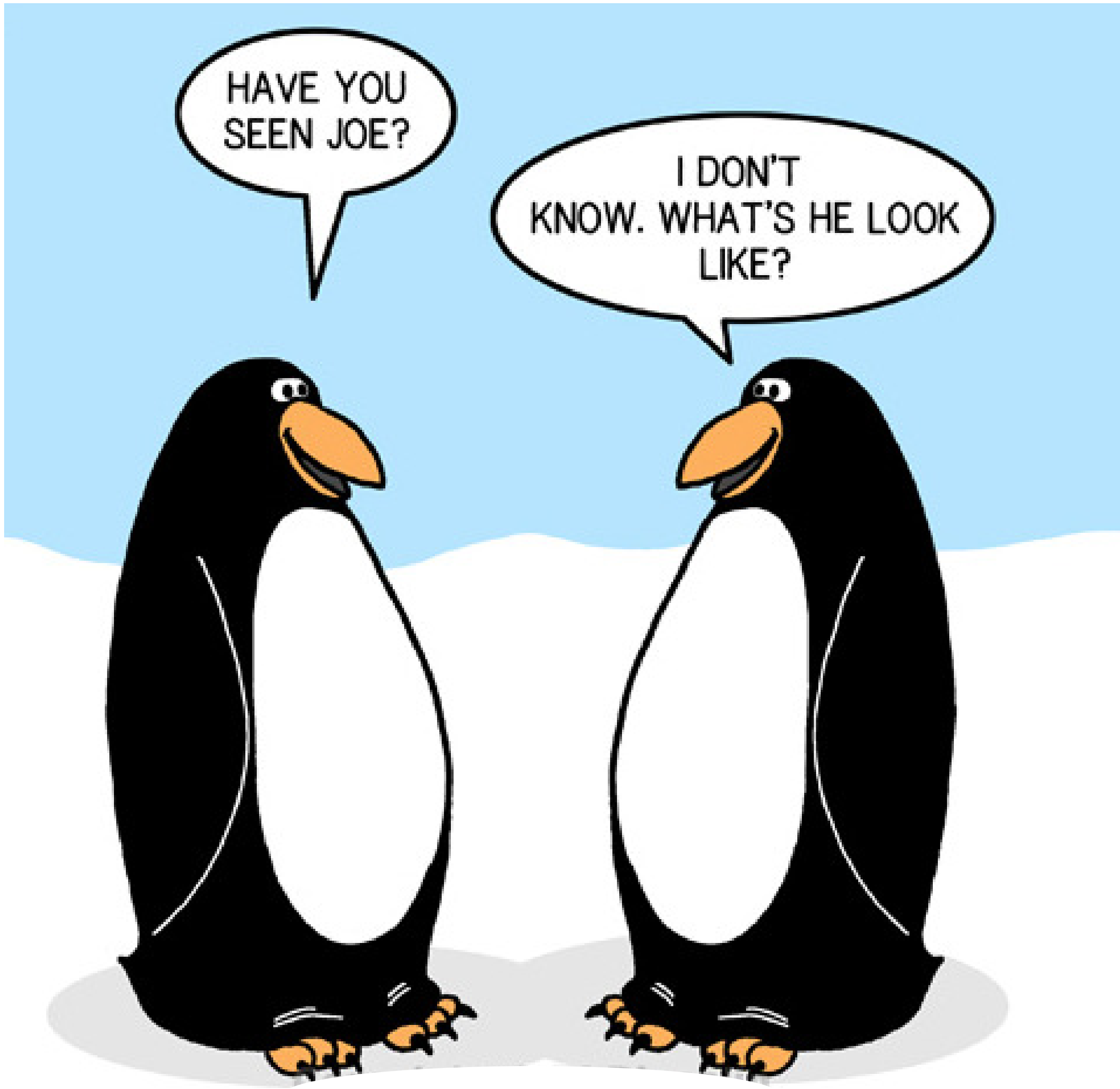
New Opportunities

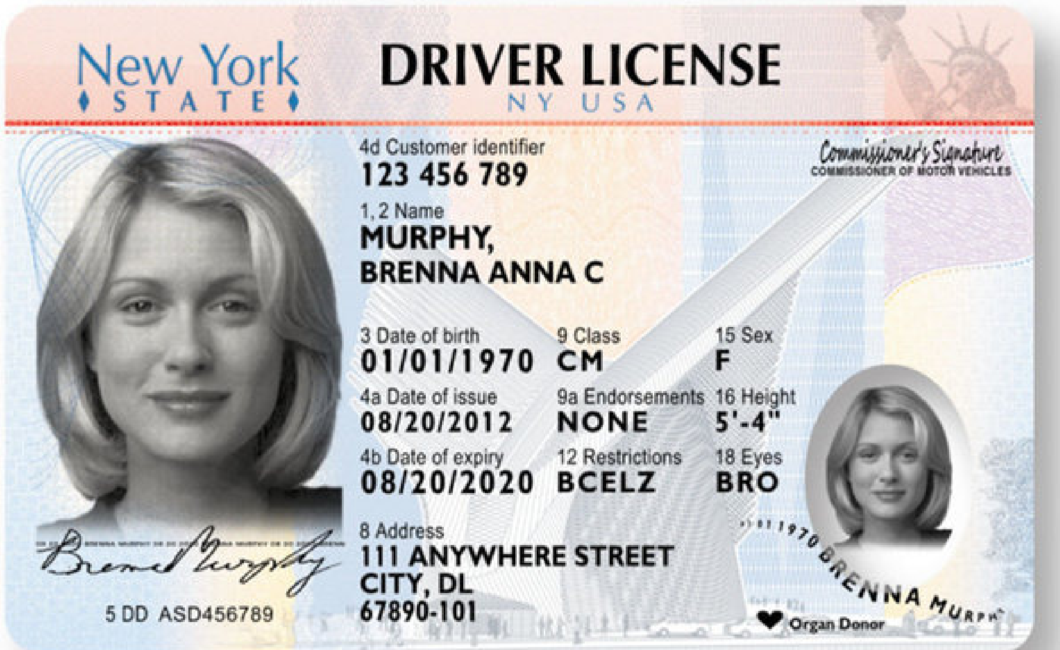
Knots



$\stackrel{?}{=}$

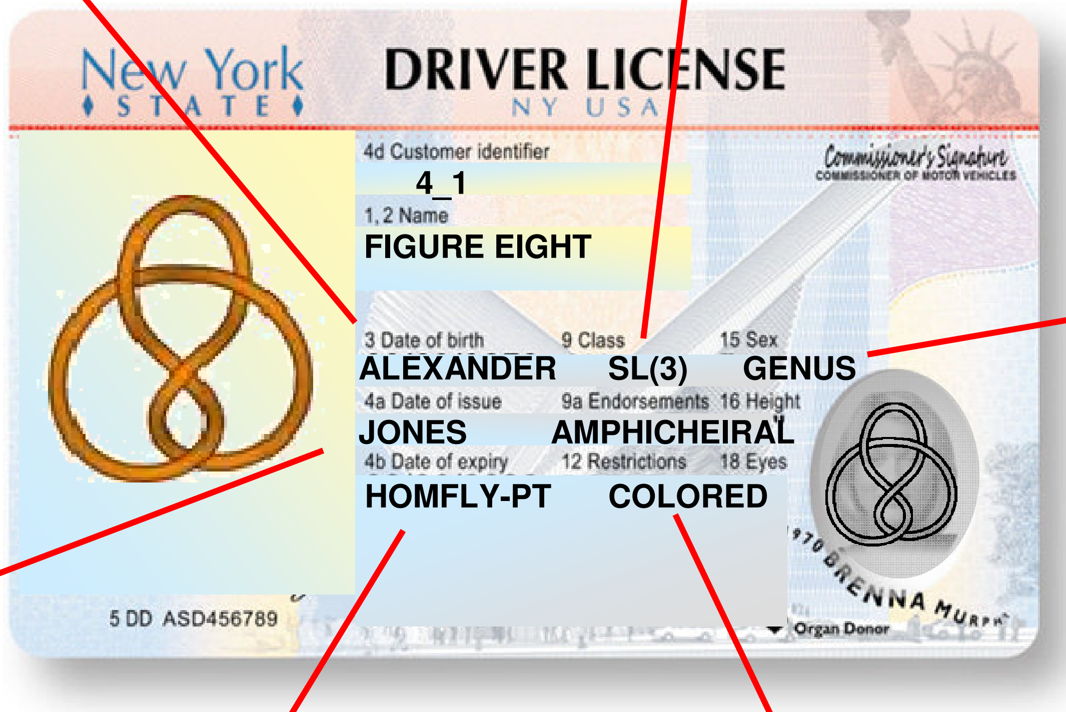



HAVE YOU SEEN JOE?
I DON'T KNOW. WHAT'S HE LOOK LIKE?


New York
STATE
DRIVER LICENSE
NY USA
4d Customer identifier
123 456 789
1, 2 Name
MURPHY,
BRENNA ANNA C
3 Date of birth
01/01/1970
9 Class
CM
15 Sex
F
4a Date of issue
08/20/2012
9a Endorsements
NONE
16 Height
5'-4"
4b Date of expiry
08/20/2020
12 Restrictions
BCELZ
18 Eyes
BRO
8 Address
111 ANYWHERE STREET
CITY, DL
67890-101
5 DD ASD456789
Commissioner's Signature
COMMISSIONER OF MOTOR VEHICLES
BRENNA MURPHY
Organ Donor

$$-t - t^{-1} + 3$$

$$q^{8} + q^{6} - 1 + q^{-6} + q^{-8}$$



$$1$$

$$q^{5} + q^{-5}$$

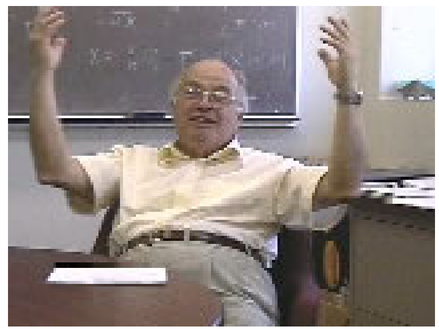
$$a^{2} + a^{-2} - z^{2} - 1$$

$$q^{14} - q^{10} + q^{2} + 1 + q^{-2} - q^{-10} + q^{-14}$$

$-t - t^{-1} + 3$

$q^8 + q^6 - 1 + q^{-6} + q^{-8}$

Why integer coefficients?



