Surface defects in 4d supersymmetric field theories via integrable lattice model

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Introduction

In this talk, we consider half-BPS surface defects in 4d supersymmetric theories, supported on 2d surface in 4d spacetime.

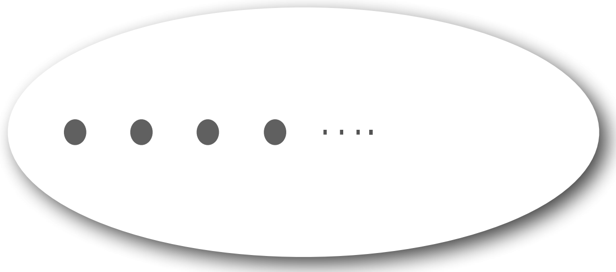
Physically, these are useful to determine the phase of the theories, like a Wilson and 't Hooft loops, and can exist as a BPS object in N=1,2,3,4 theories in 4d.

Mathematical-physically, often these defects are related with some "operation" in integrable models/CFTs behind 4d supersymmetric theories.

N=2 class S theories

Let us focus on the theories in so-called class S, which are obtained by compactifying M5-branes (6d (2,0) theory) on a Riemann surface.

[Gaiotto, Gaiotto-Moore-Neitzke]
[Pestun's and Teschner's talks]



The instanton (omega-deformed) partition function of this theory is related to the conformal block of 2d Virasoro or W algebra on this Riemann surface. **[Alday-Gaiotto-Tachikawa]**

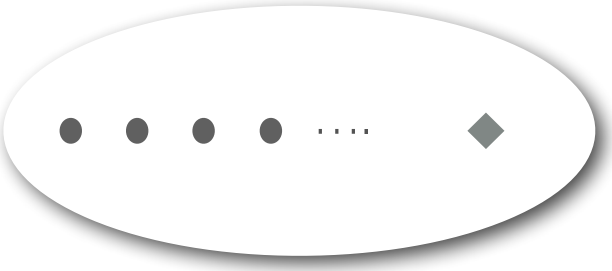
Surface defects in class S

Half-BPS surface defects are described in 2d CFT:

[Alday-Gaiotto-Gukov-Tachikawa-Verlinde, Alday-Tachikawa....]

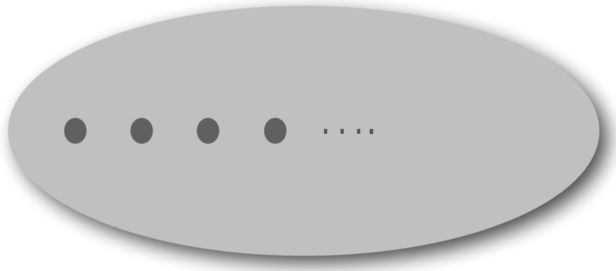
	0	1	2	3	4	5	6	7	8	9	11
M5 (6d (2,0))	×	×	×	×			×				×
M2 (codim 4)	×	×						×			
M5’ (codim 2)	×	×			×		×	×			×

M2-brane defect



insertion of degenerate field

M5’-brane defect



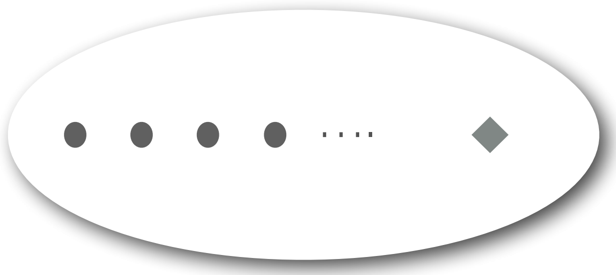
change the Virasoro to
affine Kac-Moody

Index of N=2 theories

We here consider **the supersymmetric index**, or the partition function on $S^1 \times S^3$. This quantity is a bit simpler than the previous instanton partition function, but has a rich connection to the integrable model.

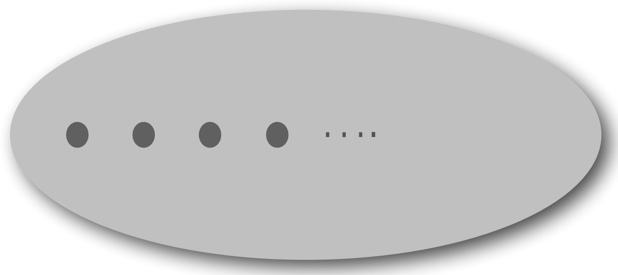
[Romelsberger, Kinney-Maldacena-Minwalla-Raju,...]

M2-brane defect



[Gaiotto-Rastelli-Razamat, Gadde-Gukov]
[Alday-Bullimore-Fluder-Hollands]
[Bullimore-Fluder-Hollands-Richmond]

M5'-brane defect



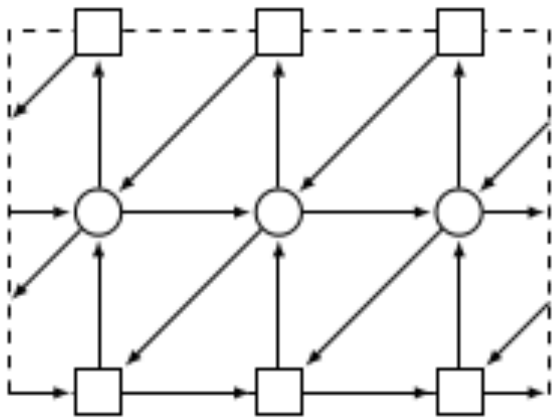
???

