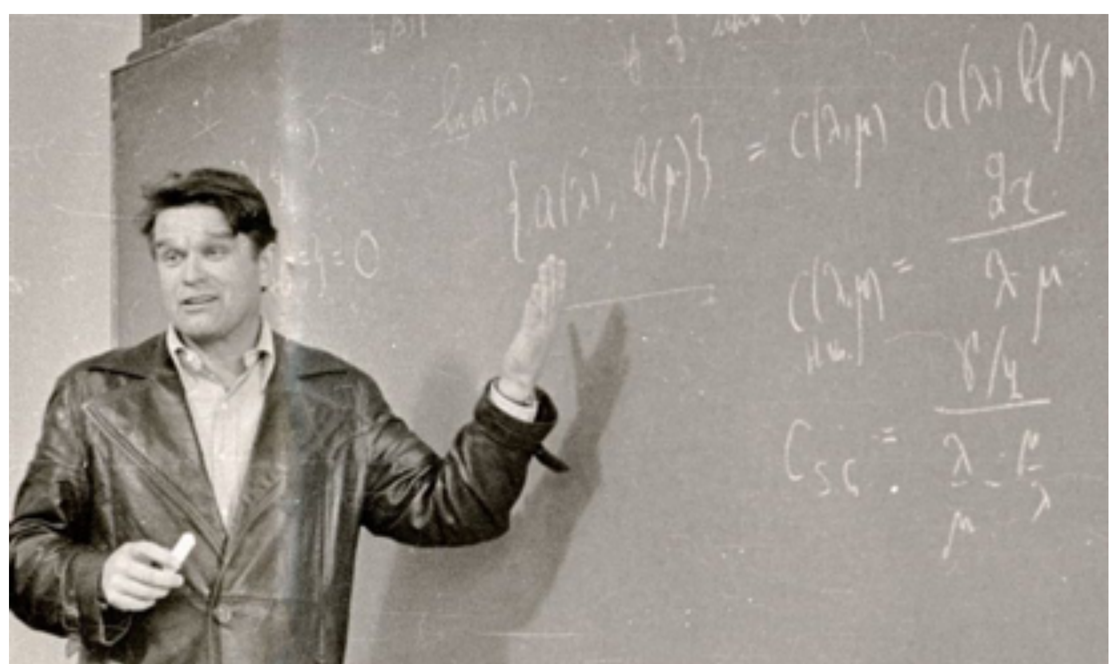
The Spectral Theory of Quantum Mirror Curves

Marcos Mariño
University of Geneva

In memoriam Ludwig Faddeev



Motivation: a simple spectral problem

Let us consider a non-relativistic particle in a confining *cosh* potential, described by the Hamiltonian

$$\mathsf{H} = \mathsf{y}^2 + \Lambda^2 \left(\mathrm{e}^{\mathsf{x}} + \mathrm{e}^{-\mathsf{x}} \right) \qquad [\mathsf{x}, \mathsf{y}] = \mathrm{i}\hbar$$

The corresponding stationary Schroedinger equation is the *modified Mathieu equation*. In addition, this is the Hamiltonian of the *N*=2 periodic Toda lattice. We have an infinite, discrete spectrum. How would you calculate it?

One can try the WKB approximation, which involves the spectral curve

$$y^2 = E - 2\Lambda^2 \cosh(x)$$

At leading order, one finds the standard Bohr-Sommerfeld quantization condition

$$\mathrm{vol}_0(E) = \oint y(x)\mathrm{d}x = 2\pi\hbar\left(n + \frac{1}{2}\right) \qquad n \in \mathbb{Z}_{\geq 0}$$

This is just the first term in the all-orders WKB method [Dunham]

$$\mathrm{vol}_{\mathrm{pert}}(E) = \sum_{k\geq 0} \mathrm{vol}_k(E)\hbar^{2k}$$

However, this is a divergent, *asymptotic* expansion, which requires Borel resummation (and maybe exponentially small corrections in $\hbar$).

Another quantization condition for this problem was found by Gutzwiller in his study of the quantum Toda lattice. However, it is a complicated one.

A completely different approach was proposed by Nekrasov and Shatashvili (NS). Let us consider the NS limit of the instanton free energy of 4d, *N*=2, *SU(2)* Yang-Mills theory

$$F_{\rm NS}(a,\hbar) = F_{\rm p}(a,\hbar) + \sum_{n\geq 1} \mathcal{F}_n(a,\hbar)\Lambda^{4n}$$

This is a well-defined function for any $\hbar$ and sufficiently large *a*. Then, the NS exact quantization condition is given by

$$\frac{\partial F_{\rm NS}}{\partial a} = 2\pi\hbar\left(n+\frac{1}{2}\right)$$

supplemented by the "quantum mirror map" $a = a(E,\hbar)$

This is in fact a *convergent resummation* of the all-orders WKB expansion!

From 4d to 5d

Let us consider now the operator

$$\mathsf{O} = \mathrm{e}^{\mathsf{x}} + \mathrm{e}^{-\mathsf{x}} + \mathrm{e}^{\mathsf{y}} + \mathrm{e}^{-\mathsf{y}}$$

This is the Hamiltonian of the *N*=2 *relativistic* Toda lattice. One could think that its spectrum can be solved by the NS limit of the instanton free energy for the SU(2), *5d* super YM theory:

$$\frac{\partial F_{\mathrm{NS}}}{\partial t} = 2\pi\hbar\left(n + \frac{1}{2}\right)$$

$$t = t(E, \hbar)$$

quantum mirror map [ACDKV]

A problem in 5d

Unfortunately, there is a dense set of poles at $\hbar \in 2\pi\mathbb{Q}$

$$F_{\rm NS}(t,\hbar) = F_{\rm p}(t,\hbar) + \cot\left(\frac{\hbar}{2}\right){\rm e}^{-t} + \cdots$$

Since this resums the perturbative all-orders WKB expansion, the cure to this disease has to be *non-perturbative in* $\hbar$
[Kallen-M.M. 2013]

What is then the right, exact quantization condition for the relativistic Toda lattice? (no analogue of Gutzwiller quantization here).