1

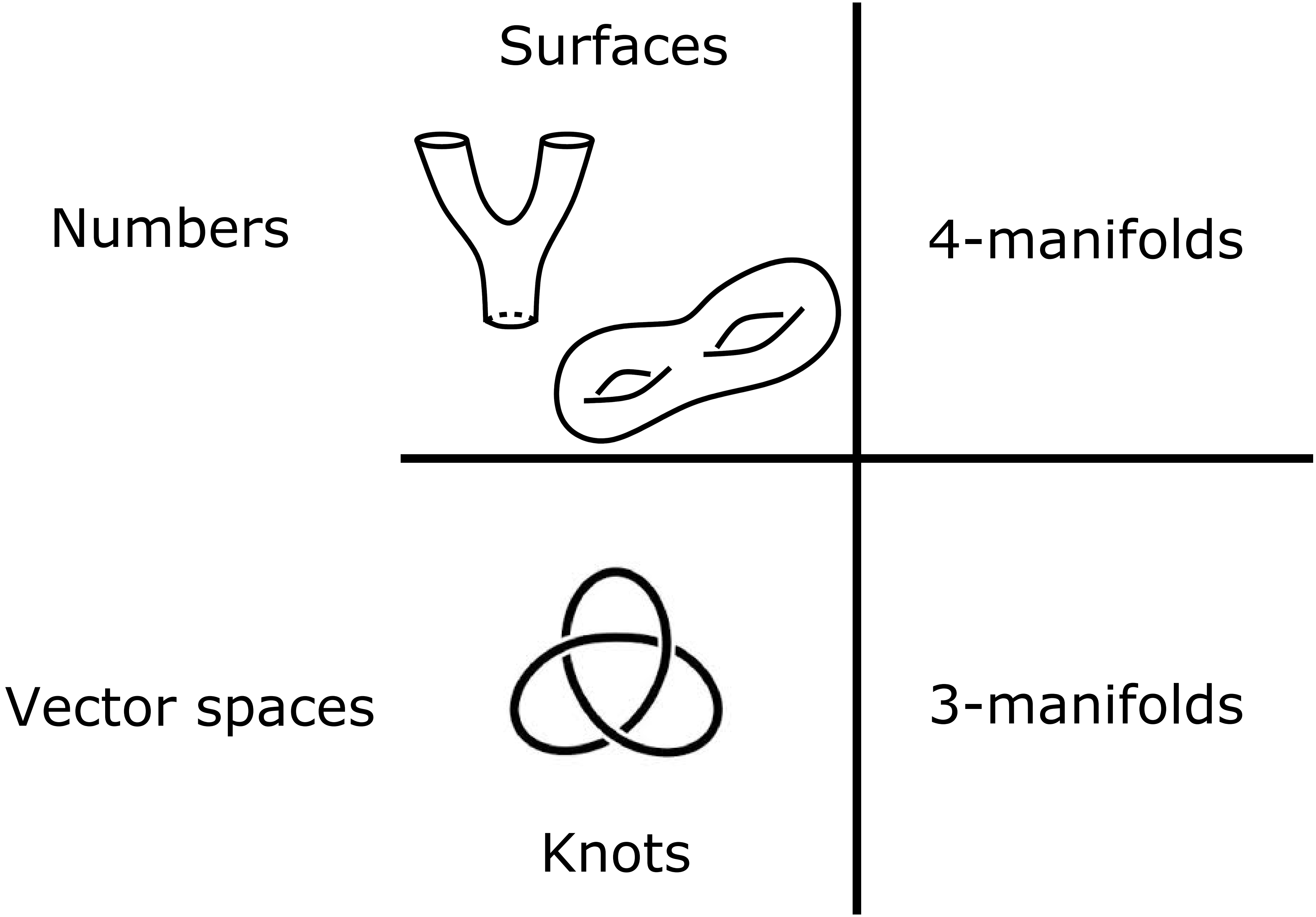
$q^5 + q^{-5}$

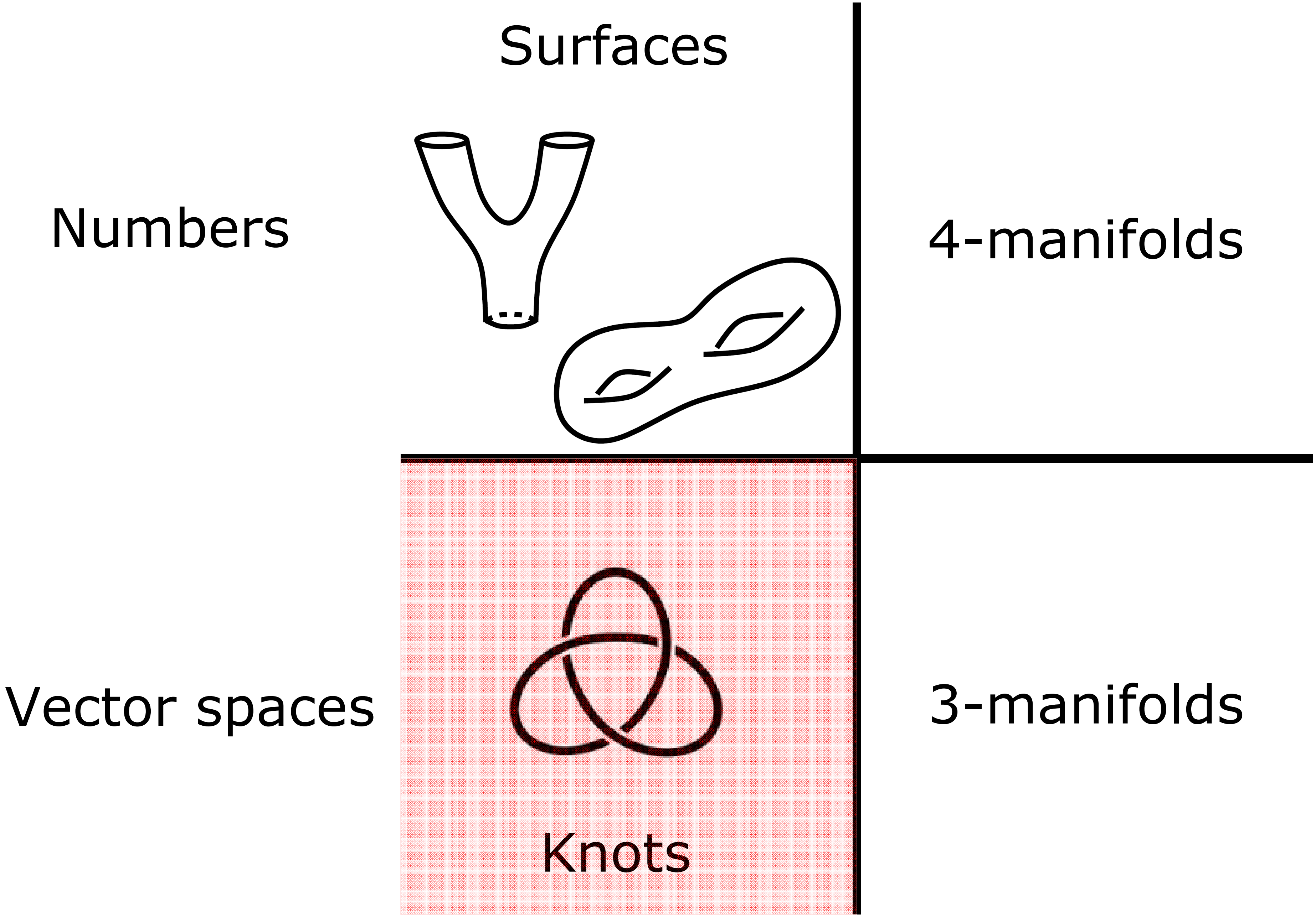
TQFT ~ 30 yrs old

$q^{14} - q^{10} + q^2 + 1 + q^{-2} - q^{-10} + q^{-14}$

$a^2 + a^{-2} - z^2 - 1$

TQFT in the 21st century


Surfaces
Numbers
4-manifolds
Vector spaces
3-manifolds
Knots


Surfaces
Numbers
4-manifolds
Vector spaces
3-manifolds
Knots

Knot homology circa 2003

- Only a few homological knot invariants:
  - Khovanov homology (N=2, combinatorial)

Kh(T(7,6)).

In 1 day

Old techniques:

~1,000 years,

~1GGb RAM.

j \ r	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
57																	1			1
55																	1	1		
53															1	2		1	1	
51													1	1		2	1			
49														3	1		1			
47												3	1		1					
45										2		1	2							
43								1		1	2									
41						1		1	2		1									
39						1	1		1											
37				1	1		1													
35					1															
33			1																	
31	1																			
29	1																			

the alternating knots zone

Knot homology circa 2003

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  - Knot Floer homology ($N=0$, symplectic)
- Categorification of HOMFLY not expected

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- No SO/Sp groups
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- Categorification of HOMFLY not expected
- Higher rank not computable
- No SO/Sp groups
- No colors
- Many unexplained patterns

“Connecting the dots”

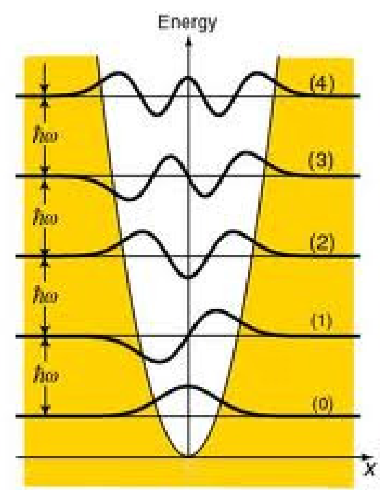
[S.G., A.Schwarz, C.Vafa, 2004]

Knot Homology
(Khovanov,...)

=

BPS spectrum
(Q-cohomology)

$$P(q) = \sum q^i (-1)^j \dim \mathcal{H}^{i,j}$$



$\mathcal{H}_{\text{BPS}}$

Fivebrane Setup

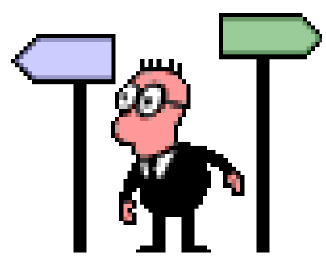
		duality	
space-time:	$\mathbb{R} \times T^* M_3 \times TN_4$	$\longleftrightarrow$	$\mathbb{R} \times X \times TN_4$
N M5-branes:	$\mathbb{R} \times \; M_3 \; \times \; \mathbb{R}^2_q$		$\mathbb{R} \times L_K \times \mathbb{R}^2_q$
M2-branes:	$\mathbb{R} \times \; \Sigma_K \; \times \{0\}$		(+ flux)

$\mathcal{H}_{\text{BPS}}$ doubly graded

fixed N

$\mathcal{H}_{\text{BPS}}$ triply graded

packages all ranks



[H.Ooguri, C.Vafa, 1999]

sl(3) link homology

Mikhail Khovanov

Abstract We define a bigraded homology theory whose Euler characteristic is the quantum sl(3) link invariant.

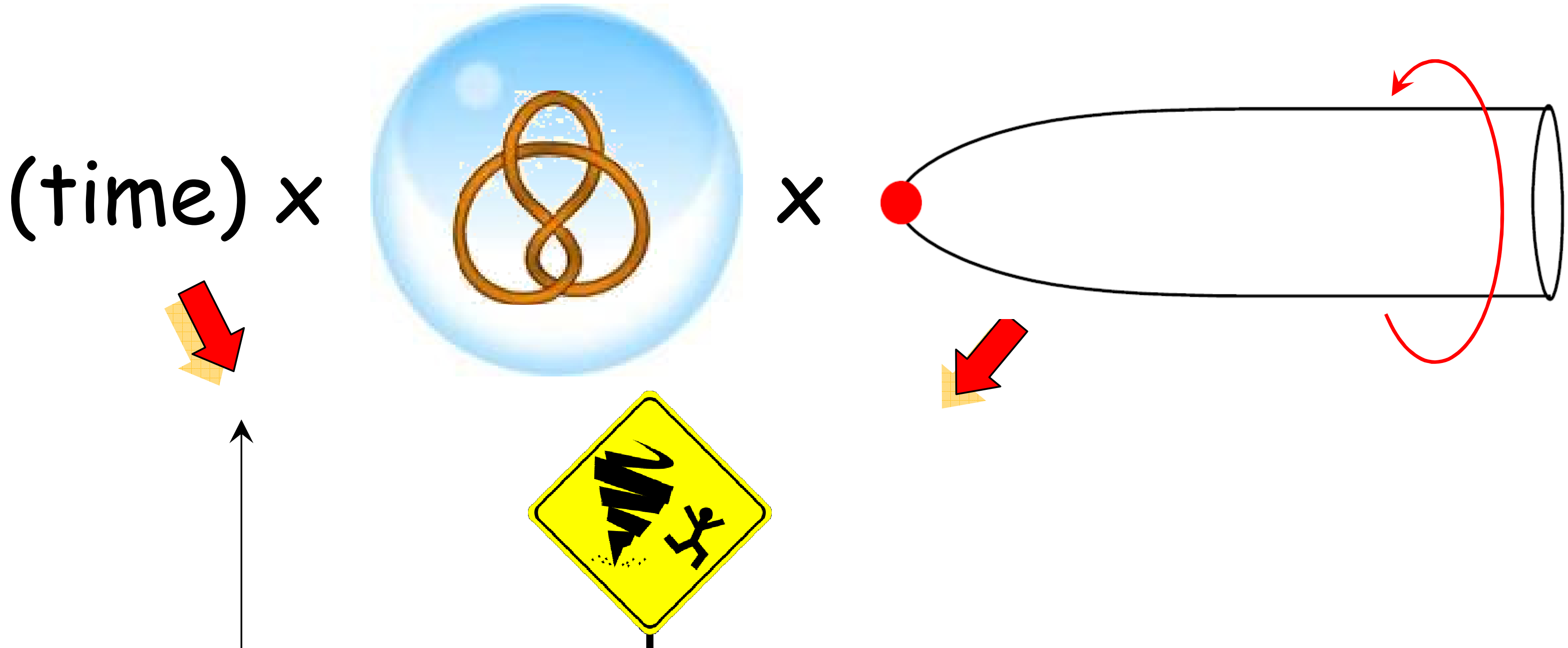
AMS Classification 81R50, 57M27; 18G60

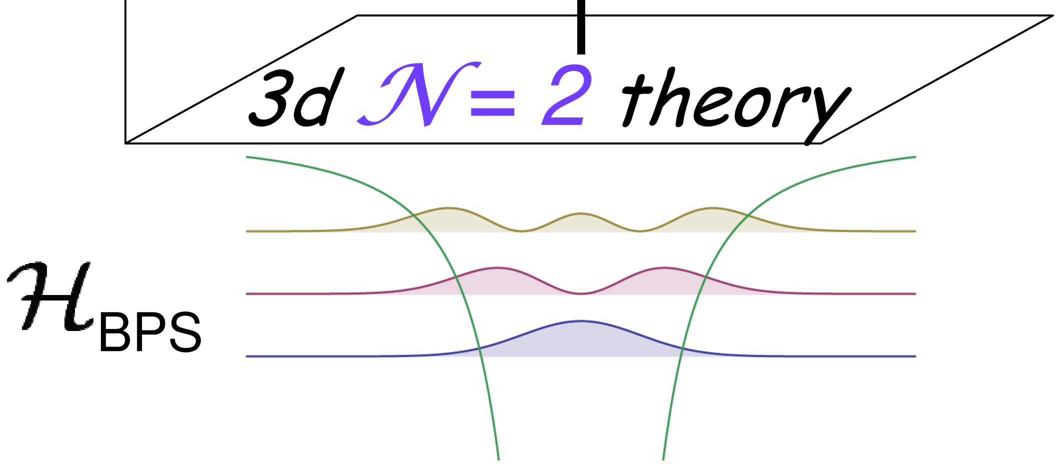
Keywords Knot, link, homology, quantum invariant, sl(3)

Figure 1: Quantum $\mathfrak{sl}(n)$ skein formula

When $\mathfrak{g} = \mathfrak{sl}(n)$ and each components of L is labelled either by the defining representation V or its dual, the invariant is determined by the skein relation in Figure 1. If we introduce a second variable $p = q^n$, the skein relation gives rise to the HOMFLY polynomial, a 2-variable polynomial invariant of oriented links [2]. We do not believe in a triply-graded homology theory categorifying the HOMFLY polynomial. Instead, for each $n \geq 0$ there should exist a bigraded theory categorifying the (q, q^n) specialization of HOMFLY. For $n = 0$

3d-3d interpretation





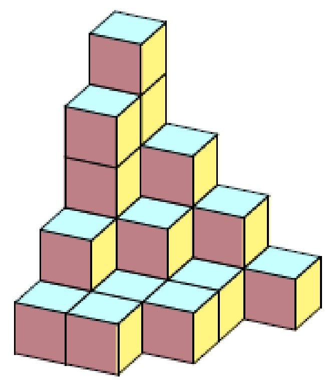
New Bridges

$$\begin{array}{rlcl} \text{space-time:} & \mathbb{R} \times T^*M_3 \times TN_4 & & \mathbb{R} \times X \times TN_4 \\ N \text{ M5-branes:} & \mathbb{R} \times M_3 \times \mathbb{R}^2_q & \overset{\text{duality}}{\longleftrightarrow} & \mathbb{R} \times L_K \times \mathbb{R}^2_q \\ \text{M2-branes:} & \mathbb{R} \times \Sigma_K \times \{0\} & & \end{array}$$

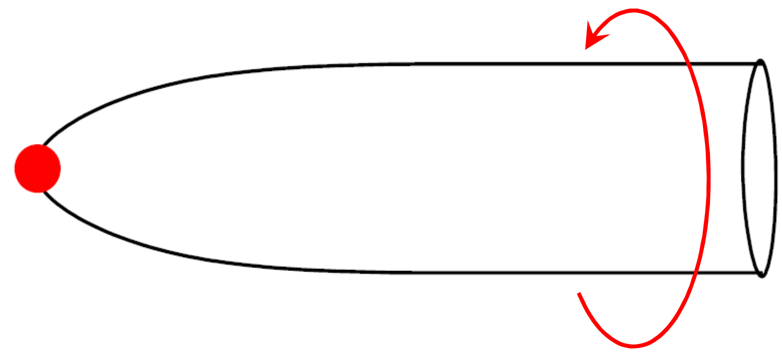
Refined Chern-Simons, DAHA, ...