Brane tiling model

To tackle this, we argue another class of N=1 theories, Brane tiling models which have some overlap with N=2 class S theories:

[Franco-Hanany-Vegh-Wecht-Kennaway]

This is realized on intersecting D5-NS5 system in Type IIB (or D3-branes on toric CY), and generally is 4d N=1 theory whose matter content is specified by a quiver diagram, like

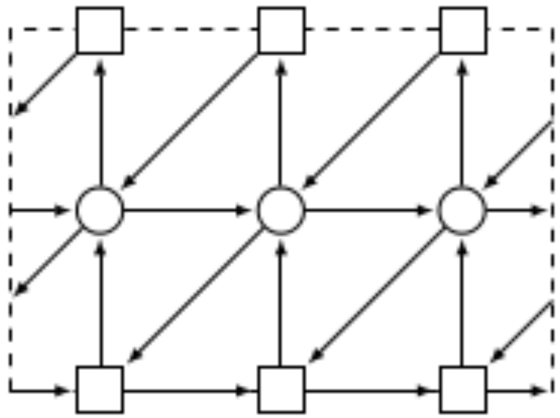


Brane tiling/integrable lattice model

These N=1 models are known to be related to integrable lattice model on the same 2d surface on which the tiling is defined **[Spiridonov, Yamazaki]**

$S^3 x S^1$ partition function (or superconformal index) of Brane tiling model

Partition function of integrable lattice model by **[Bazhanov-Sergeev]**



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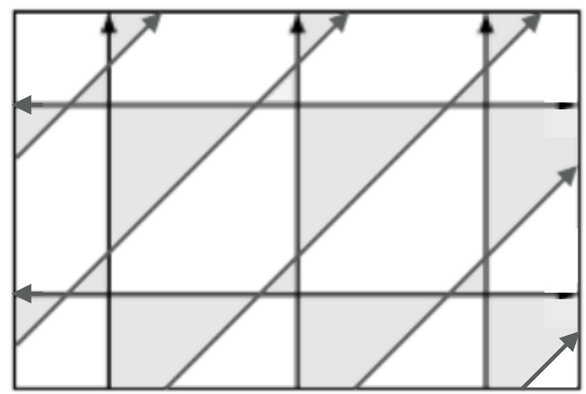
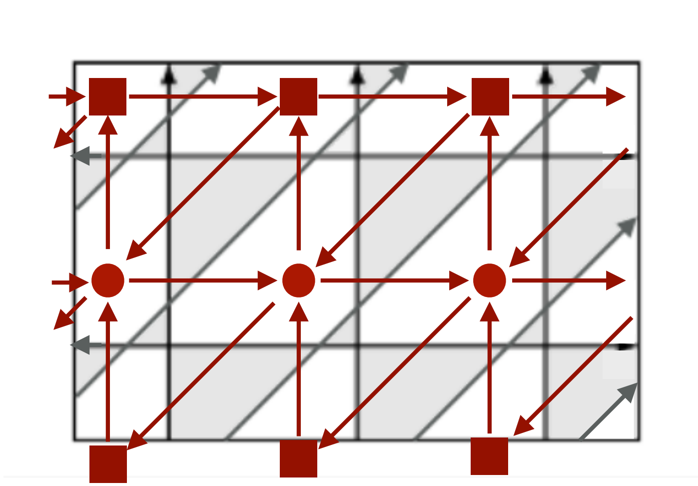


A model on a lattice

Let us consider a 2d lattice specified by rapidity lines: **a** ⟶ (**spectral parameter**)

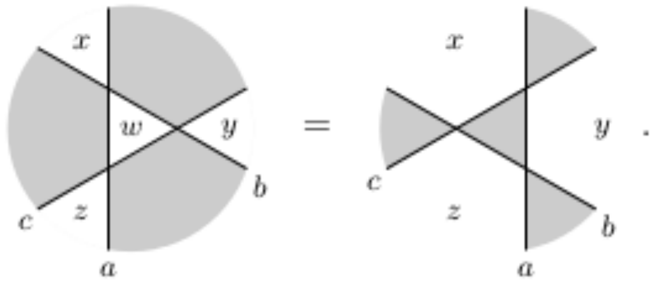


- **to a white face, we associate continuous spin variables z_i**
- **a crossing of lines gives the interaction of spins**

Integrable lattice model

[Bazhanov-Sergeev]

They show that the following star-triangle relation is solved



by assigning a particular Boltzmann weight to a interaction and self-interaction, written by elliptic gamma function:

$$\Gamma(a;p,q) = \prod_{i,j>0} \frac{1 - a^{-1}p^{i+1}q^{j+1}}{1 - ap^iq^j}$$

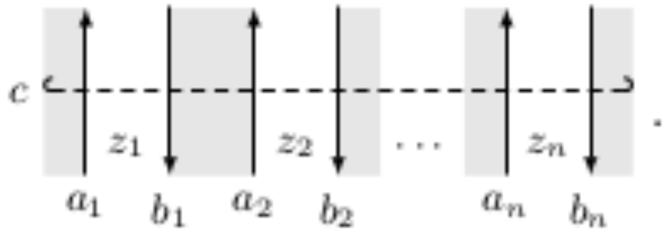
This leads to integrability of the model.