To answer this (and other) questions, we will construct a general (spectral) theory of *quantum mirror curves for toric CYs*

Toric CYs and their mirrors

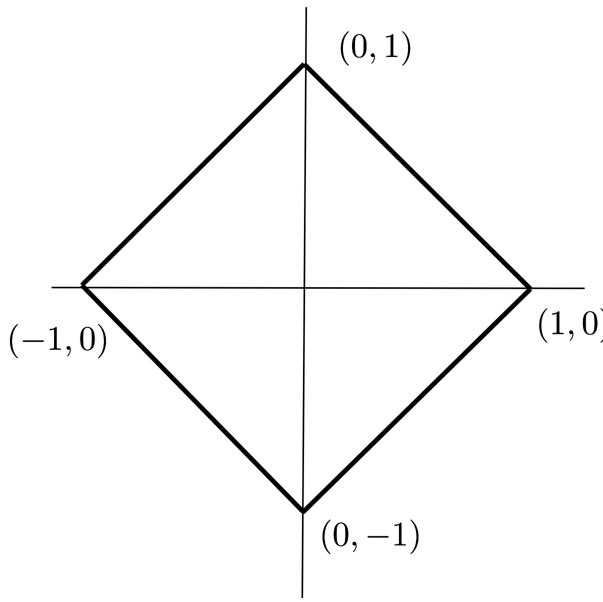
The simplest yet non-trivial CY threefolds are *toric CYs*, which are noncompact. They can be described by *convex lattice polygons.* Their mirror manifolds reduce to algebraic curves

$$W_X(\mathrm{e}^x, \mathrm{e}^y) = 0$$

given by the Newton polynomial of the polygon.

Example: the canonical bundle over $\mathbb{F}_0$

"local $\mathbb{F}_0$"



inner point of the polygon

$$W_X(\mathrm{e}^x, \mathrm{e}^y) = \mathrm{e}^x + \mathrm{e}^{-x} + \mathrm{e}^y + \mathrm{e}^{-y} + \kappa = 0$$

Topological strings

We will be focus on the (conventional, unrefined) topological string on these geometries. Their genus g free energies can be put together in a formal (and asymptotically divergent) series

$$F(t, g_s) = \sum_{g=0}^{\infty} F_g(t) g_s^{2g-2}$$

moduli of the CY

string coupling constant

When computed in the so-called large radius frame, the genus g free energies $F_g(t)$ are generating functionals for the Gromov-Witten invariants of the CY

Quantum curves

$$\text{Curve} \longrightarrow \text{Operator} \longrightarrow \text{Wavefunction}$$

There are by now many theories of "quantum curves". They do not really use Hilbert spaces and they focus on *formal* WKB-like solutions. They tend to use fancy math (opers, D-modules, ...).

Our approach will be different. We will use an old-fashioned framework (compact operators on $L^2(\mathbb{R})$, Fredholm theory), and we will look for *actual* solutions (spectra, eigenfunctions).

Operators from mirror curves

It was first proposed in [ADKMV] that mirror curves can be "quantized" by promoting *x, y* to canonically conjugate Heisenberg operators

$$[\mathsf{x}, \mathsf{y}] = \mathrm{i}\hbar \qquad \hbar \in \mathbb{R}_{>0}$$

For simplicity, we will focus on mirror curves of genus one. Weyl quantization of the mirror curve (without the inner point) produces a *self-adjoint operator* on $L^2(\mathbb{R})$

$$W_X(\mathrm{e}^x, \mathrm{e}^y) \to \mathsf{O}_X$$

For example, for local $\mathbb{F}_0$ we find

$$\mathsf{O} = \mathrm{e}^{\mathsf{x}} + \mathrm{e}^{-\mathsf{x}} + \mathrm{e}^{\mathsf{y}} + \mathrm{e}^{-\mathsf{y}}$$

Examples from local surfaces

Many examples from *local del Pezzo CYs*

$$\mathcal{O}(K_S) \to S$$

where *S* is a del Pezzo complex algebraic surface

S	$\mathcal{O}_S(x,y)$
$\mathbb{P}^2$	$e^x + e^y + e^{-x-y}$
$\mathbb{F}_0$	$e^x + \zeta e^{-x} + e^y + e^{-y}$
$\mathbb{F}_1$	$e^x + \zeta e^{-x} + e^y + e^{-x-y}$
$\mathbb{F}_2$	$e^x + \zeta e^{-x} + e^y + e^{-2x-y}$
$\mathcal{B}_2$	$e^x + e^y + e^{-x-y} + \zeta_1 e^{-y} + \zeta_2 e^{-x}$
$\mathcal{B}_3$	$e^x + e^y + e^{-x-y} + \zeta_1 e^{-x} + \zeta_2 e^{-y} + \zeta_3 e^{x+y}$

Trace class operators

Theorem
[Grassi-Hatsuda-M.M.,
Kashaev-M.M.,
Laptev-Schimmer-Takhtajan]