[M.Aganagic, S.Shakirov, 2011]
[I.Cherednik, 2011]

Enumerative invariants of Calabi-Yau 3-folds and combinatorics of skew 3-dimensional partitions

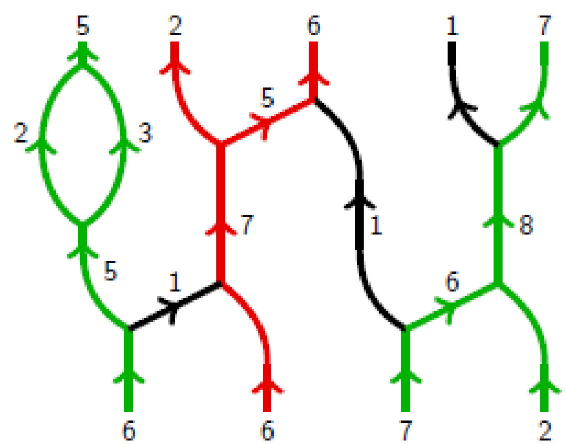


LG interfaces and KLR algebras

(time) x  x 

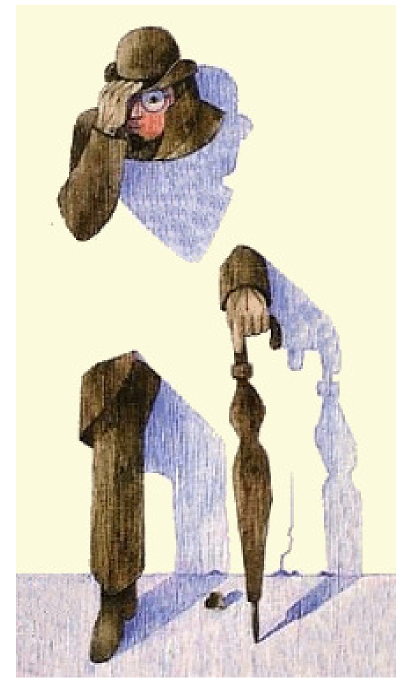
"brane intersection" perspective on (time) x (knot):
2d Landau-Ginzburg models & interfaces

D.Roggenkamp



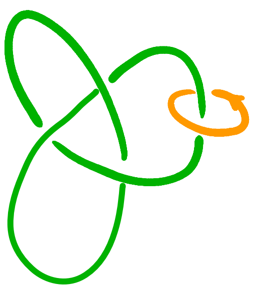
J.Walcher

Reduction to gauge theory



Singly-graded, familiar PDE's,
new wall crossing phenomena

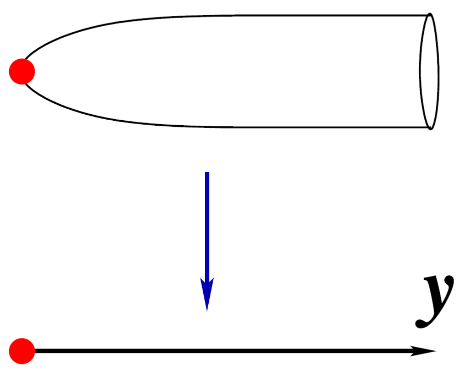
[S.G., 2007]



$$\rho(\gamma_m) = \begin{pmatrix} x & * \\ 0 & x^{-1} \end{pmatrix}$$

[P.Kronheimer, T.Mrowka, 2008]

Singly-graded, familiar PDE's, unknot-detector



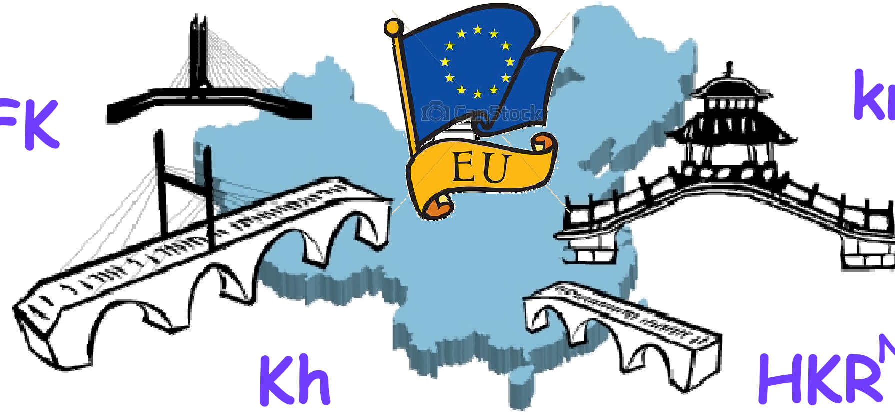
Doubly-graded,
Haydys-Witten PDE's in five dimensions

[E.Witten, 2011]

What physics gives us?

- HOMFLY homology & new bridges:

HFK

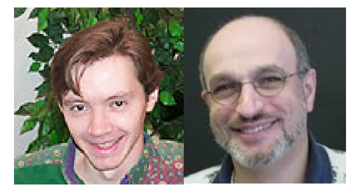


knot contact homology

A-polynomial

Kh

HKR^N



What physics gives us?

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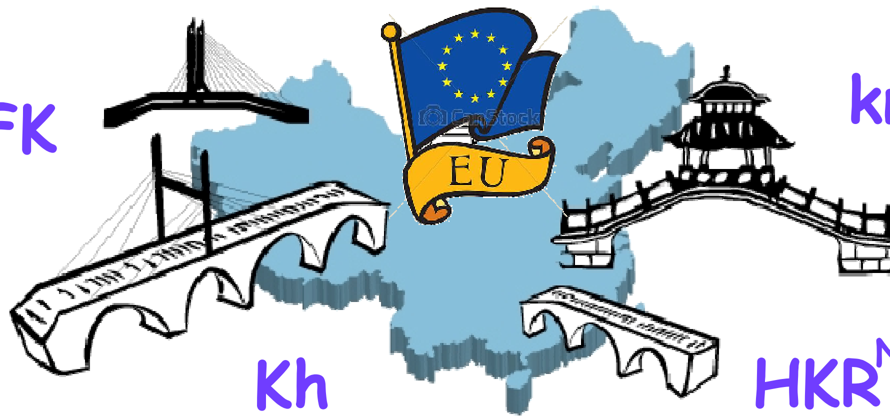
HFK



knot contact homology

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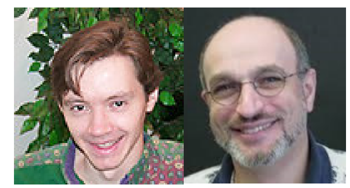
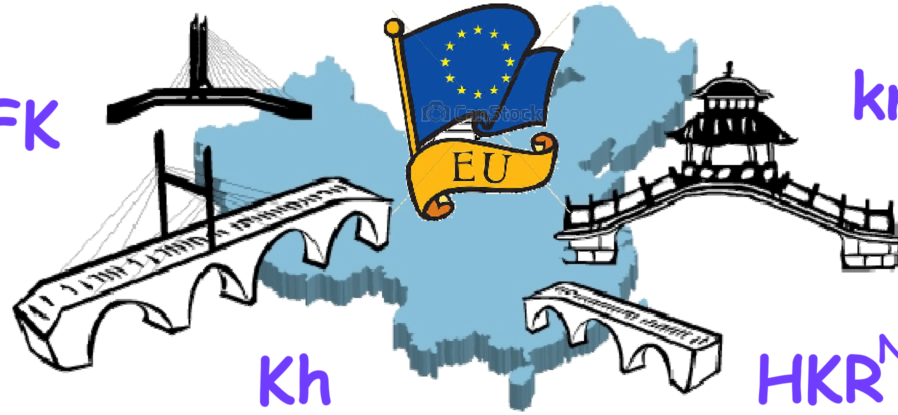


- New structural properties (differentials, recursion relations, …)

What physics gives us?

- HOMFLY homology & new bridges:

HFK



knot contact homology

A-polynomial

Kh

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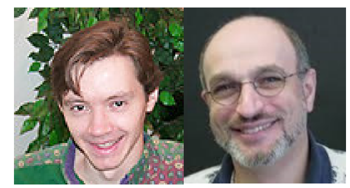


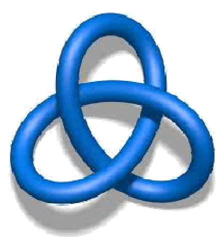
- New structural properties (differentials, recursion relations, …)
- New computational techniques

Example: trefoil

$$\mathscr{H}^{\mathfrak{e}_6,\mathbf{27}}(\mathbf{3}_1) = 1 + q^2 t^2 + q^5 t^2 + q^{10} tu + q^{13} tu + q^{10} t^4 + q^{15} t^3 u + q^{18} t^3 u + q^{23} t^2 u^2$$

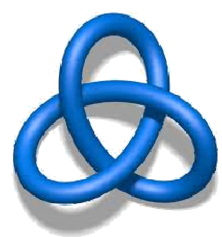
All knots up to 10 crossings
in one day!

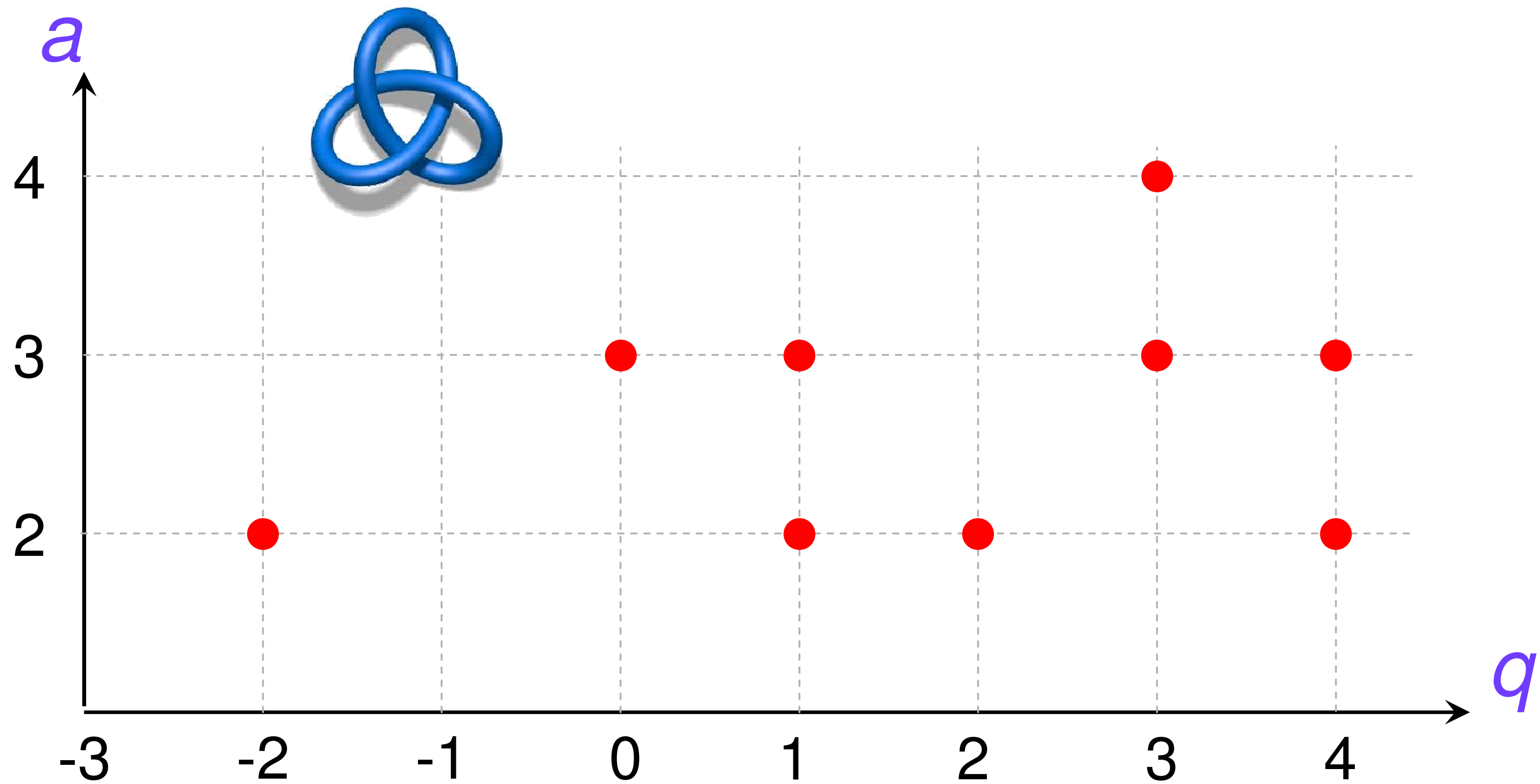
$$P^{\square\square} = q^2 + q^5 - q^7 + q^8 - q^9 - q^{10} + q^{11}$$



$$sl(2) \longrightarrow sl(N)$$

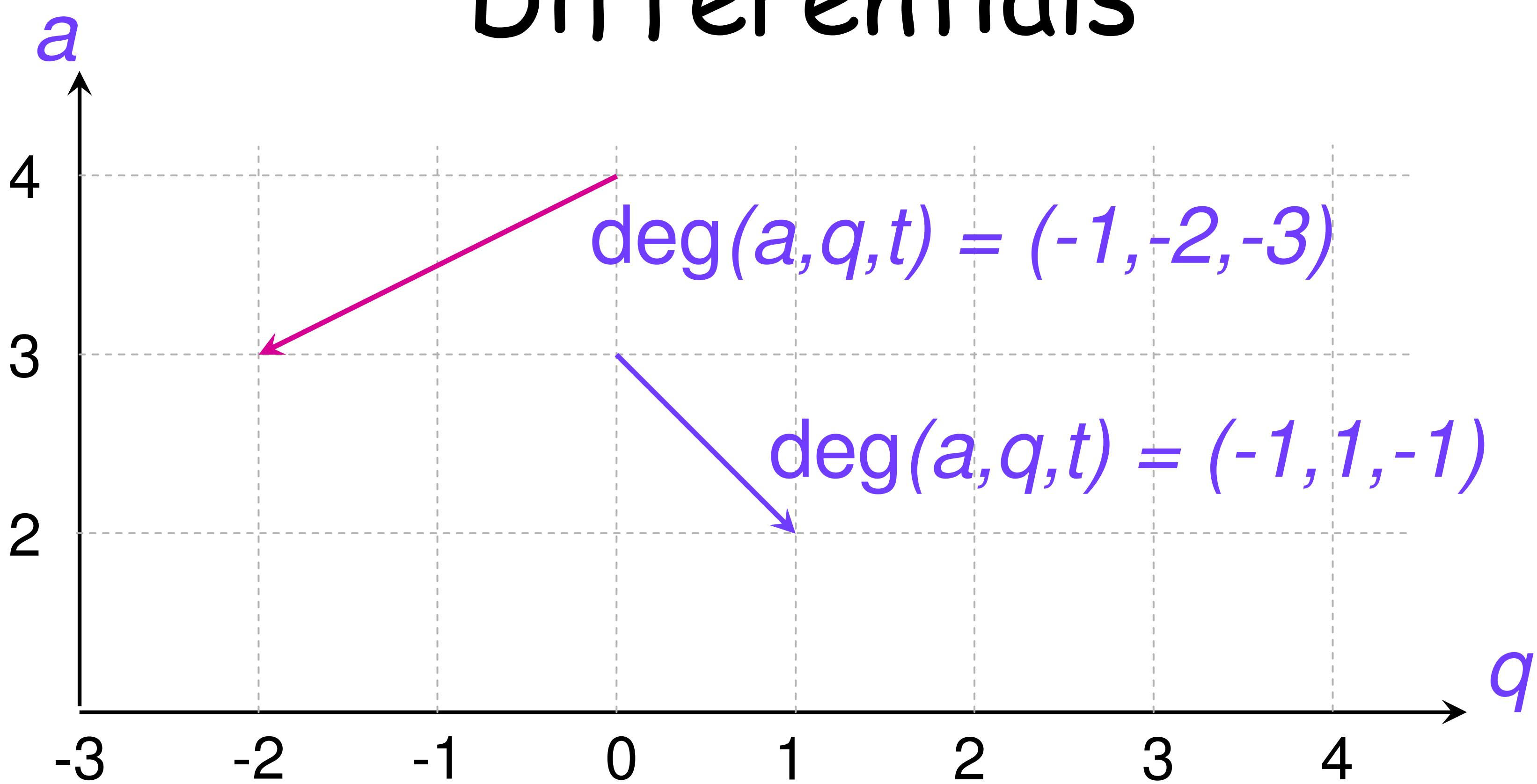
$$P^{\square\square} = q^{4N+3} - q^{3N+4} - q^{3N+3} - q^{3N+1} - q^{3N} + q^{2N+4} + q^{2N+2} + q^{2N+1} + q^{2N-2}$$