Surface defects

Actually one can introduce **a different kind of rapidity line** in integrable model. From the intersection of this and the ordinary rapidity line is called L-operator (or Lax operator)

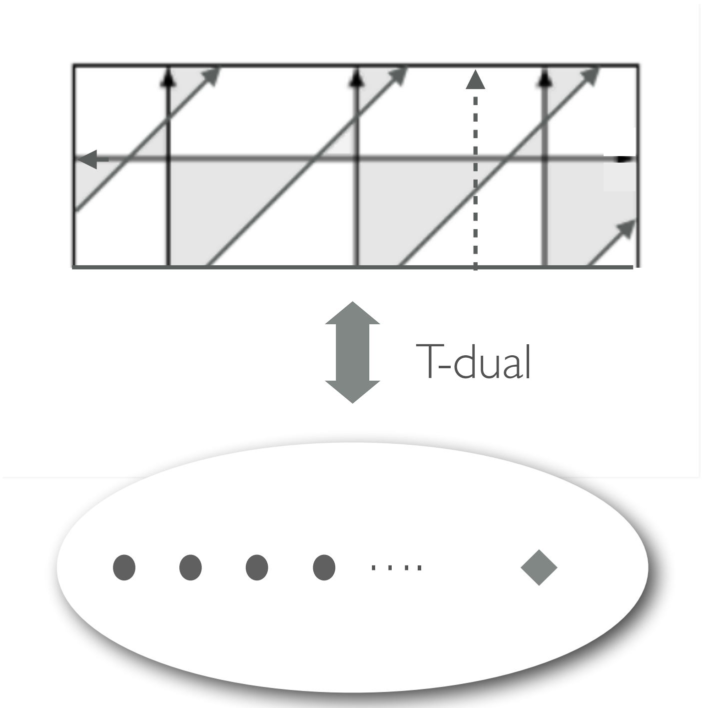
$$L_{ij} =$$

<u>Proposal</u>: a class of surface defects is represented by the transfer matrix of L-operators in the corresponding integrable lattice model.

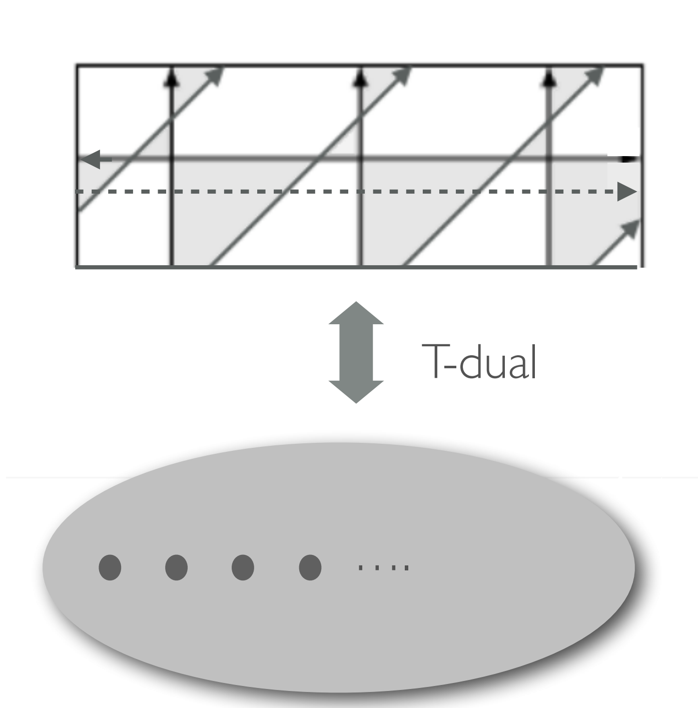


Surface defects in class S

We consider the special case of the brane-tiling model where the supersymmetry in 4d is N=2, then introduce the surface defects. This is associated with L-operator:



M2-brane defect as difference operator in elliptic RS model



M5'-brane defect

1. **Brane tiling and integrable model**

2. **Surface defect as transfer matrix**

3. **Surface defects in class S theories**

Brane tiling

and integrable model

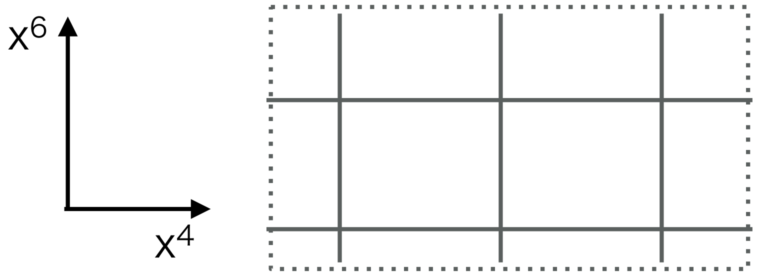
Type IIB brane configuration

Let us consider the following brane configuration in Type IIB:

	0	1	2	3	4	5	6	7	8	9
N D5	×	×	×	×	×		×			
NS5	×	×	×	×	×	×				
NS5'	×	×	×	×			×	×		

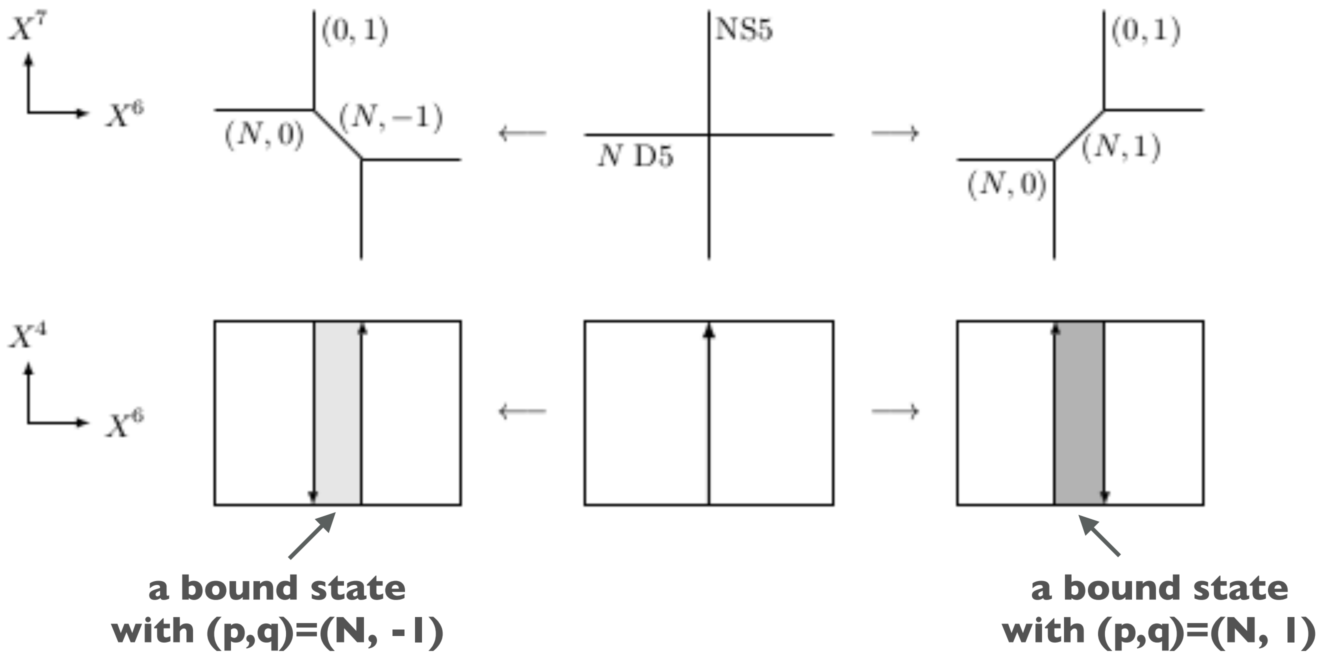
which preserves 4 supercharges (N=1 in 4d).

Let x^4 and x^6 coordinates parametrize T^2. Then, NS5-branes intersect with D5-branes **along curves in x^4-x^6 plane**.



Brane intersection

At the intersection of D5-brane and NS5-brane we can resolve as

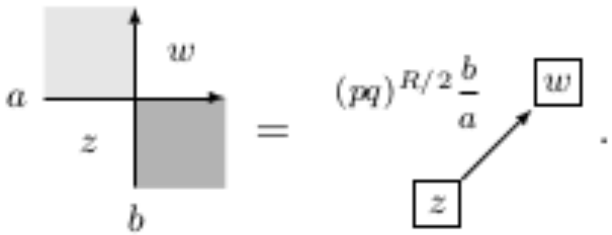


We call the lines representing NS5-branes along half-line as rapidity lines.

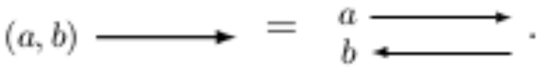
In general, the diagram from rapidity lines on T^2 or other surface is called as **brane tiling**.

To a brane tiling without |q|>1 region there is an associated an N=1 4d quiver gauge theory which is a worldvolume theory on D5-branes:

- **to a face with q=0, we associate an SU(N) gauge group** with fugacity z_i.
- **a crossing of rapidity lines**, each of which has continuous parameter, **gives rise to a bi-fundamental chiral multiplet**

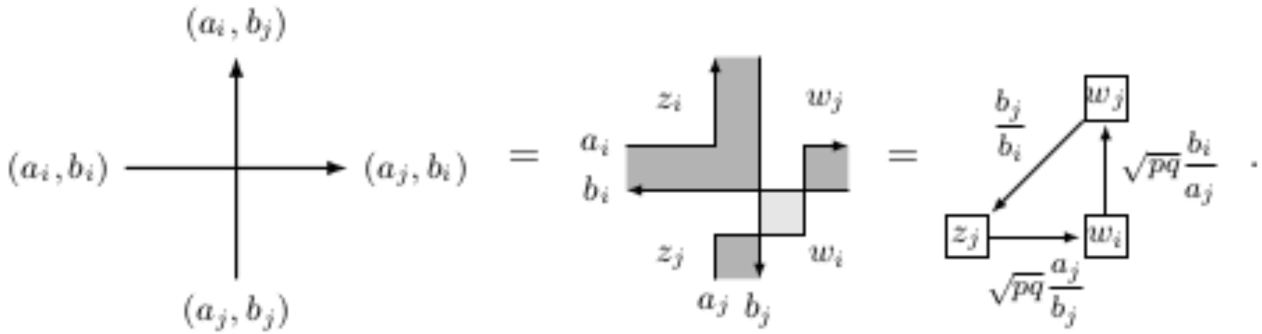


Let us take a pair of rapidity lines. By using this doubled notation we can avoid the region with |q|>1