The operator $\rho_X = \mathsf{O}_X^{-1}$ on $L^2(\mathbb{R})$ is positive definite and of trace class

discrete spectrum! $\mathrm{e}^{-E_n}, \qquad n = 0, 1, \cdots$

Example: $\mathsf{O} = \mathrm{e}^{\mathsf{x}} + \mathrm{e}^{-\mathsf{x}} + \mathrm{e}^{\mathsf{y}} + \mathrm{e}^{-\mathsf{y}}$ with $\hbar = 2\pi$

$$E_0 = 2.881815429926296...$$

$$E_1 = 4.25459152858199...$$

$$E_2 = 5.28819530714418...$$

similar to confining potentials in Schrödinger theory

Spectral theory of trace class operators

The spectral information of these operators can be collected in various ways. We have the *spectral traces*

$$Z_\ell = \mathrm{Tr}\, \rho_X^\ell = \sum_{n\geq 0} \mathrm{e}^{-\ell E_n}, \qquad \ell = 1, 2, \cdots$$

and the *Fredholm (or spectral) determinant*

$$\Xi_X(\kappa) = \det(1+\kappa\rho_X) = \exp\left\{ \sum_{\ell=1}^{\infty} \frac{(-1)^\ell}{\ell} Z_\ell \kappa^\ell \right\}$$

$$= 1 + \sum_{N=1}^{\infty} Z_X(N, \hbar) \kappa^N$$

"fermionic" spectral traces

Once the Fredholm determinant is known, the spectrum can be in principle obtained by looking at its zeros

$$\Xi_X\left(\kappa = -\mathrm{e}^{E_n}\right) = 0$$

This can be interpreted in terms of the StatMech of non-interacting fermions:

$Z_X(N,\hbar)$ ⟶ canonical partition function for a *Fermi gas* of *N* particles with one-particle density matrix ρ_X

$\Xi_X(\kappa)$ ⟶ grand canonical partition function

$\kappa = \mathrm{e}^{\mu}$ fugacity

Extracting the invariants

How do we extract geometric information from spectral theory? Let us consider the following *'t Hooft limit* for the fermionic spectral traces $Z_X(N, \hbar)$

$$N \to \infty \qquad \frac{N}{\hbar} = \lambda \quad \text{fixed}$$
$$\hbar \to \infty$$

We conjecture that the asymptotic expansion of the fermionic spectral traces, in the above 't Hooft limit, is the genus expansion of the topological string (in the conifold or "magnetic" frame):

$$F_X(N, \hbar) = \log Z_X(N, \hbar) \sim \sum_{g \geq 0} F_g(\lambda) \hbar^{2-2g}$$

This is a falsifiable statement which has been tested in massive detail in the last two years. No counterexamples found.

Note that the conventional topological string coupling constant is related to the Planck constant by

$$g_s = \frac{4\pi^2}{\hbar}$$

This result provides in particular *a non-perturbative completion* of the topological string in terms of a one-dimensional quantum mechanical model.

A matrix model realization

The integral kernel of the operator ρ_X can be written explicitly, for many geometries, in terms of Faddeev's quantum dilogarithm [Kashaev-M.M.]. This leads to *analytic* computations of the (fermionic) spectral traces in many cases - a rare luxury in Quantum Mechanics!

We can write the fermionic spectral traces as explicit *matrix integrals*

[Fredholm, 1903]

$$Z_X(N,\hbar) = \frac{1}{N!}\int \mathrm{d}x_1 \cdots \mathrm{d}x_N \, \mathrm{det}_{i,j}\rho_X(x_i, x_j)$$

This can be used to extract their large *N* asymptotics and verify our conjecture [M.M.-Zakany,Kashaev-M.M.-Zakany]

A conjecture for the Fredholm determinant

We have a stronger conjecture which gives an *exact* formula for the Fredholm determinant itself. We first introduce the "grand potential" of the toric CY

$$J_X(\mu,\hbar) = \underbrace{J^{\rm WKB}(\mu,\hbar)}_{\text{resummation of WKB, calculable from NS limit}} + \underbrace{F^{\rm GV}\left(\frac{2\pi}{\hbar}t(\mu,\hbar), \frac{4\pi^2}{\hbar}\right)}_{\text{non-perturbative in } \hbar}$$

$$F^{\rm GV}(t,g_s) = \sum_{g\geq 0}\sum_{d,w=1}^{\infty} \frac{1}{w} n_g^d \left(2\sin\frac{w g_s}{2}\right)^{2g-2} {\rm e}^{-wdt}$$

n_g^d: Gopakumar-Vafa invariants

J_X can be calculated systematically, as an expansion at large radius, by using *BPS invariants* of the toric CY.

Conjecture: The Fredholm determinant is a Zak transform of the "grand potential" of *X*:

$$\Xi_X(\kappa) = \sum_{n \in \mathbb{Z}} \mathrm{e}^{J_X(\mu + 2\pi \mathrm{i} n)}$$

This type of Zak transform has been ubiquitous in topological strings/gauge theory [Nekrasov-Okounkov, ADKMV, DHSV, Eynard-M.M.]