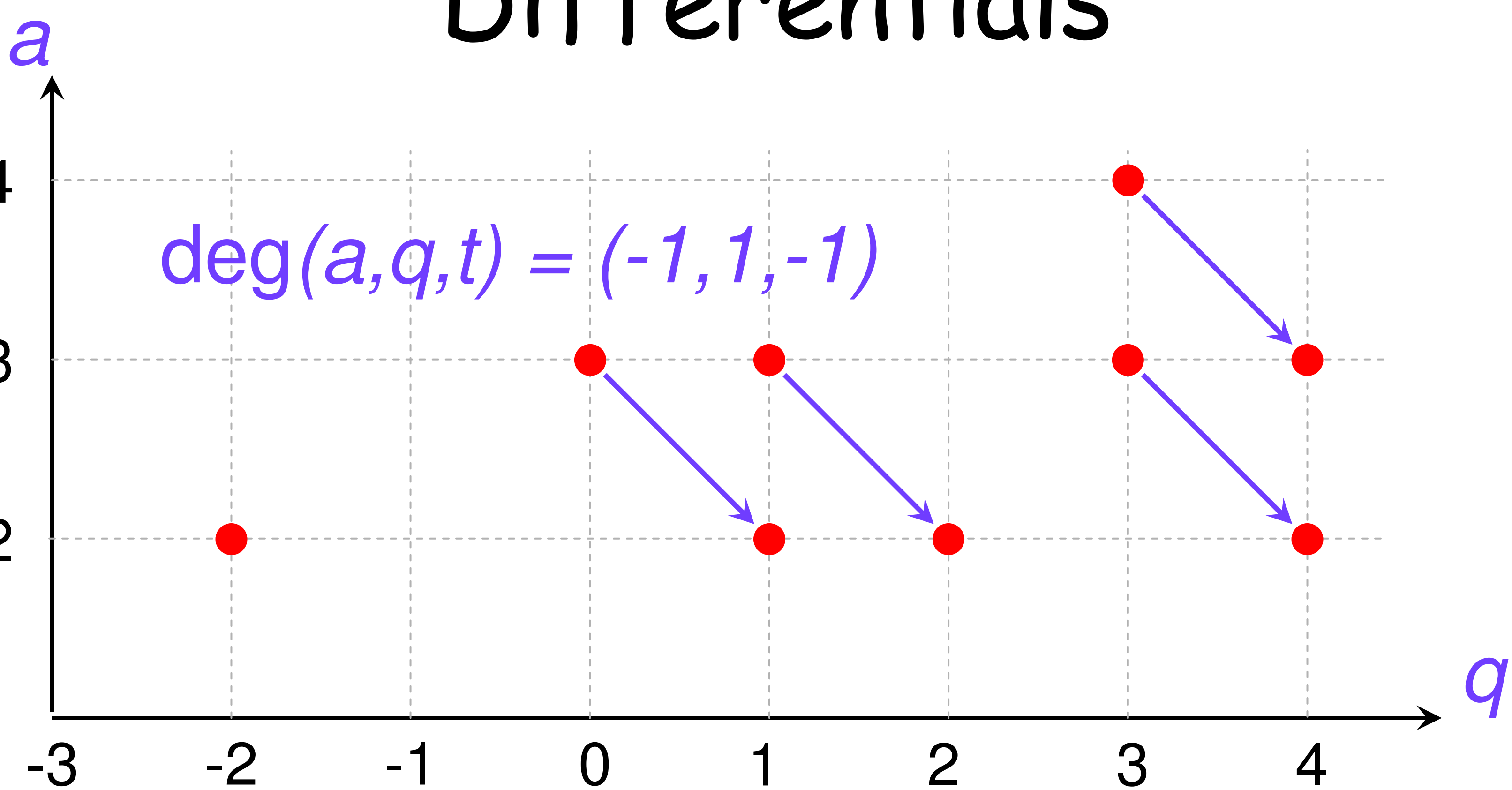


$$P^{\square\square} = q^{4N+3} - q^{3N+4} - q^{3N+3} - q^{3N+1} - q^{3N} + q^{2N+4} + q^{2N+2} + q^{2N+1} + q^{2N-2}$$

Differentials

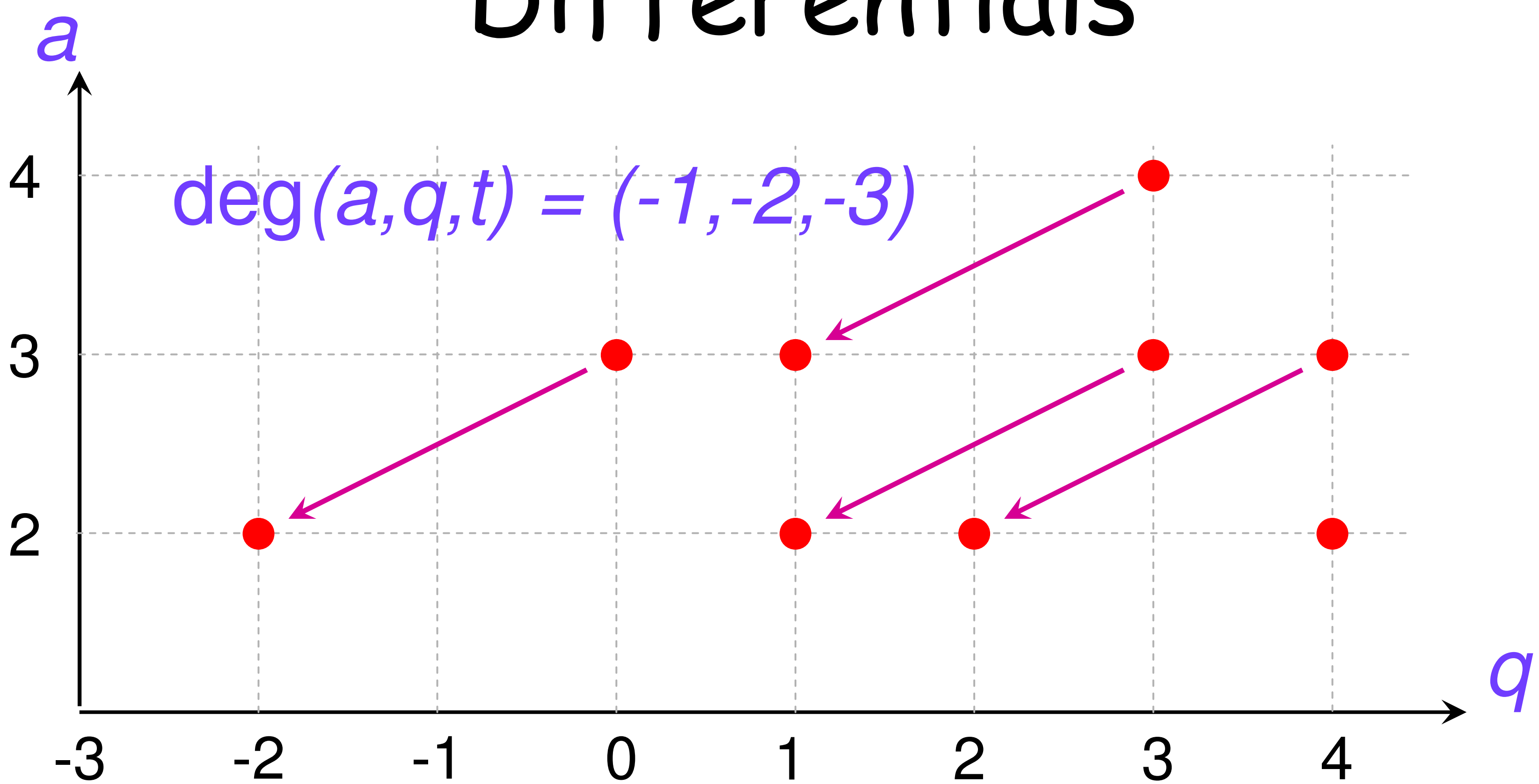


Differentials



$$P^{\square\square} = q^{4N+3} - q^{3N+4} - q^{3N+3} - q^{3N+1} - q^{3N} + q^{2N+4} + q^{2N+2} + q^{2N+1} + q^{2N-2}$$

Differentials



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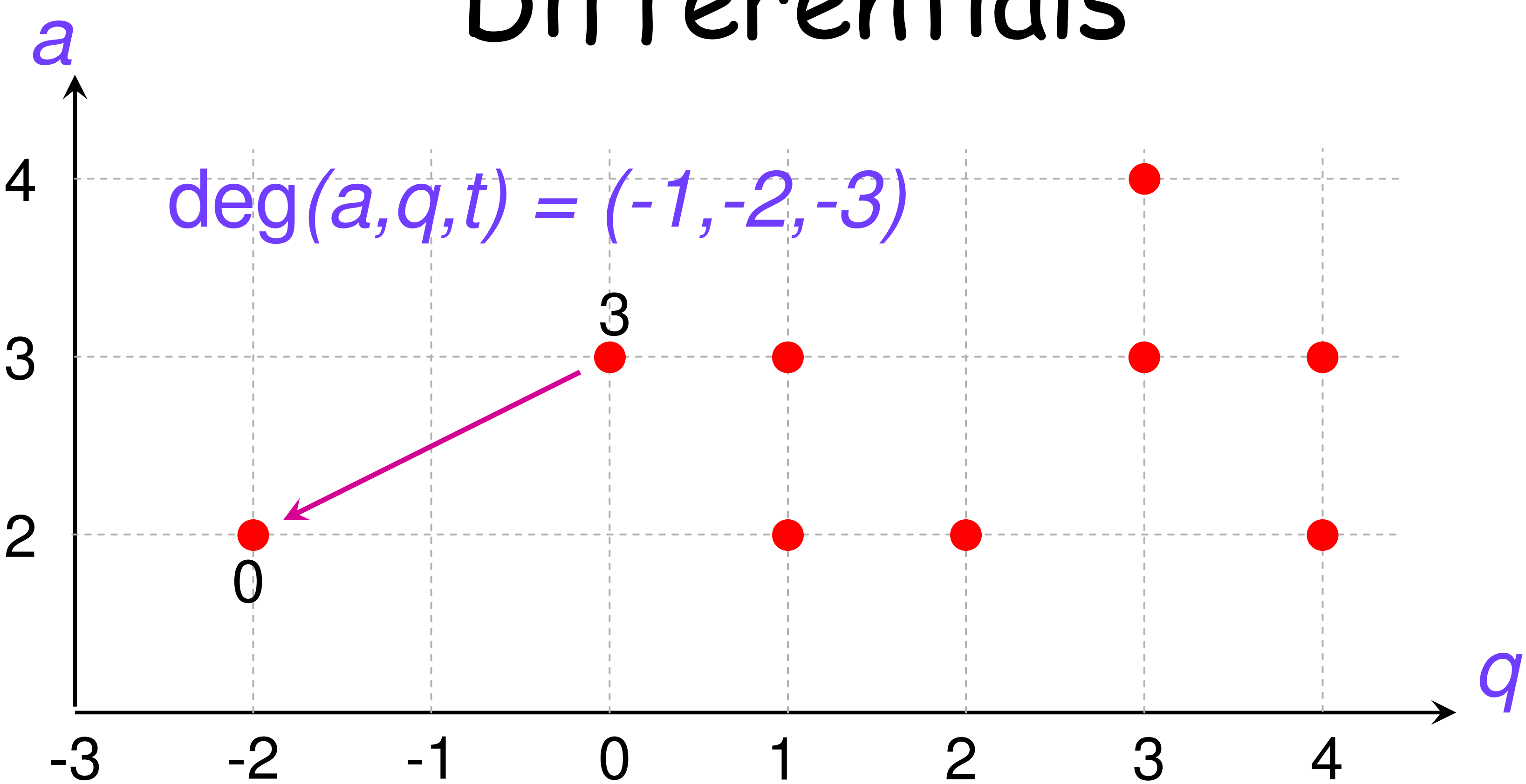
Differentials