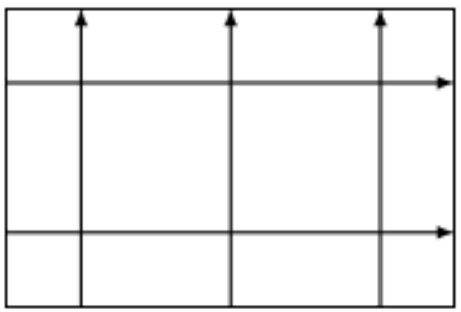


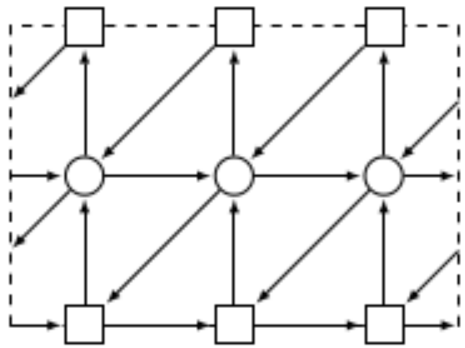
A quiver gauge theory

We choose the (N,0) background, which leads to the following:



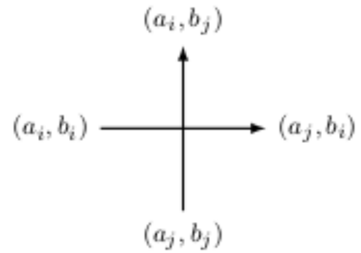
For example, this produces the following quiver gauge theory





R-matrix and Yang-Baxter eq

We associate an R-matrix to the crossing of lines:

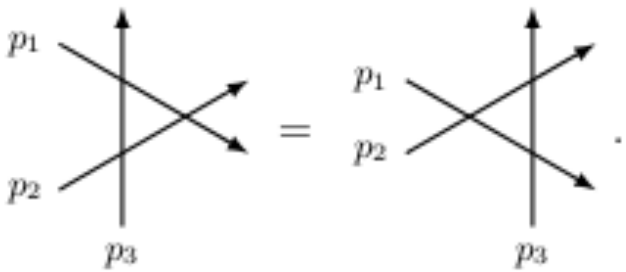
Index of $\Bigg($ 

$\Bigg) = R^{\diamond}_{ij}(a_i, b_i; a_j, b_j)^{w_i, w_j}_{z_i, z_j}$

This defines a map $R^{\diamond}_{ij} : V^{\diamond}_i \otimes V^{\diamond}_j \to V^{\diamond}_j \otimes V^{\diamond}_i$, where $V^{\diamond}_i$ is the space of symmetric meromorphic functions of z_i or w_i.

The R-matrix satisfies **the Yang-Baxter equation**

$$R^{\diamond}_{12}(p_1, p_2) R^{\diamond}_{13}(p_1, p_3) R^{\diamond}_{23}(p_2, p_3) = R^{\diamond}_{23}(p_2, p_3) R^{\diamond}_{13}(p_1, p_3) R^{\diamond}_{12}(p_1, p_2)$$



$$p_i = (a_i, b_i)$$

This is indeed satisfied because of the brane construction after uplift to M-theory.

Surface defect

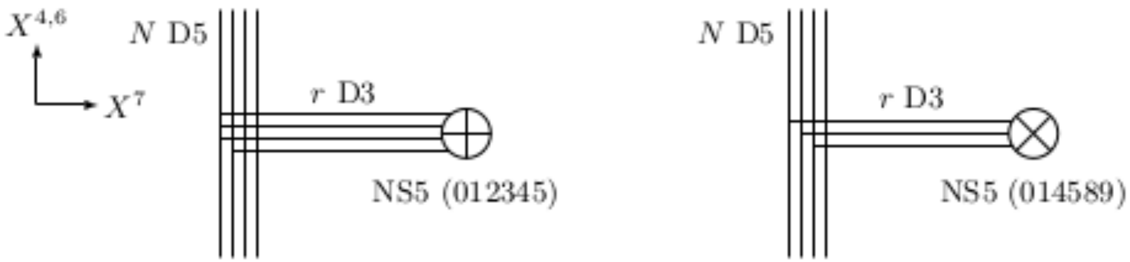
as transfer matrix

Half-BPS surface defect

We consider **D3-brane insertion** in the previous Type IIB brane configuration which preserves 2 supercharges:

	0	1	2	3	4	5	6	7	8	9
N D5	×	×	×	×	×		×			
NS5	×	×	×	×	×	×				
NS5’	×	×	×	×			×	×		
D3	×	×			×			×		

Depending on the NS5-brane worldvolume direction, we have, for example:



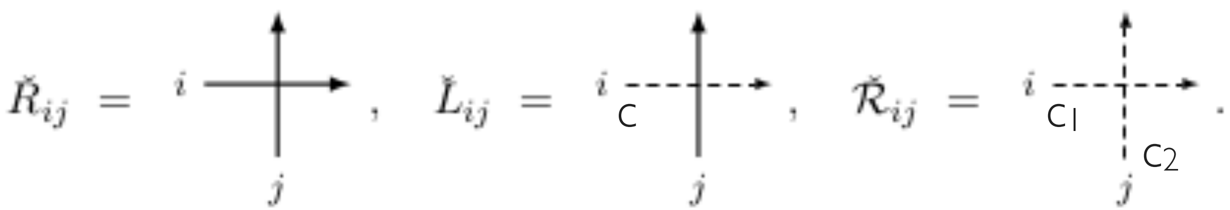
These can be regarded as **symmetric** and **antisymmetric** representation of SU(N).

“L-operator”

We denote the projection of D3-branes in x^4-x^6 surface by the dashed arrow:

Similar to the line from NS5, the dashed line associates **the fugacity c**, and **the representation space W_R of rep. R**.

By using this and zigzag path, we now have the following three types of intersection:

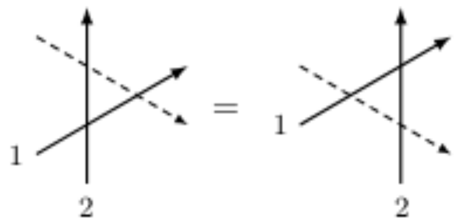


$$L(c,(a_j,b_j)) : W_i \otimes V_j \to V_j \otimes W_i$$

“Yang-Baxter equations”

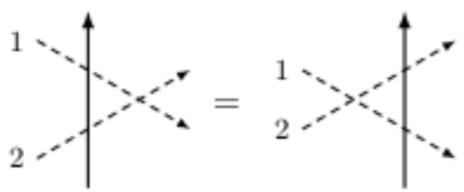
Accordingly, three types of Yang-Baxter like equations can be considered:

<u>One dashed line:</u>



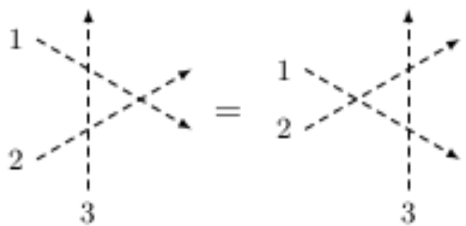
$$\Longleftrightarrow \check{L}_1\check{L}_2\check{R}_{12} = \check{R}_{12}\check{L}_2\check{L}_1$$

<u>Two dashed lines:</u>



$$\Longleftrightarrow \check{\mathcal{R}}_{12}\check{L}_1\check{L}_2 = \check{L}_2\check{L}_1\check{\mathcal{R}}_{12},$$

<u>Three dashed lines:</u>



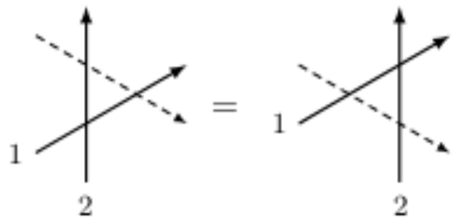
$$\Longleftrightarrow \check{\mathcal{R}}_{12}\check{\mathcal{R}}_{13}\check{\mathcal{R}}_{23} = \check{\mathcal{R}}_{23}\check{\mathcal{R}}_{13}\check{\mathcal{R}}_{12},$$

Again these are indeed satisfied because of the brane construction after uplift to M-theory.

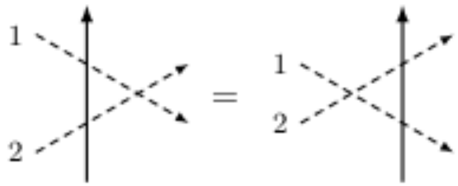
L-operator (N=2 case)

Let us focus on N=2 (SU(2)) case and the representation of surface defect is the fundamental one. **It is known that there is an operator in integrable model**, so-called **L-operator** L: $W_i \otimes V_j \to V_j \otimes W_i$
This can be represented by 2x2 matrix whose entries are operators acting on function in V_i.

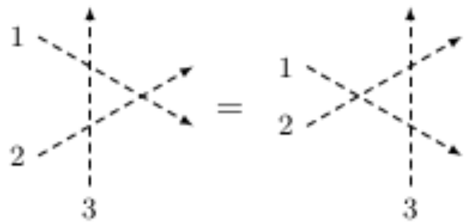
Yang-Baxter equations:

One dashed line: 

$\Longleftrightarrow \check{L}_1\check{L}_2\check{R}_{12} = \check{R}_{12}\check{L}_2\check{L}_1$ **[Derkachov-Spiridonov]**

Two dashed lines: 

$\Longleftrightarrow \check{\mathcal{R}}_{12}\check{L}_1\check{L}_2 = \check{L}_2\check{L}_1\check{\mathcal{R}}_{12},$ **[Sklyanin]**

Three dashed lines: 

$\Longleftrightarrow \check{\mathcal{R}}_{12}\check{\mathcal{R}}_{13}\check{\mathcal{R}}_{23} = \check{\mathcal{R}}_{23}\check{\mathcal{R}}_{13}\check{\mathcal{R}}_{12},$ **[Baxter]**

Proposal

[Yagi-KM]

Sklyanin found that the following operator solves the RLL relation:

$$L = \sum_{a=0}^{3} w_a \ \sigma_a \otimes \mathrm{S}_a$$

where S_a satisfies the so-called Sklyanin algebra and is represented by a difference operator.