Many, many analytic and numerical tests of this conjecture. Numerical tests typically with precision

$$10^{-40} - 10^{-500}$$

The spectral determinant is an entire function of κ

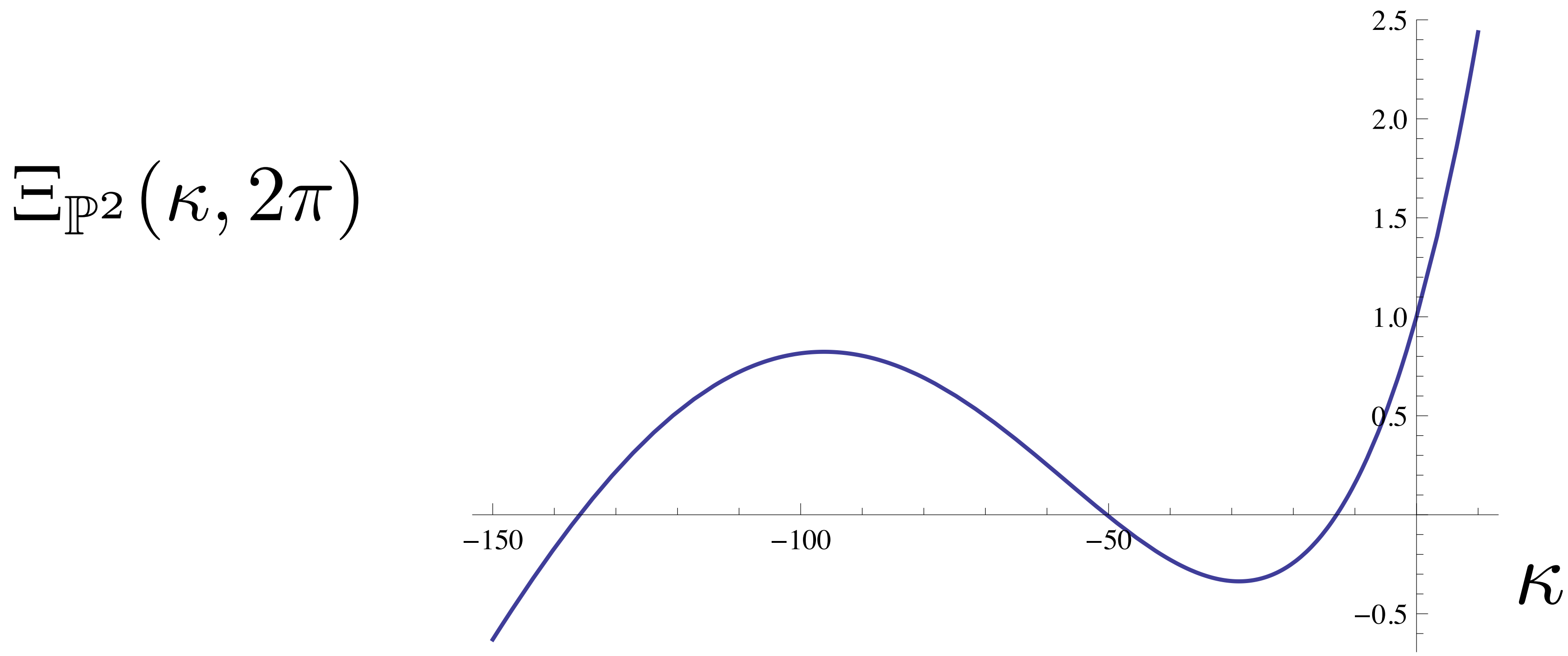
Therefore, we have constructed an *entire function on the moduli space of the CY*. In the 't Hooft limit

$$\hbar \to \infty \qquad \frac{\mu}{\hbar} = t \text{ fixed}$$
$$\mu \to \infty$$

has the asymptotics

$$\Xi(\kappa) \sim \exp\left(F_0(t)/\hbar^2\right)$$

Is this the "wavefunction on CY moduli space" conjectured by Witten in 1993?



We can now use the zeros of the spectral determinant to obtain an exact quantization condition for the spectrum

[Grassi-Hatsuda-M.M.]

$$\frac{\partial F_{\rm NS}}{\partial t} + \mathcal{O}(\mathrm{e}^{-1/\hbar}, \mathrm{e}^{-t/\hbar}) = 2\pi\hbar\left(n+\frac{1}{2}\right)$$

WKB

GV: Non-perturbative corrections which *cancel* the poles!

Modular duality

In fact, the non-perturbative corrections can be written in closed form as [Wang-Zhang-Huang, Grassi-Gu]

$$\frac{\partial F^{\rm NS}}{\partial t}(t(\hbar),\hbar) + \boxed{\frac{\hbar}{2\pi}\frac{\partial F^{\rm NS}}{\partial t}\left(\frac{2\pi}{\hbar}t(\hbar), \frac{4\pi^2}{\hbar}\right)} = 2\pi\hbar\left(n+\frac{1}{2}\right)$$

There is an "almost" modular duality in the theory

$$\hbar \rightarrow \frac{4\pi^2}{\hbar}$$

This is related to the modular double structure typical of Weyl (exponentiated) operators in QM [Faddeev].

The self-dual point

In particular, the theory simplifies enormously at the self-dual or "maximally supersymmetric" point

$$\hbar = 2\pi$$

Here, the theory is in a sense semiclassical, or one-loop exact. Everything can be written down in terms of

$$F_0(t), \; F_1(t), \; F_1^{\rm NS}(t)$$

Example:

$$\Xi(\kappa) = \exp\left(\frac{F_0(t)}{4\pi^2} + \cdots \right) \boxed{\vartheta_3(\xi|\tau)}$$

$$\xi \propto t \partial_t^2 F_0 - \partial_t F_0$$

In general, we have a "quantum" theta function

A 4d corollary with a proof

Interestingly, our 5d results have new consequences in 4d [Bonelli-Grassi-Tanzini]: the (unrefined!) Nekrasov-Okounkov dual partition function for SU(2) SYM

$$Z^{\rm NO}(a,\Lambda) = \sum_{n\in\mathbb{Z}} Z(a+n,\Lambda)$$

is in fact the Fredholm determinant of the following trace class operator

$$\rho(x_1,x_2) = \frac{{\rm e}^{-4\Lambda^2(\cosh(x_1)+\cosh(x_2))}}{\cosh\left(\frac{x_1-x_2}{2}\right)}$$