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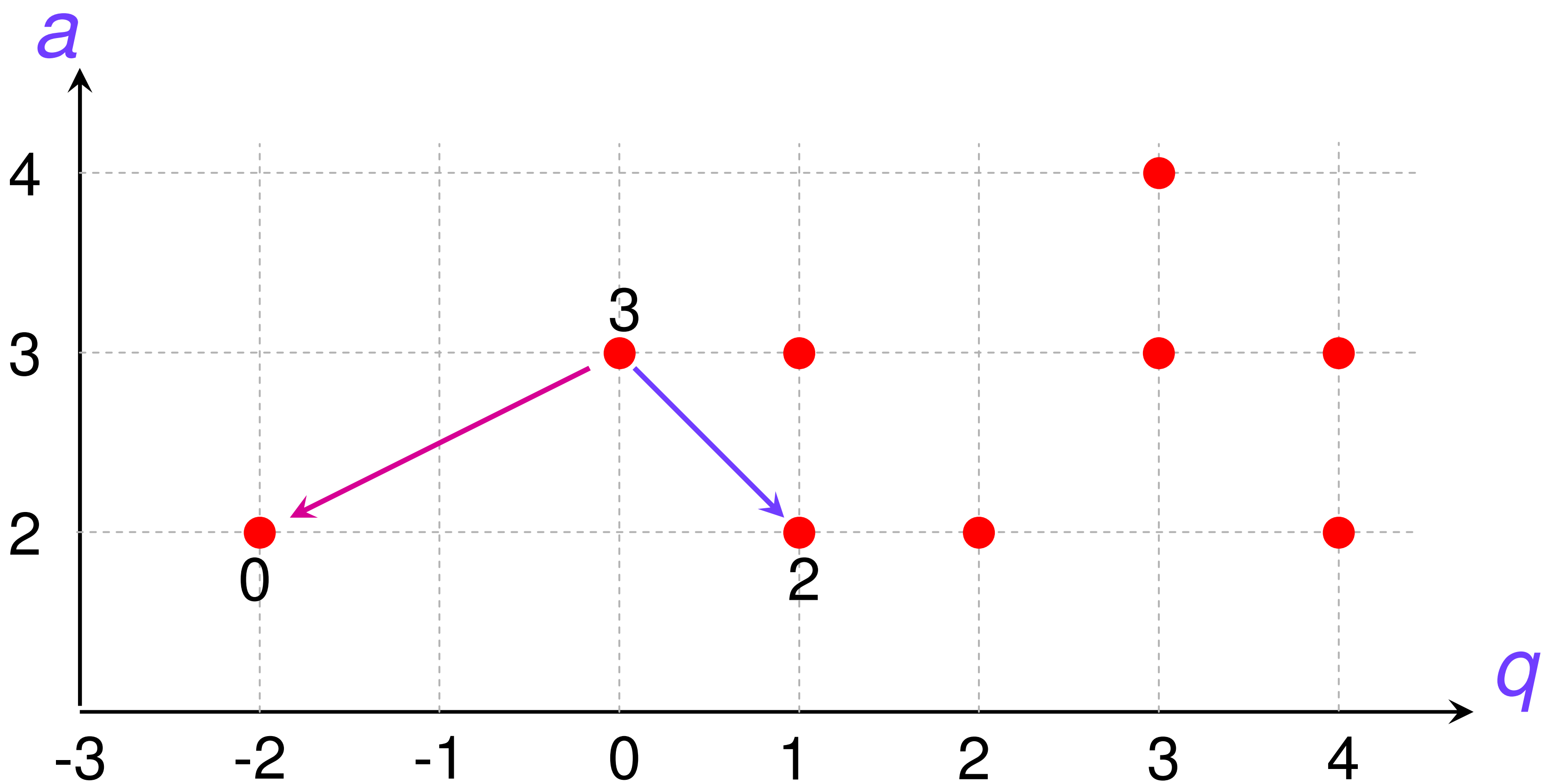
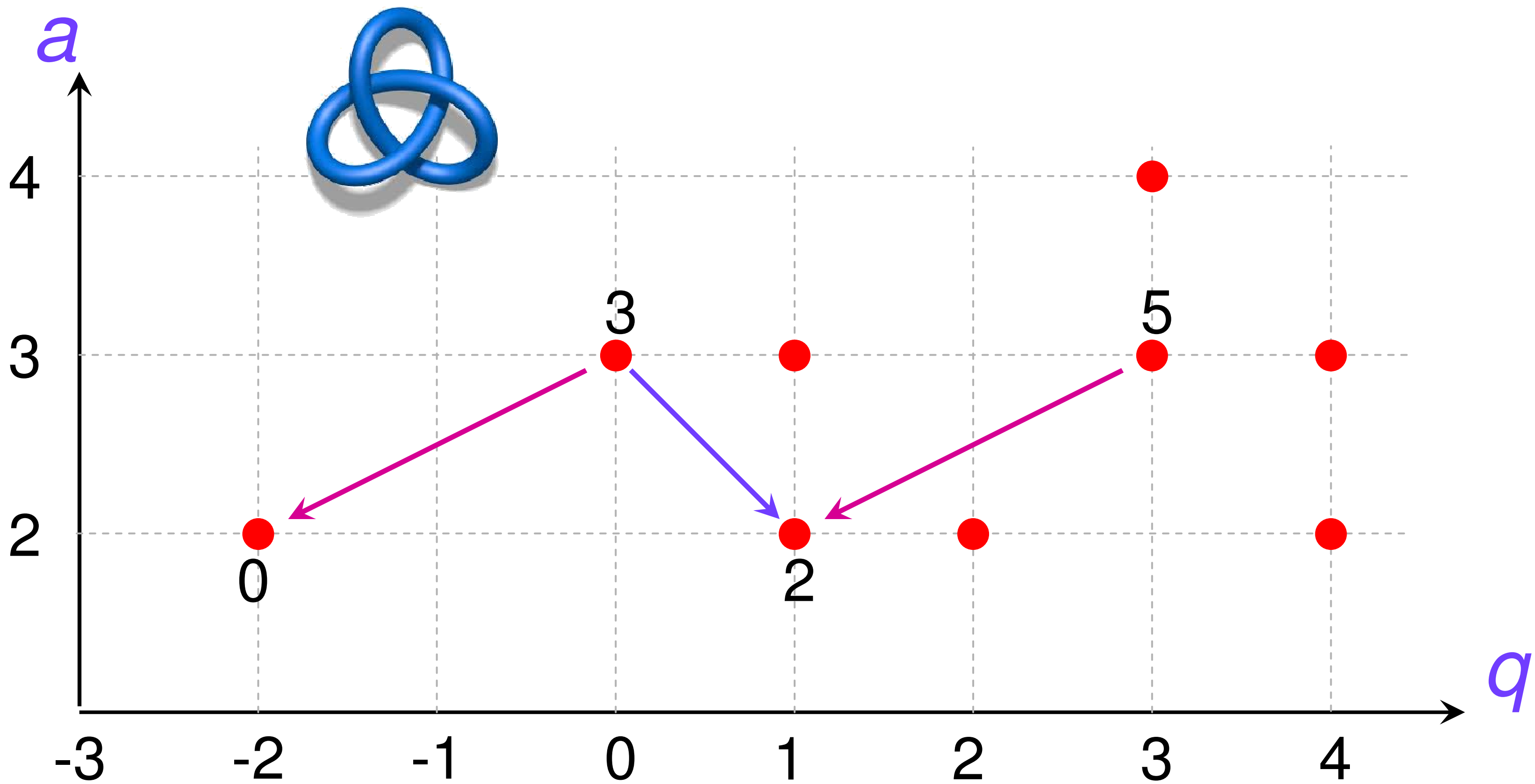
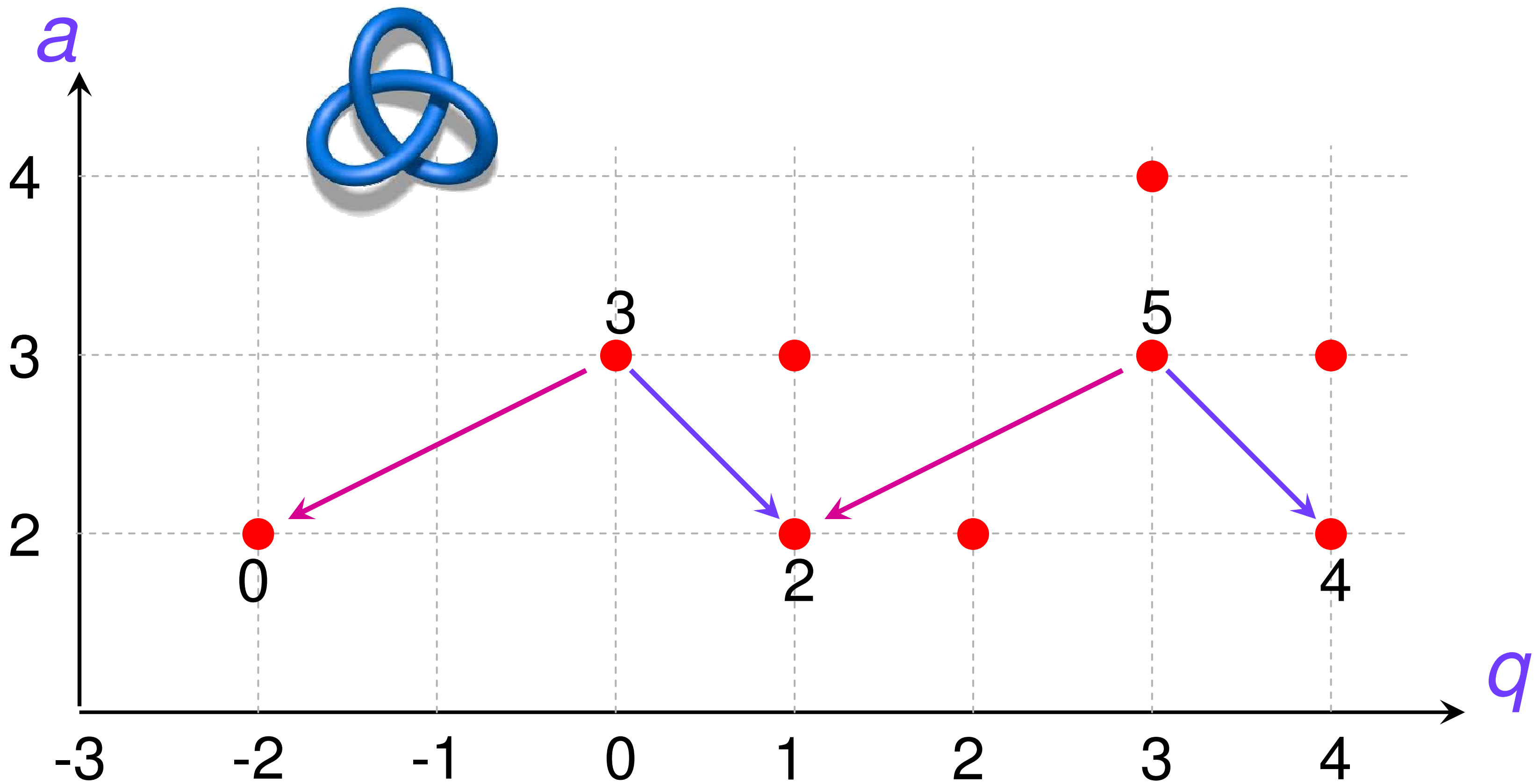


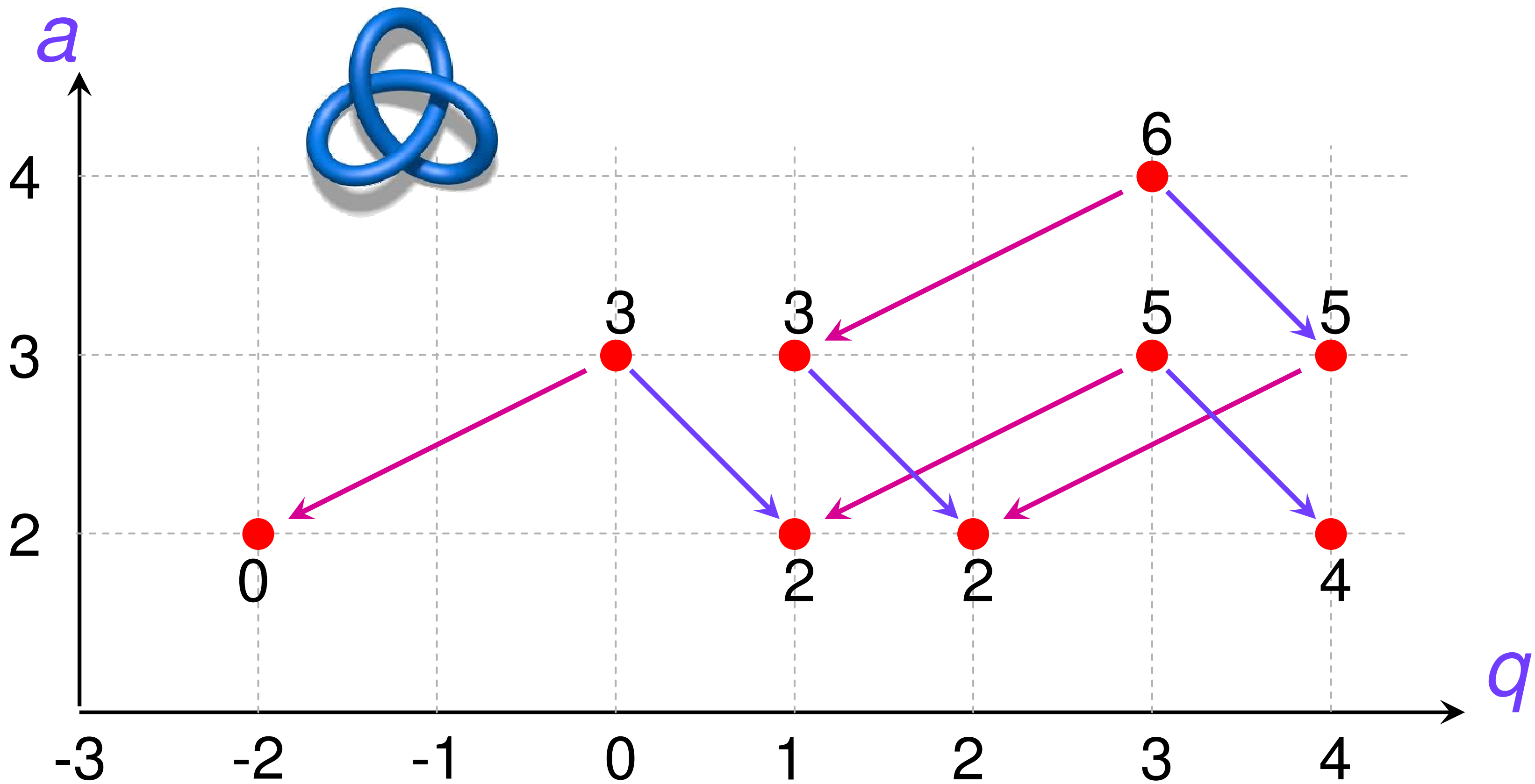
$$P^{\square\!\square} = q^{4N+3} - q^{3N+4} - q^{3N+3} - q^{3N+1} - q^{3N} + q^{2N+4} + q^{2N+2} + q^{2N+1} + q^{2N-2}$$