We propose that **the transfer matrix from these L-operators is identified with the index of the surface defect**

$$\begin{aligned}&\mathrm{Tr}(L(c,(a_n,b_n))\dots L(c,(a_1,b_1)))\\ &\qquad = \sum_{s_1=\{\pm 1\}} \cdots \sum_{s_n=\{\pm 1\}} \prod_{i=1}^{n} l(z_i^{s_i}, z_{i+1}^{s_{i+1}}; \frac{b_i}{c}, \frac{a_{i+1}}{c}) \prod_{j=1}^{n} \Delta_j^{s_j/2}.\end{aligned}$$

where

$$(\Delta_i^{\pm 1/2} f)(z_i) = f(q^{\pm 1/2} z_i) \qquad l(z_i, z_{i+1}; \frac{b_i}{c}, \frac{a_{i+1}}{c}) = \frac{1}{\theta(z_i^2)} \theta(\sqrt{pq}\frac{c^2}{b_i a_{i+1}} \frac{z_{i+1}}{z_i}) \theta(\sqrt{pq} \frac{b_i}{a_{i+1}} z_i z_{i+1}).$$

Generalizations

- We can consider the surface defects supported on $S^1 x S^1$. The S^1 can be one of two circles in S^3 ($|\xi_1|^2 + |\xi_2|^2 = 1$). Having two choices corresponds to modular double of Sklyanin algebra. (p↔q)

- General N and general representations have been studied in **[Yagi, '17]**

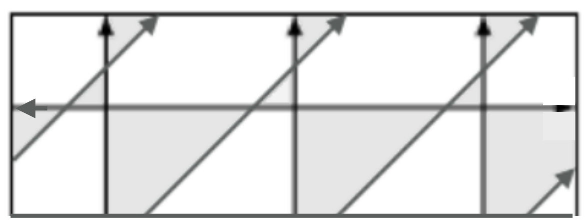
Surface defect

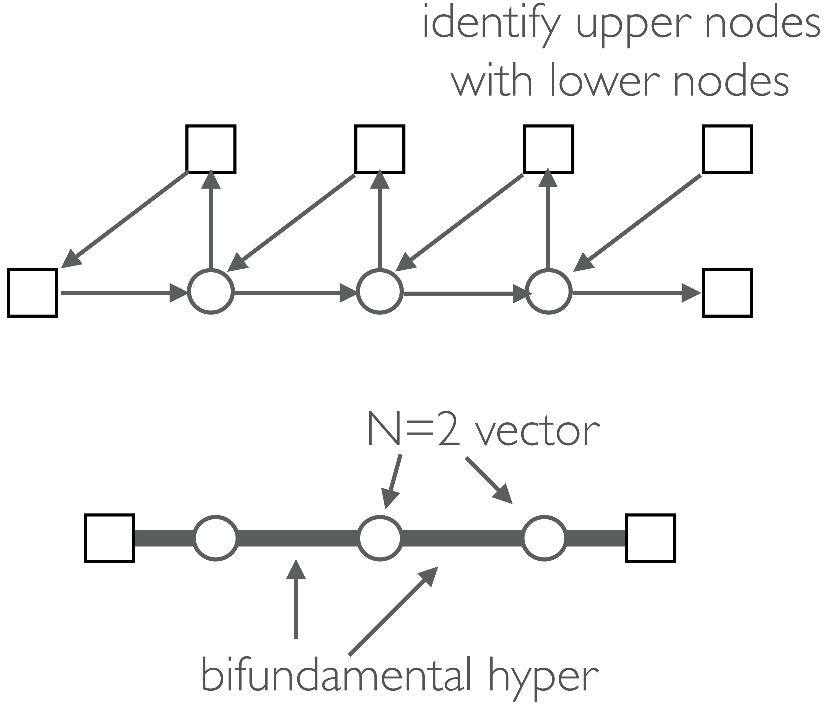
in class S theory

Class S theories

Now we come back to class S theory. Actually some theories in class S are a special case of brane tiling models.