This can be proved by using a common connection to Painleve III

Wavefunctions

What about wavefunctions? In the context of quantum curves/ topological recursion, one can construct a "canonical" wavefunction, which in the case of mirror curves encodes *open* GW invariants

$$\psi_{\rm top}(x,t,\hbar) = \exp\left(\frac{\rm i}{\hbar}\int^x y(x')\mathrm{d}x' + \cdots\right)$$

This is *not* a solution (even formal) of the difference equation

$$(\mathsf{O}_X + \kappa)\psi(x) = 0$$

It agrees with the formal WKB solution $\psi_{\rm WKB}(x,t,\hbar)$ at leading order, but the subleading orders are different

A conjecture for the wavefunctions

We have conjectured recently that the exact wavefunctions of the trace class operators are given by

$$\psi_{\rm exact}(x,\mu,\hbar)=\sum_{\sigma=\pm}\sum_{n\in\mathbb{Z}}\Psi_\sigma(x,\mu+2\pi {\rm i} n,\hbar)$$

$$\Psi_\sigma(x,\mu,\hbar)=\psi^\sigma_{\rm WKB}(x,t(\mu),\hbar)\,\psi^\sigma_{\rm top}\left(\frac{2\pi}{\hbar}x,\frac{2\pi}{\hbar}t(\mu),\frac{4\pi^2}{\hbar}\right)$$

σ labels the two sheets of the Riemann surface

The resulting function is *entire* on the *x*-plane and belongs to $L^2(\mathbb{R})$ when $\kappa=-{\rm e}^{E_n}$

Wavefunctions in the self-dual case

In the self-dual case, the exact wavefunction is closely related to a *Baker-Akhiezer function* on the mirror curve. Once evaluated at the physical energies, it is very similar to a (one-loop) WKB wavefunction

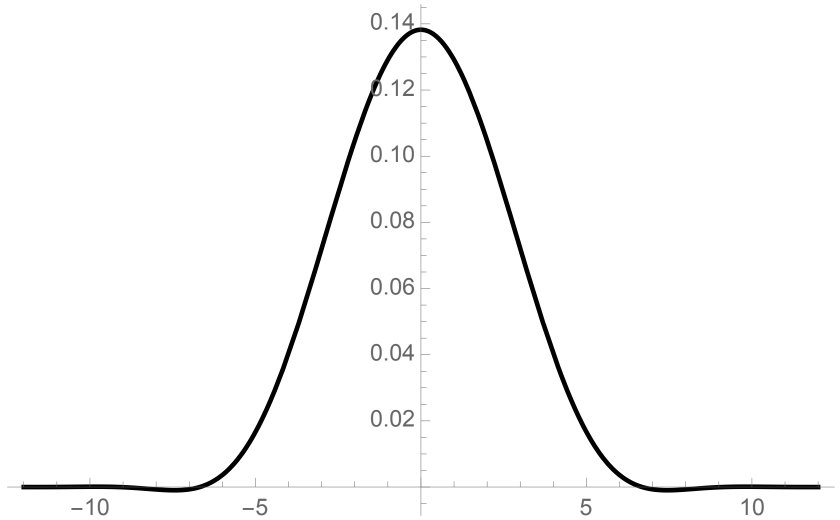
$$\psi_{\text{exact}}(x, \kappa = -\mathrm{e}^E, 2\pi) = \frac{1}{\Delta(x)^{1/2}} \sum_{\sigma=\pm} \exp\left[\frac{\mathrm{i}\sigma}{2\pi}\left(\int^x x' \mathrm{d}y(x') - t(E) u(x)\right)\right]$$