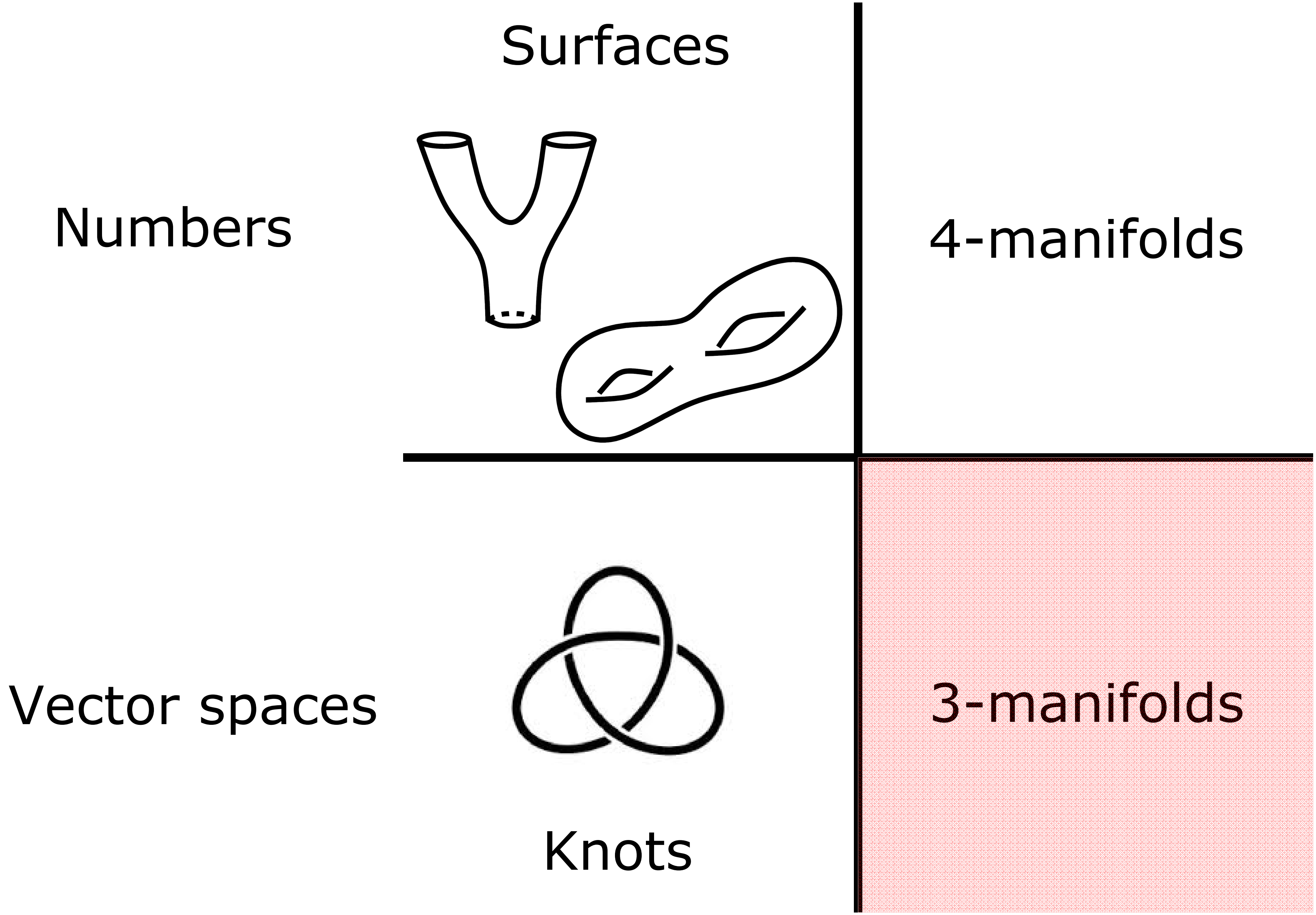
$$P^{\square\!\square} = q^{4N+3} - q^{3N+4} - q^{3N+3} - q^{3N+1} - q^{3N} + q^{2N+4} + q^{2N+2} + q^{2N+1} + q^{2N-2}$$



$$P^{\square\!\square} = q^{4N+3} - q^{3N+4} - q^{3N+3} - q^{3N+1} - q^{3N} + q^{2N+4} + q^{2N+2} + q^{2N+1} + q^{2N-2}$$



$$P^{\square\!\square} = q^{4N+3} - q^{3N+4} - q^{3N+3} - q^{3N+1} - q^{3N} + q^{2N+4} + q^{2N+2} + q^{2N+1} + q^{2N-2}$$


Surfaces
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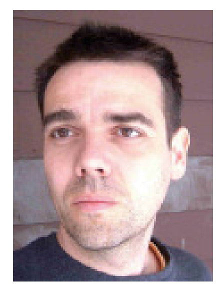
Knots

3-manifold homology in 2015

1 "monopole Floer homology"
based on Seiberg-Witten equations

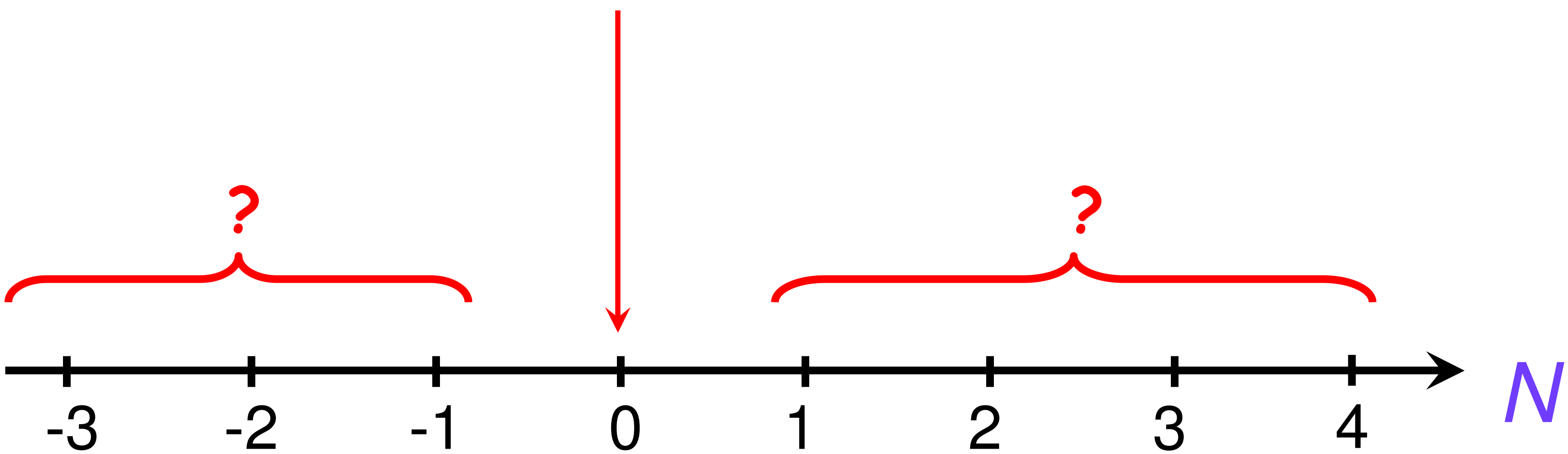

2 "embedded contact homology"
homology version of *SW=Gr*

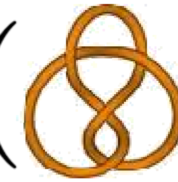

3 "Heegaard Floer homology"
("Atiyah-Floer conjecture")

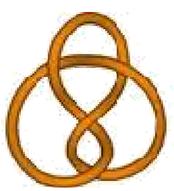
1 + 2 + 3 = 1

$$HF(M_3) \cong HM(M_3) \cong ECH(M_3)$$



Knot Homology $\mathcal{H}($$)$

$$1+\frac{1}{q^4t^2}+\frac{1}{qt}+qt+q^4t^2$$

WRT()

$$1+\frac{1}{q^4}-\frac{1}{q}-q+q^4$$

Homology

3-manifold $\longrightarrow$ q-series

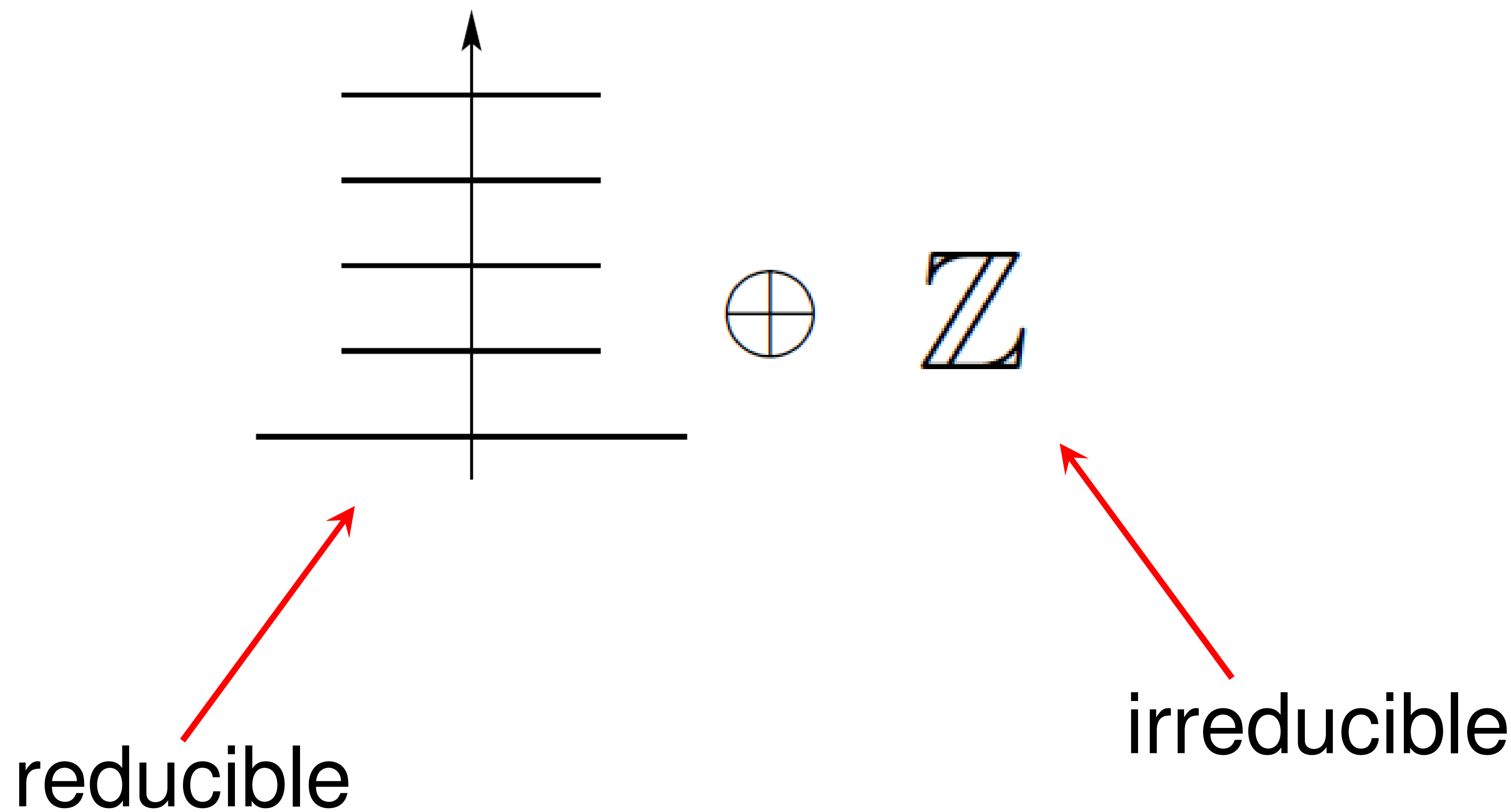
$$\mathcal{H}_{a}(M_3)$$

labeled by
Abelian flat
connection on M_3

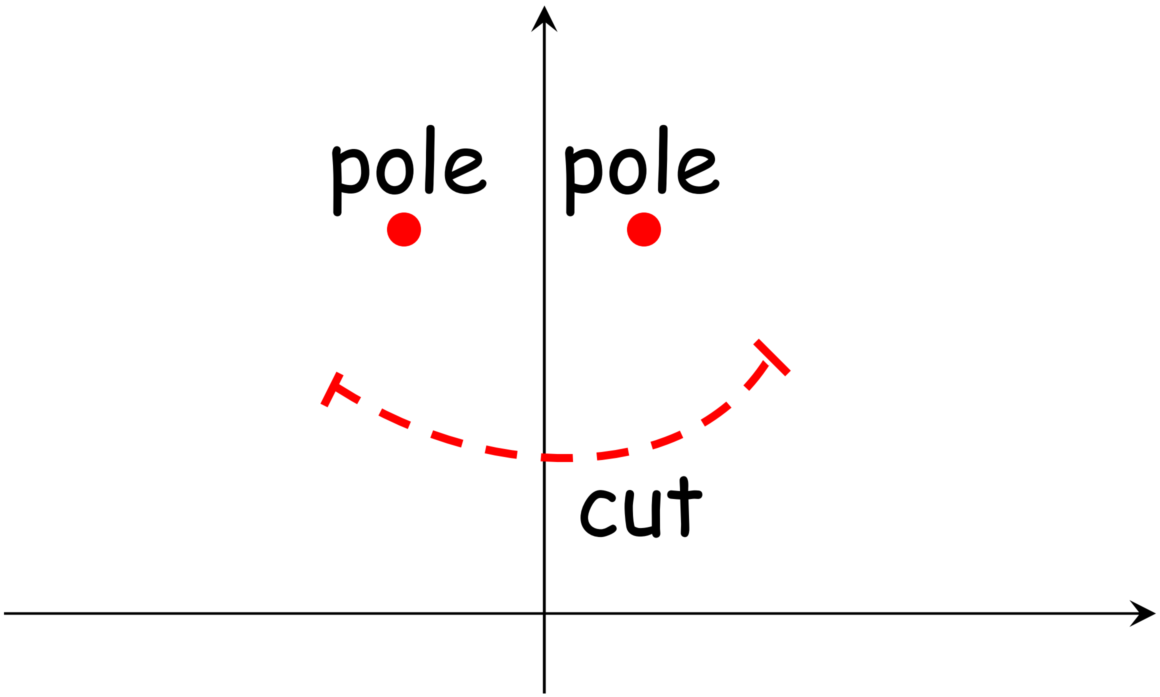
3-manifold $\longrightarrow$ $\widehat{Z}_a(q)$

$$DW = \sum_{\substack{a \\ \text{basic classes}}} SW(a)$$

$$\widecheck{HM}(\Sigma(2,3,7))$$



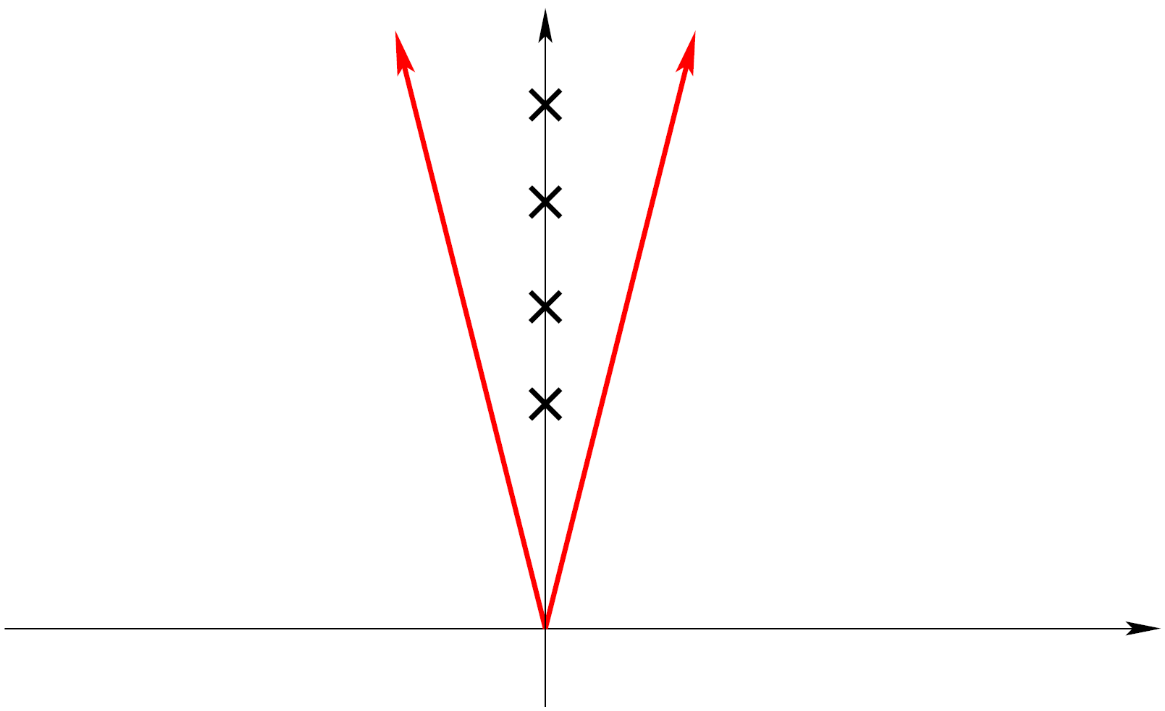
Resurgence



Emile Borel
(1871-1956)

$$Z(\hbar) = \sum_{k=0}^{\infty} p_k \hbar^k \quad \Rightarrow \quad BZ(t) = \sqrt{2\pi} \sum_{k=0}^{\infty} \frac{p_k}{k!} t^k$$