Let us consider the following tiling:





M5 on cylinder

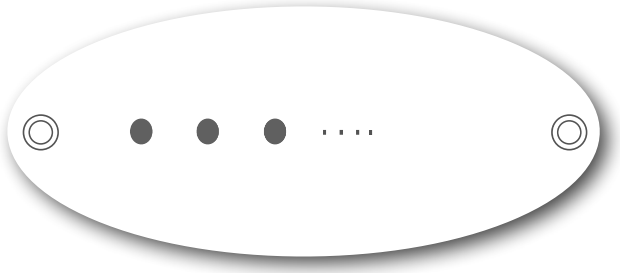
The previous brane tiling is produced by the brane system like:

	0	1	2	3	4	5	6	7	8	9
N D5	×	×	×	×	×		×			
NS5	×	×	×	×	×	×				

T-dual

	0	1	2	3	4	5	6	7	8	9
N D4	×	×	×	×			×			
NS5	×	×	×	×	×	×				

N M5-branes on



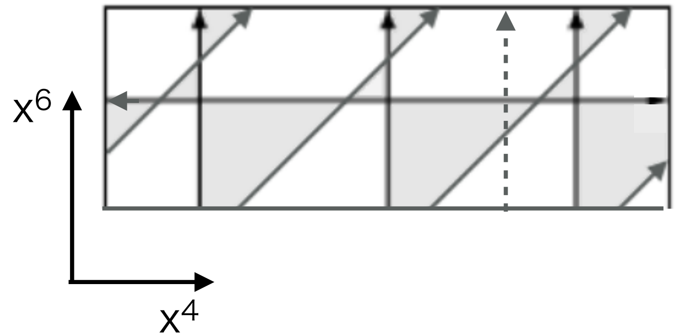
A surface defect

We add to this a D3 brane to describe a surface defect.

Depending on the directions, we have two types of defects:

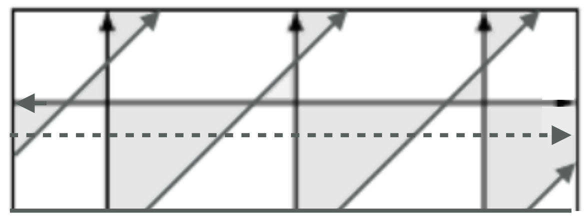
1.

	0	1	2	3	4	5	6	7	8	9
N D5	×	×	×	×	×		×			
NS5	×	×	×	×	×	×				
D3	×	×			×			×		



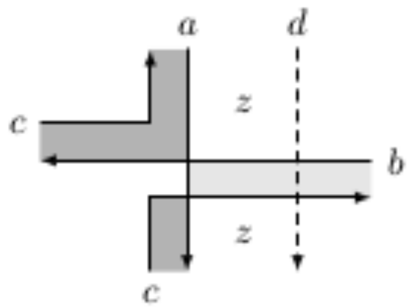
2.

	0	1	2	3	4	5	6	7	8	9
N D5	×	×	×	×	×		×			
NS5	×	×	×	×	×	×				
D3'	×	×					×	×		



The first case:

From the proposal from integrable lattice model, the inclusion of the surface defect corresponds just to the trace of single L-operator:

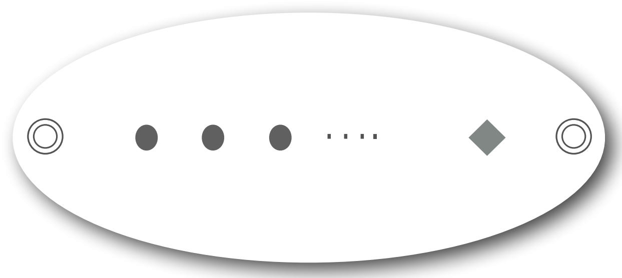


$$\mathrm{Tr}\left(\check{L}^{\diamond}(d,(c,b))\right) \propto \sum_{s=\pm 1} \frac{1}{\theta(z^{2s})}\theta\left(\sqrt{pq}\frac{d^2}{bc}\right)\theta\left(\sqrt{pq}\frac{b}{c}z^{2s}\right)\Delta^{s/2}.$$

The difference operator acts on the flavor or gauge fugacity z

	0	1	2	3	4	5	6	7	8	9
N D4	×	×	×	×			×			
NS5	×	×	×	×	×	×				
D2	×	×						×		

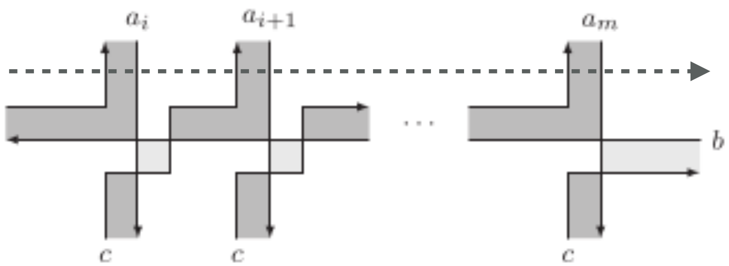
N M5-branes on



[Gaiotto-Rastelli-Razamat]

The second case:

This corresponds to the trace of multiple L-operators:



$$= \left(\prod_i \sum_{s_i = \pm 1} \right) \prod_i g_i(z_i, a_i, b, c) \Delta^{s_i/2}$$

	0	1	2	3	4	5	6	7	8	9
N D4	×	×	×	×	×		×			
NS5	×	×	×	×	×	×				
D4	×	×			×		×	×		

N M5-branes + M5 on

Conclusion

We propose that the index of the half-BPS surface defect from D3-branes in brane tiling model can be identified with the transfer matrix of L-operators in the corresponding integrable model.

This gives the unified picture of the indices of the M2- and M5-branes defects in class S theories.

Outlook

- ✪ M2-M5' defect from Baxter's R matrix
- ✪ another T-dual gives line defects in class S
- ✪ 3d reduction