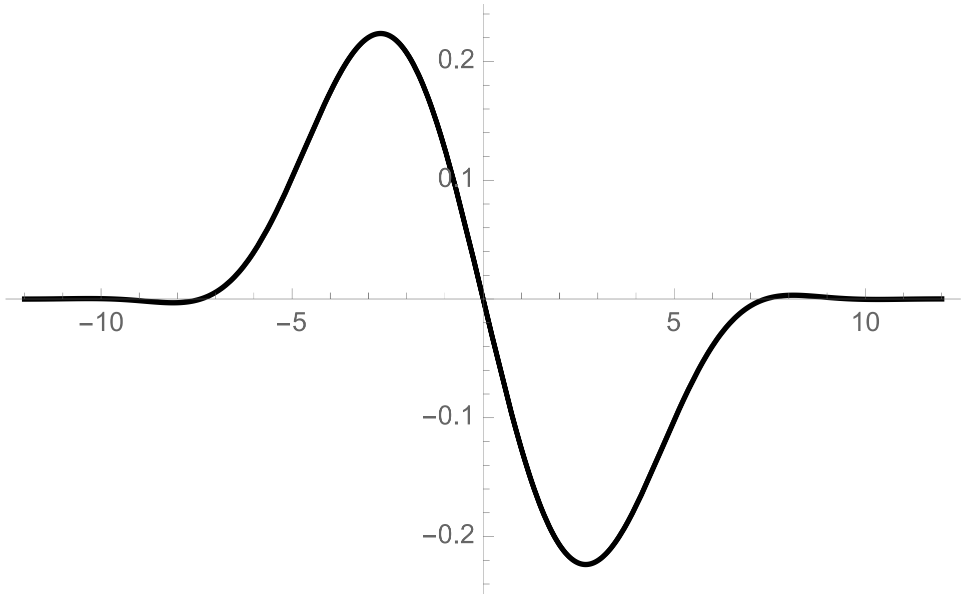
$\Delta(x)$: discriminant of the curve

$u(x)$: Abel-Jacobi map

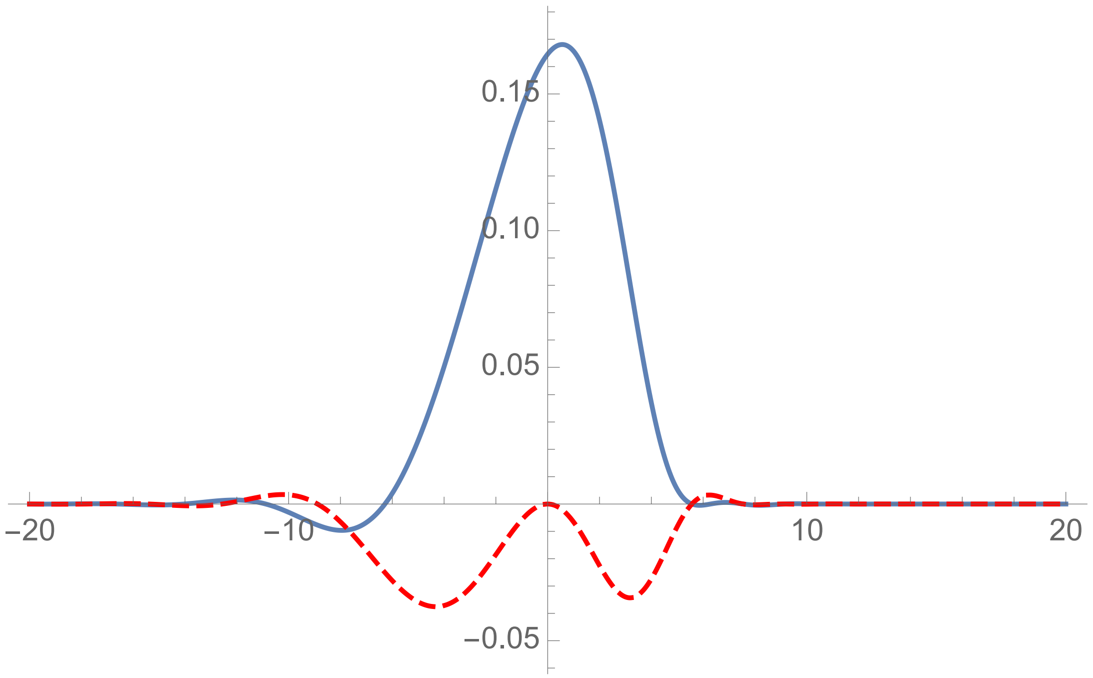
but has no singularities at turning points!