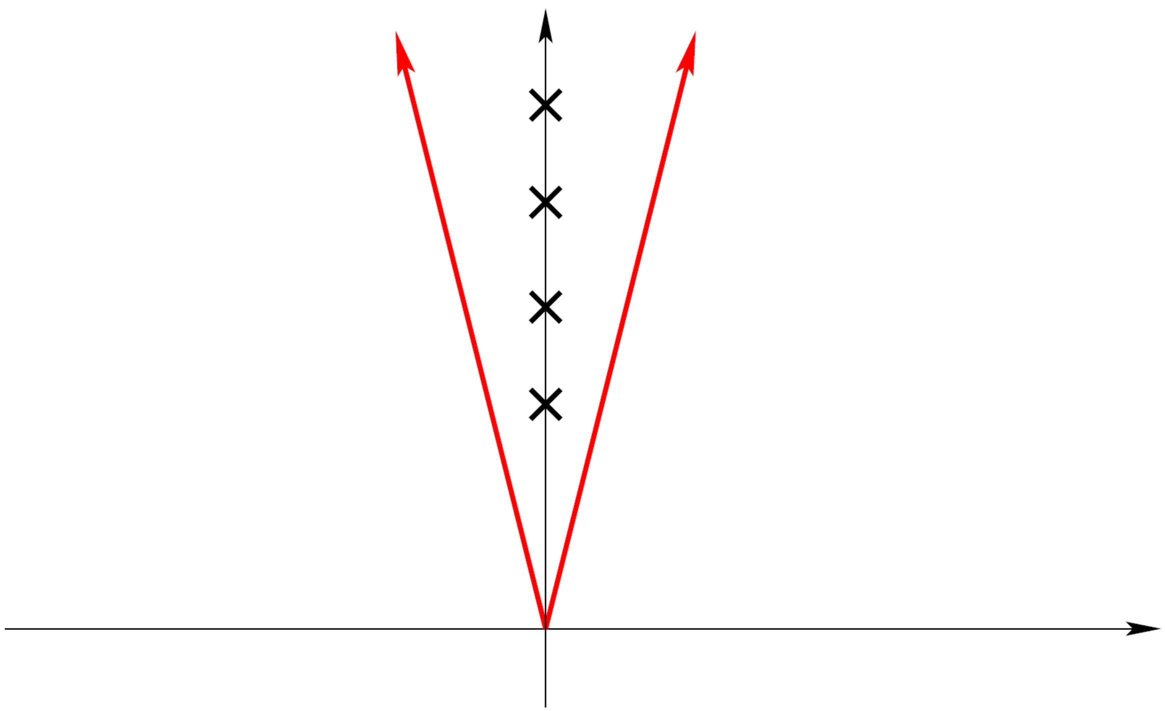
Borel plane



Emile Borel
(1871-1956)

$$\mathcal{S}_\theta Z(\hbar) = \frac{1}{\hbar}\int_0^{+\infty e^{i\theta}} dt e^{-t/\hbar}\widetilde{BZ}(t)$$

Borel plane



Emile Borel
(1871-1956)

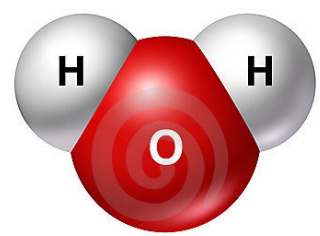
1) position of singularities   2) “residues”

position of singularities: $\ell_{\alpha\beta} = S_\beta - S_\alpha$

$$\mathcal{S}_\theta Z_\alpha^{\text{pert}}(\hbar) = Z_\alpha^{\text{pert}}(\hbar) + \sum_\beta n_\beta^\alpha e^{-\frac{\ell_{\alpha\beta}}{\hbar}} Z_\beta^{\text{pert}}(\hbar)$$

<u>Theorem [G-Marino-Putrov]:</u>

$$n_\beta^\alpha = 0 \qquad \begin{aligned} \alpha &= \text{irreducible} \\ \beta &= \text{abelian} \end{aligned}$$

<u>Corollary:</u> $\text{WRT}(M_3) =$ 

From knots to 3-manifolds

$$M_3 = S^3_p(K)$$

$$H_1(M_3) = \mathbb{Z}_p$$

$$\widehat{Z}_a = \int_{|x|=1} \frac{dx}{2\pi i x} \frac{(x^2;q)_\infty}{(-x^2tq;q)_\infty} \frac{(x^{-2};q)_\infty}{(-x^{-2}tq;q)_\infty} \sum_{n\in\mathbb{Z}} q^{\frac{(pn+a)^2}{p}} x^{2(pn+a)}$$

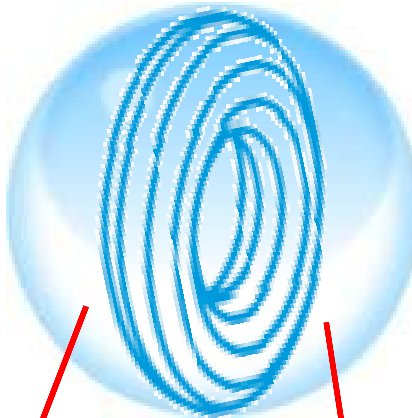
“Laplace transform”

cf.
A.Beliakova,
C.Blanchet,
T.T.Q.Le

$$F(q,x) = \sum_{m,n} N_{m,n}\, q^m x^n$$

$$\widehat{Z}_a = \int_{|x|=1} \frac{dx}{2\pi i x} \overbrace{\frac{(x^2;q)_\infty}{(-x^2tq;q)_\infty} \frac{(x^{-2};q)_\infty}{(-x^{-2}tq;q)_\infty}} \sum_{n\in\mathbb{Z}} q^{\frac{(pn+a)^2}{p}} x^{2(pn+a)}$$

$$\widehat{Z}_a(q) \;=\; \mathcal{L}_p^{(a)}\left[F(q,x)\right] \;=\; \sum_{m,n=pj+a} N_{m,n}\, q^{m+\frac{j^2}{4p}}$$



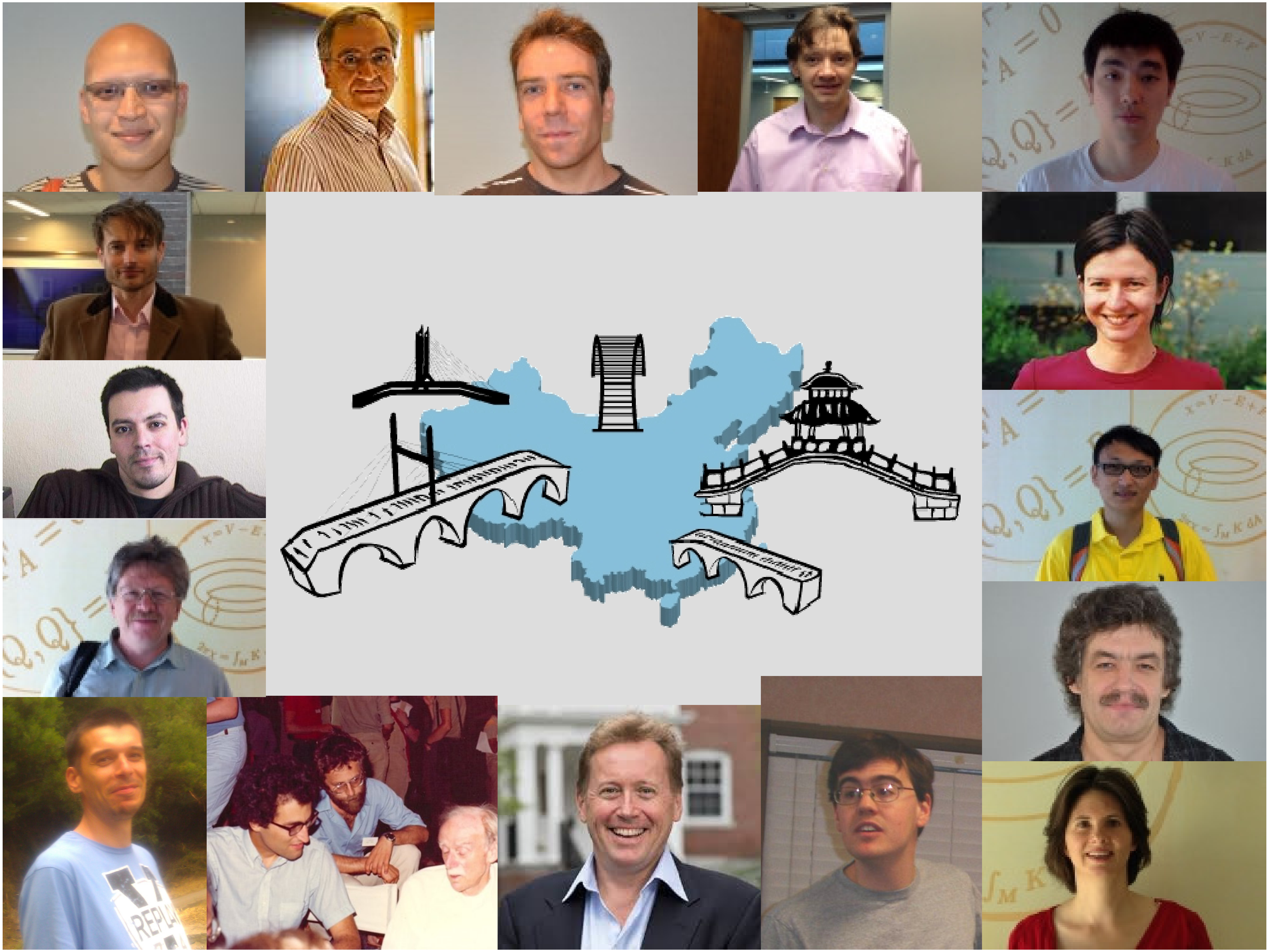
$$\widehat{Z}_a = \int_{|x|=1} \frac{dx}{2\pi i x} \underbrace{\frac{(x^2;q)_\infty}{(-x^2tq;q)_\infty}}_{F(\text{unknot};\,x)} \underbrace{\frac{(x^{-2};q)_\infty}{(-x^{-2}tq;q)_\infty}}_{F(\text{unknot};\,x^{-1})} \sum_{n\in\mathbb{Z}} q^{\frac{(pn+a)^2}{p}} x^{2(pn+a)}$$

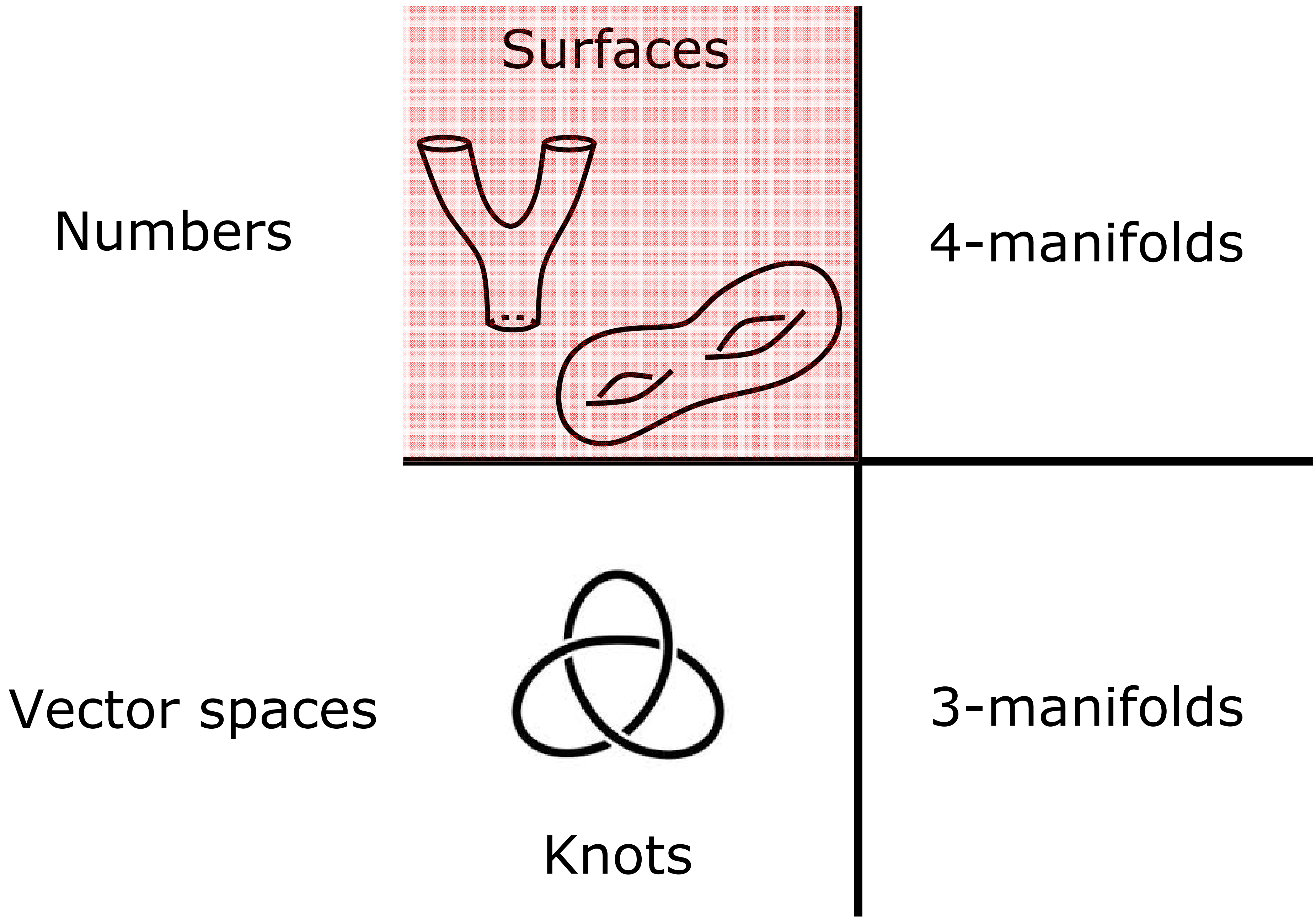
$$\hat{A}^{\text{super}}(\hat{x},\hat{y})\ F(\text{unknot};x) = 0$$