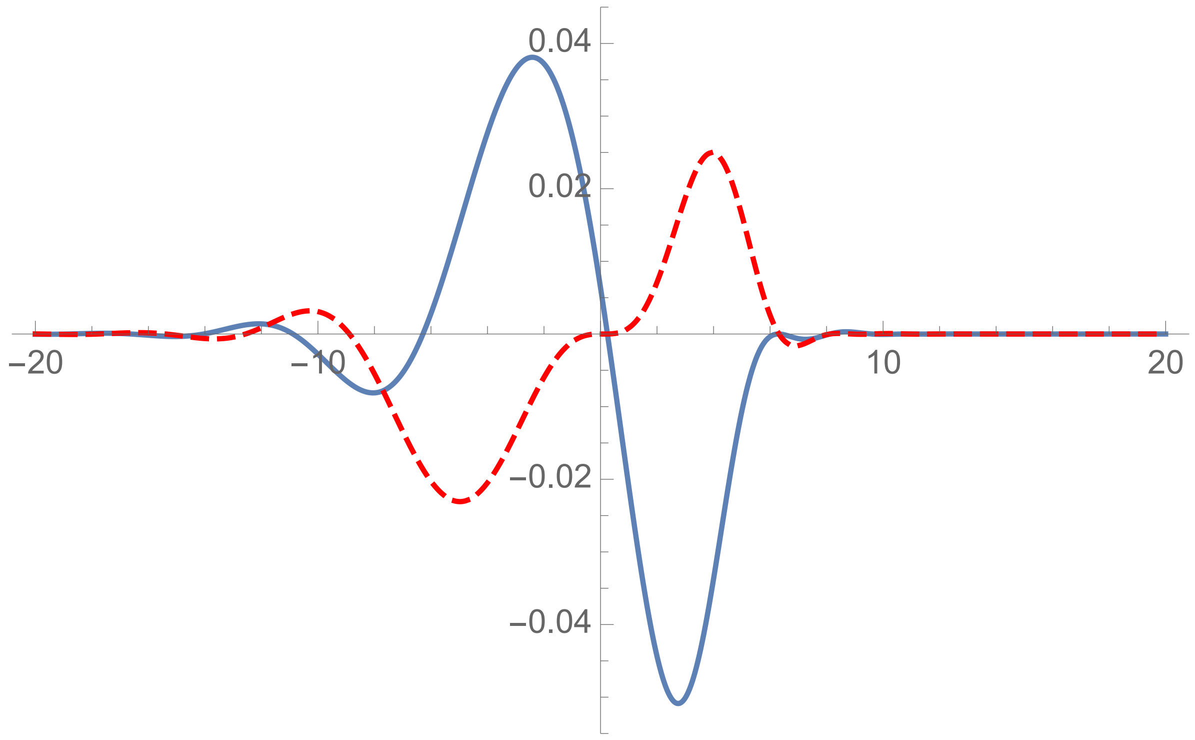
local $\mathbb{F}_0$





local $\mathbb{P}^2$





The higher genus case

A higher genus mirror curve has g_Σ moduli $\kappa_1, \cdots, \kappa_{g_\Sigma}$

We can always write the curve as

$$\mathcal{O}^{(0)} + \sum_{i=1}^{g_\Sigma} \kappa_i \mathcal{P}_i = 0$$

Weyl quantization $\swarrow$

$$\mathsf{O}^{(0)} \left(1 + \sum_{i=1}^{g_\Sigma} \kappa_i \mathsf{A}_i \right) \longrightarrow \mathsf{A}_1, \cdots, \mathsf{A}_{g_\Sigma}$$

trace class operators

Generalized Fredholm determinant

$$\Xi(\kappa_1, \cdots, \kappa_{g_\Sigma}) = \det \left(1 + \sum_{i=1}^{g_\Sigma} \kappa_i \mathsf{A}_i \right)$$

$$\Xi(\kappa_1, \cdots, \kappa_{g_\Sigma}) = \sum_{N_i \geq 0} Z(N_1, \cdots, N_{g_\Sigma}) \kappa_1^{N_1} \cdots \kappa_{g_\Sigma}^{N_{g_\Sigma}}$$

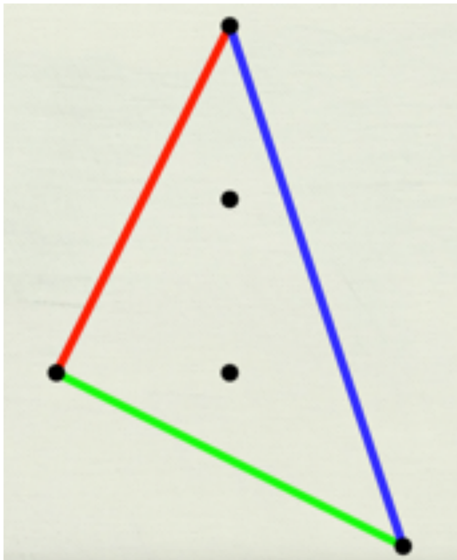
↑ fermionic traces

The generalized Fredholm determinant can be determined from the BPS invariants of the toric CY. In addition, the fermionic traces reconstruct the topological string genus expansion:

$$\log Z(N_1, \cdots, N_g) \sim \sum_{g \geq 0} F_g(\lambda_1, \cdots, \lambda_g) \hbar^{2-2g}$$

in the 't Hooft limit $\quad N_i \to \infty, \quad \hbar \to \infty \qquad \frac{N_i}{\hbar} = \lambda_i$

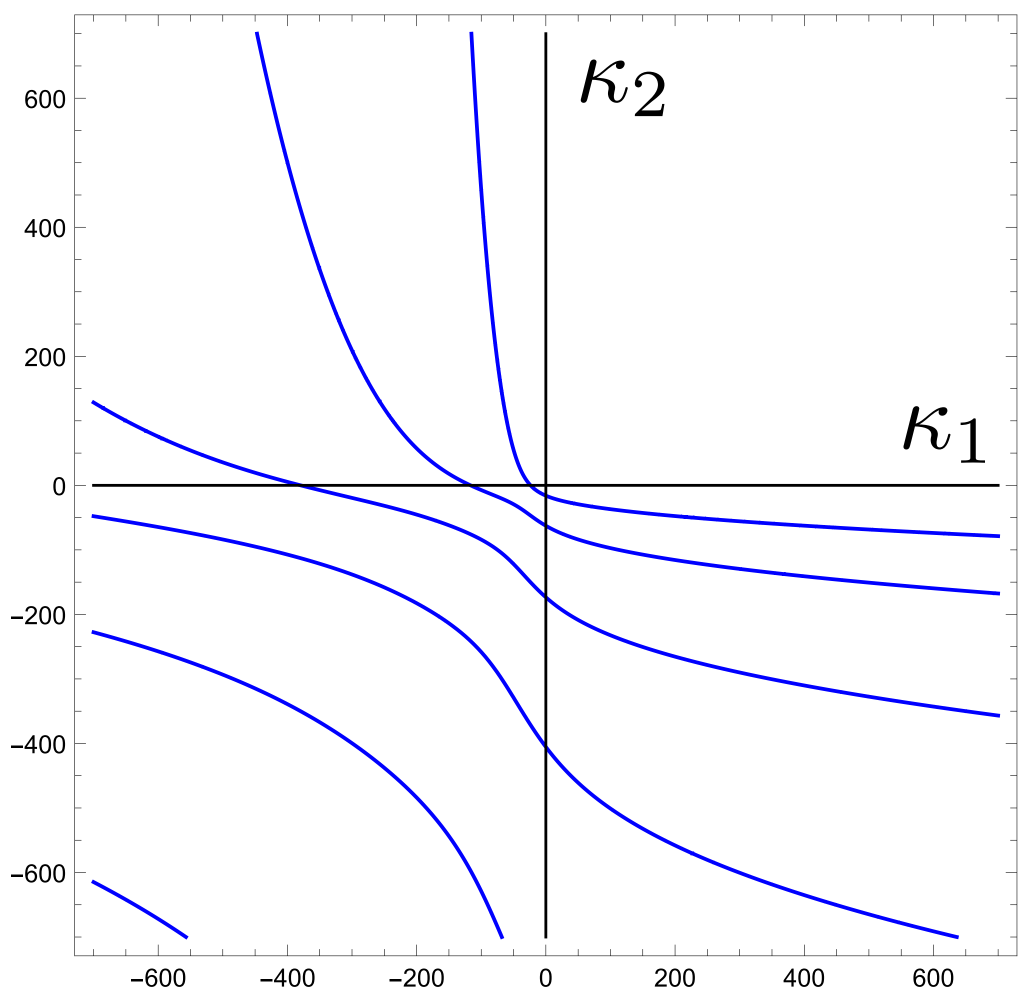
A higher genus example



Resolved $\mathbb{C}^3/\mathbb{Z}^5$ orbifold

$$A_1 = \left(e^x + e^y + e^{-3x-y}\right)^{-1} e^{-x}$$

$$A_2 = \left(e^x + e^y + e^{-3x-y}\right)^{-1}$$



Vanishing real locus of the generalized Fredholm determinant for

$\hbar = 2\pi$

Cluster integrable systems

The previous quantization scheme produces g_Σ trace class operators on $L^2(\mathbb{R})$. We reconstruct the topological string from their spectral traces.

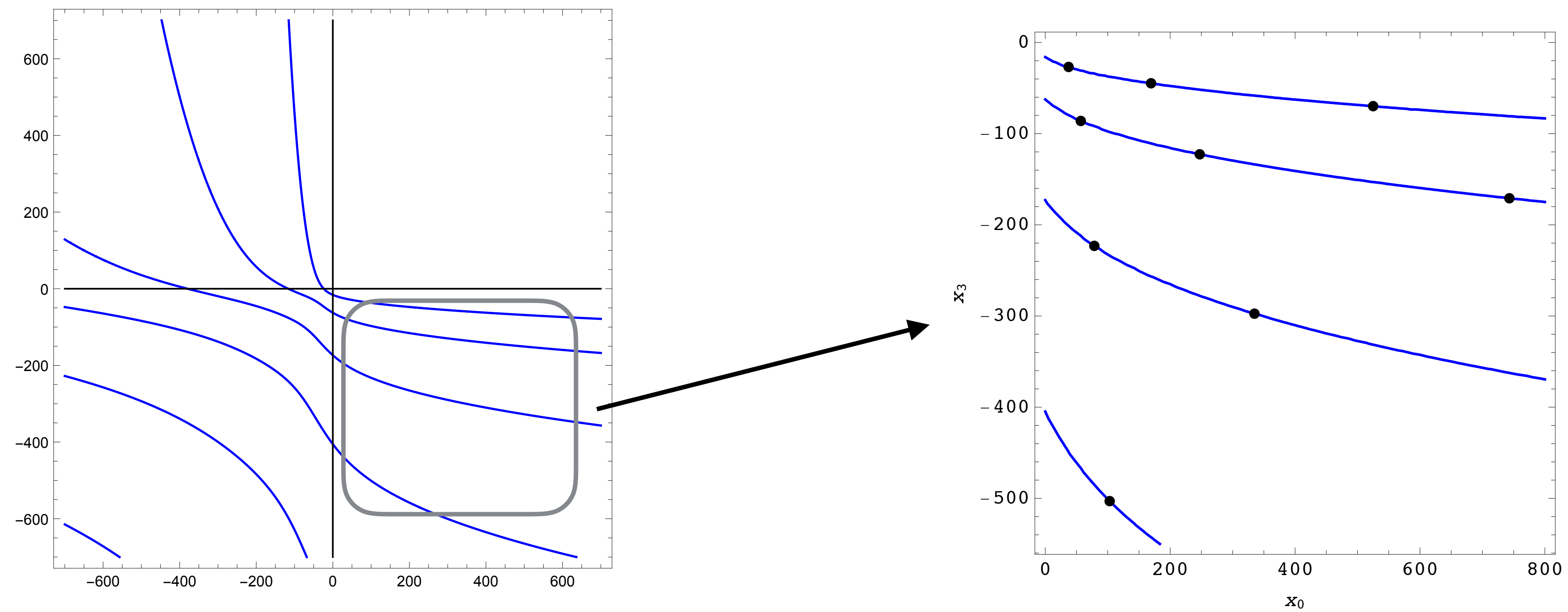
However, the same CY geometry leads to a *GK or cluster integrable system* with g_Σ mutually commuting Hamiltonians [Goncharov-Kenyon], corresponding to the moduli

$$\kappa_1, \cdots, \kappa_{g_\Sigma}$$

When the CY engineers a 5d U(N) gauge theory, the cluster integrable system is the relativistic Toda lattice of *N* particles.

The eigenvalues of the Hamiltonians form a discrete set in moduli space (associated to g_Σ integer quantum numbers)

One can write down exact quantization condition for these eigenvalues, generalizing the results for genus one [Franco-Hatsuda-M.M.]. This solves (conjecturally) the spectrum of cluster integrable systems.



Conclusions

The spectral theory of quantum mirror curves solves two problems simultaneously. On one hand, it uses spectral theory to provide a well-defined *non-perturbative completion of topological strings* on arbitrary toric CYs. On the other hand, it uses topological string theory to solve *a new, infinite family of trace class operators associated to mirror curves*.

As a bonus, we obtain exact (and explicit) quantization conditions for cluster integrable systems.

Many open problems, in particular concerning the eigenfunctions.

Obvious open problem: Prove all this!