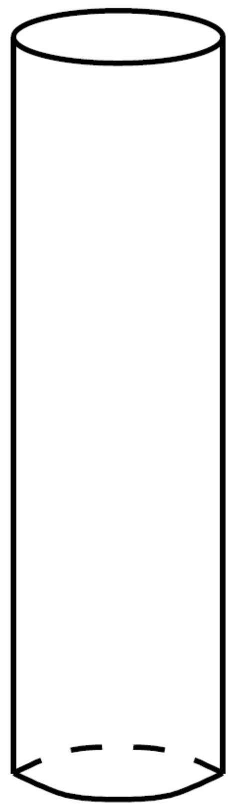
JOURNAL OF KNOT THEORY AND ITS RAMIFICATIONS

Special Issue

World Scientific




Surfaces
Numbers
4-manifolds
Vector spaces
3-manifolds
Knots



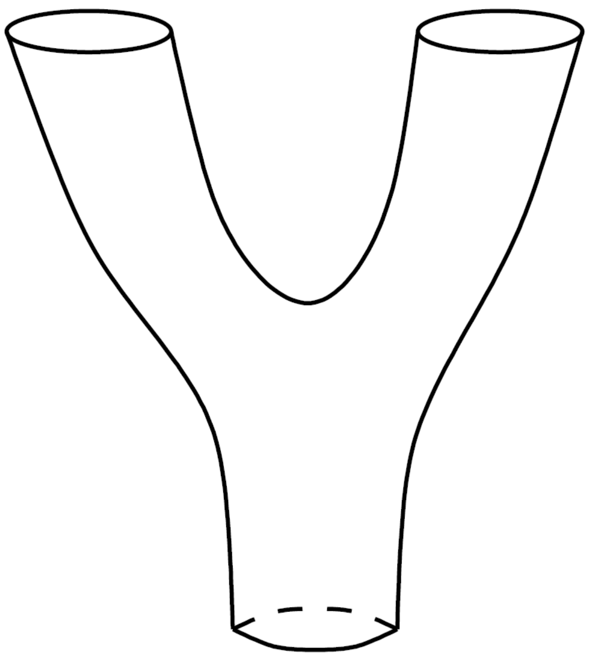
$\dim(\mathcal{H}) = 2$

$\mathbb{CP}(1)$ A-model

LG model $W = e^{x} + e^{-x}$

LG model $W = x^3$

LG model $W = x^3 + x$



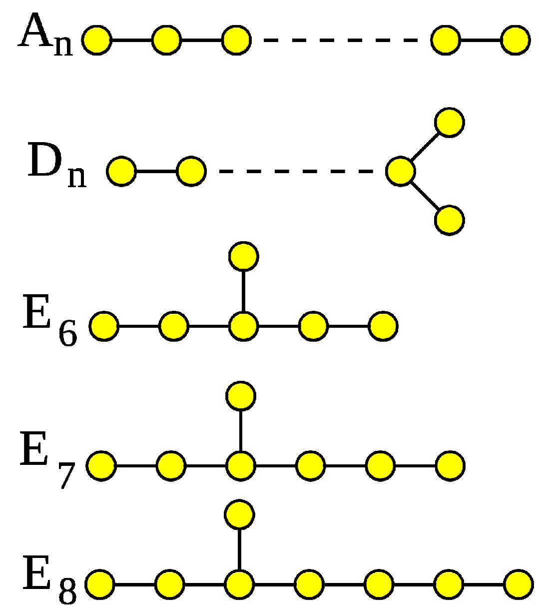
$\mathbb{CP}(1)$ A-model

LG model $W = e^{x} + e^{-x}$

LG model $W = x^3$

LG model $W = x^3 + x$

$$W_{sl(N),\Lambda^k}(z_1,\ldots,z_k) = x_1^{N+1} + \cdots + x_k^{N+1}$$



$$W_{so(N)} = x^{N-1} + xy^2$$

$$W_{E_6,27} = z_1^{13} - \frac{25}{169} z_1 z_4^3 + z_4 z_1^9$$

$$W_{E_7,56} = \frac{2791}{19} z_1^{19} + 37 z_1^{14} z_5 - 21 z_1^{10} z_9 + z_5^2 z_9 + z_1 z_9^2$$

$$W_{sl(N),\Lambda^k}(z_1,\ldots,z_k) = x_1^{N+1} + \cdots + x_k^{N+1}$$

$$W_{so(N)} = x^{N-1} + xy^2$$

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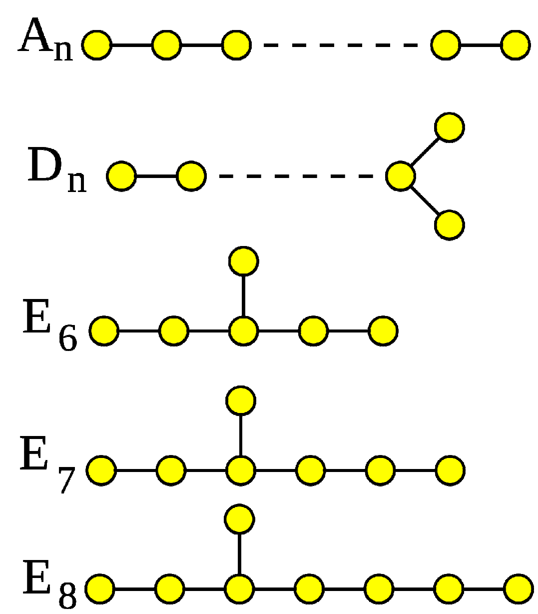
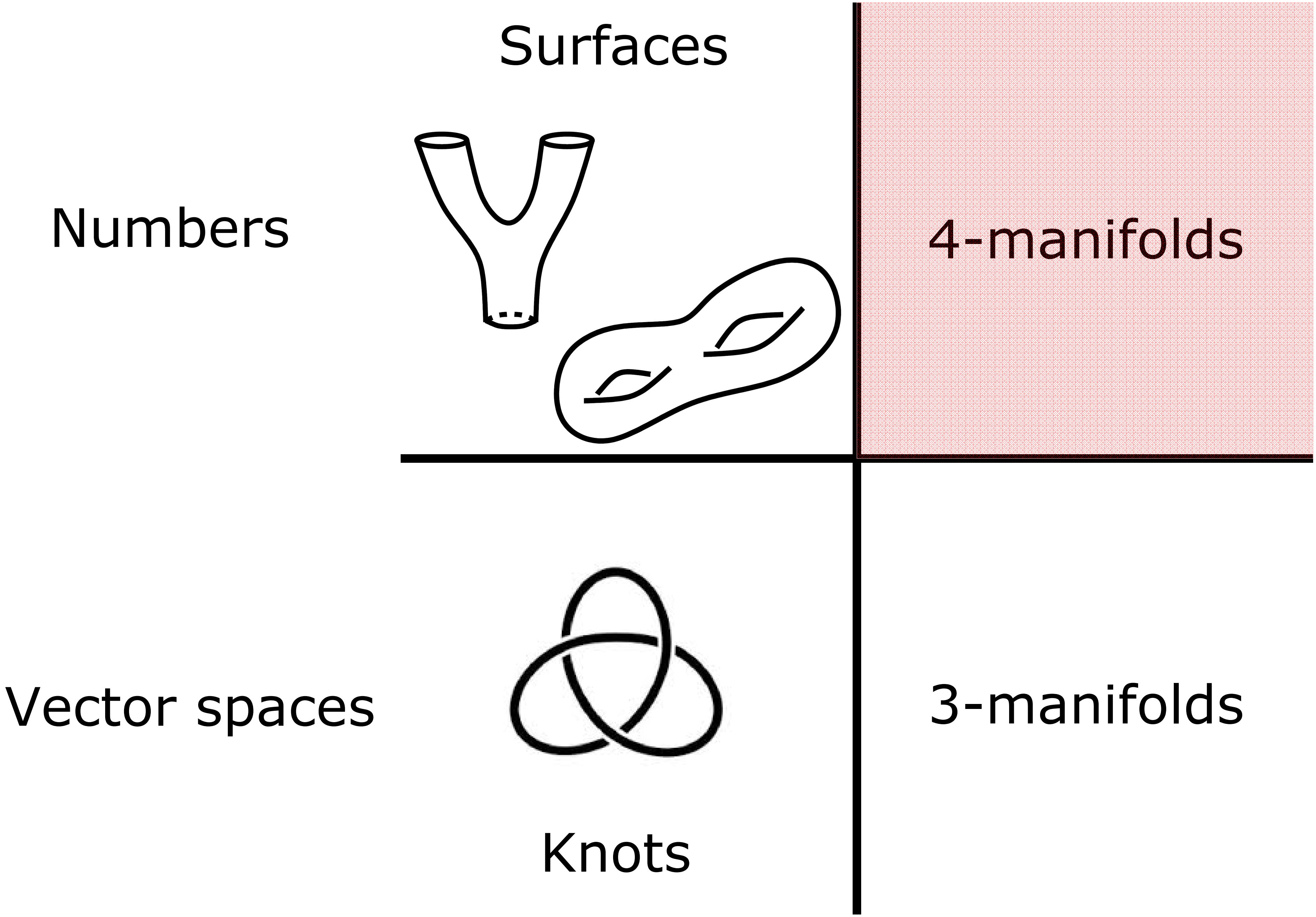


Example: trefoil

$$\mathscr{H}^{\mathfrak{e}_6,\mathbf{27}}(\mathbf{3}_1) = 1 + q^2t^2 + q^5t^2 + q^{10}tu + q^{13}tu + q^{10}t^4 + q^{15}t^3u + q^{18}t^3u + q^{23}t^2u^2$$



Surfaces
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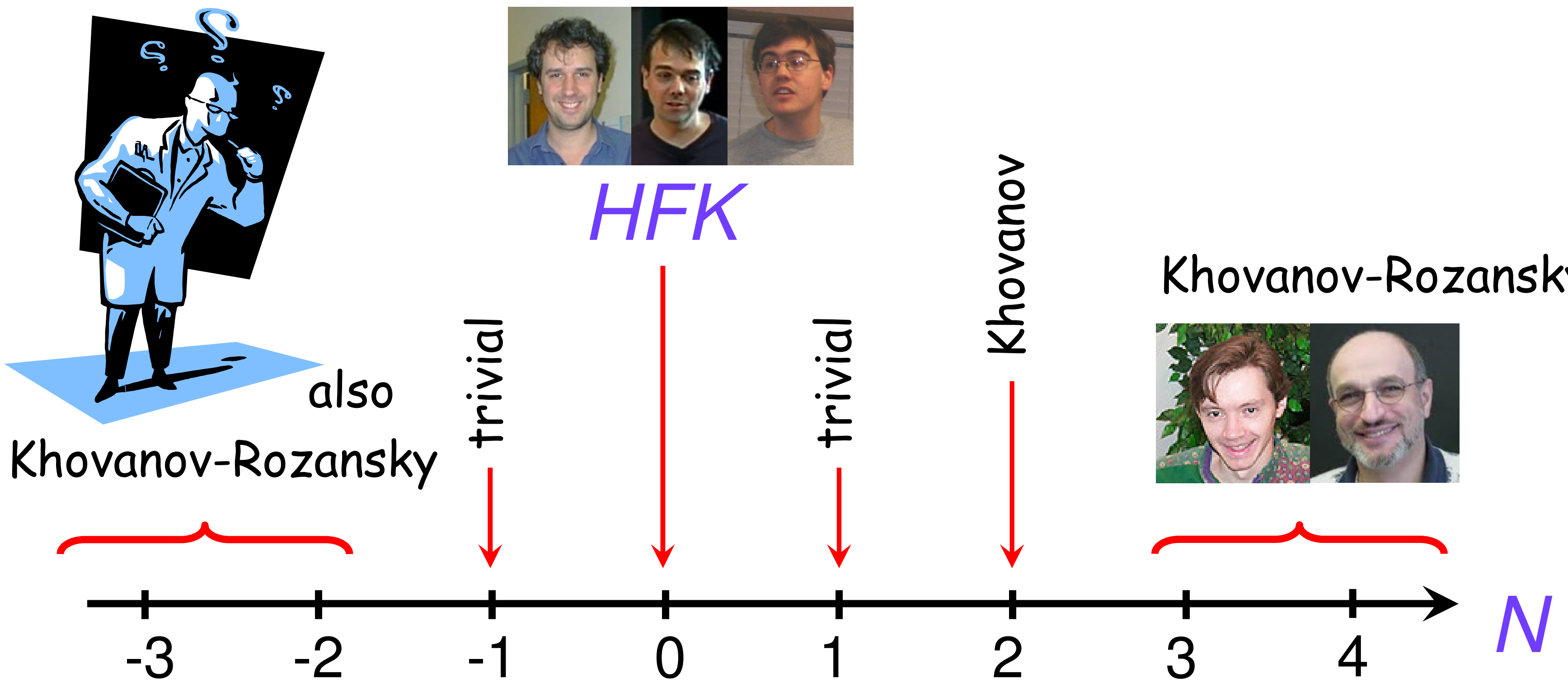
Unification of different theories



In categorification of quantum group invariants of knots, *HFK* is an oddball ... Will play an important role in categorification of 3-manifold invariants.