Surface defects in 4d supersymmetric field theories via integrable lattice model

Kazunobu Maruyoshi
(Seikei University)

w/ Junya Yagi, 1606.01041 + more

“Workshop on Quantum Fields, Knots and Integrable Systems”
Edinburgh, March 2nd, 2017
In this talk, we consider half-BPS surface defects in 4d supersymmetric theories, supported on 2d surface in 4d spacetime.

**Physically**, these are useful to determine the phase of the theories, like a Wilson and ’t Hooft loops, and can exist as a BPS object in N=1,2,3,4 theories in 4d.

**Mathematical-physically**, often these defects are related with some “operation” in integrable models/CFTs behind 4d supersymmetric theories.
Let us focus on the theories in so-called class $S$, which are obtained by compactifying M5-branes (6d (2,0) theory) on a Riemann surface.

The instanton (omega-deformed) partition function of this theory is related to the conformal block of 2d Virasoro or $W$ algebra on this Riemann surface.
Surface defects in class S

Half-BPS surface defects are described in 2d CFT:

*Alday-Gaiotto-Gukov-Tachikawa-Verlinde, Alday-Tachikawa....*

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>11</th>
</tr>
</thead>
<tbody>
<tr>
<td>M5 (6d (2,0))</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td></td>
<td>X</td>
<td></td>
<td>X</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>M2 (codim 4)</td>
<td>X</td>
<td>X</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>M5’ (codim 2)</td>
<td>X</td>
<td>X</td>
<td></td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

*M2-brane defect*

insertion of degenerate field

*M5’-brane defect*

change the Virasoro to affine Kac-Moody
We here consider the supersymmetric index, or the partition function on $S^1 \times S^3$. This quantity is a bit simpler than the previous instanton partition function, but has a rich connection to the integrable model.

[Romelsberger, Kinney-Maldacena-Minwalla-Raju,...]

### Index of $N=2$ theories

- **M2-brane defect**
  - [Gaiotto-Rastelli-Razamat, Gadde-Gukov]
  - [Alday-Bullimore-Fluder-Hollands]
  - [Bullimore-Fluder-Hollands-Richmond]

- **M5’-brane defect**
  - ???
To tackle this, we argue another class of N=1 theories, Brane tiling models which have some overlap with N=2 class S theories:

[Franco-Hanany-Vegh-Wecht-Kennaway]

This is realized on intersecting D5-NS5 system in Type IIB (or D3-branes on toric CY), and generally is 4d N=1 theory whose matter content is specified by a quiver diagram, like
Brane tiling/integrable lattice model

These N=1 models are known to be related to integrable lattice model on the same 2d surface on which the tiling is defined \([\text{Spiridonov, Yamazaki}]\).

\(S^3 \times S^1\) partition function (or superconformal index) of Brane tiling model

\begin{equation*}
\text{Partition function of integrable lattice model by [Bazhanov-Sergeev]}
\end{equation*}
A model on a lattice

Let us consider a 2d lattice specified by rapidity lines:

- to a white face, we associate continuous spin variables $z_i$
- a crossing of lines gives the interaction of spins

(a \rightarrow (spectral parameter))
They show that the following star-triangle relation is solved

by assigning a particular Boltzmann weight to an interaction and self-interaction, written by elliptic gamma function:

\[ \Gamma(a; p, q) = \prod_{i, j > 0} \frac{1 - a^{-1}p^{i+1}q^{j+1}}{1 - ap^i q^j} \]

This leads to integrability of the model.
Surface defects

Actually one can introduce a different kind of rapidity line in integrable model. From the intersection of this and the ordinary rapidity line is called L-operator (or Lax operator).

Proposal: a class of surface defects is represented by the transfer matrix of L-operators in the corresponding integrable lattice model.
We consider the special case of the brane-tiling model where the supersymmetry in 4d is $N=2$, then introduce the surface defects. This is associated with L-operator:

Surface defects in class S

M2-brane defect as difference operator in elliptic RS model

M5’-brane defect
1. Brane tiling and integrable model
2. Surface defect as transfer matrix
3. Surface defects in class S theories
Brane tiling
and integrable model
Let us consider the following brane configuration in Type IIB:

\[
\begin{array}{cccccccccc}
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
\text{N D5} & \times & \times & \times & \times & \times & \times & \times & & \\
\text{NS5} & \times & \times & \times & \times & \times & & & & \\
\text{NS5'} & \times & \times & \times & \times & & \times & \times & & \\
\end{array}
\]

which preserves 4 supercharges (N=1 in 4d).

Let \(x^4\) and \(x^6\) coordinates parametrize \(T^2\). Then, NS5-branes intersect with D5-branes along curves in \(x^4\)-\(x^6\) plane.
Brane intersection

At the intersection of D5-brane and NS5-brane we can resolve as

We call the lines representing NS5-branes along half-line as rapidity lines.

In general, the diagram from rapidity lines on $T^2$ or other surface is called as brane tiling.
To a brane tiling without \(|q| > 1\) region there is an associated an \(N=1\) 4d quiver gauge theory which is a worldvolume theory on D5-branes:

- **to a face with \(q=0\), we associate an \(SU(N)\) gauge group with fugacity \(z_i\).**

- **a crossing of rapidity lines**, each of which has continuous parameter, **gives rise to a bi-fundamental chiral multiplet**

Let us take a pair of rapidity lines. By using this doubled notation we can avoid the region with \(|q| > 1\)
A quiver gauge theory

We choose the (N,0) background, which leads to the following:

For example, this produces the following quiver gauge theory
R-matrix and Yang-Baxter eq

We associate an R-matrix to the crossing of lines:

\[
\text{Index of } \begin{pmatrix} (a_i, b_i) \ar (a_j, b_i) \\ (a_j, b_j) \end{pmatrix} = R_{ij}^w(a_i, b_i; a_j, b_j)_{w_i, w_j}
\]

This defines a map \( R_{ij}^w : V_i^w \otimes V_j^w \to V_j^w \otimes V_i^w \), where \( V_i^w \) is the space of symmetric meromorphic functions of \( z_i \) or \( w_i \).

The R-matrix satisfies the **Yang-Baxter equation**

\[
R_{12}^\circ(p_1, p_2)R_{13}^\circ(p_1, p_3)R_{23}^\circ(p_2, p_3) = R_{23}^\circ(p_2, p_3)R_{13}^\circ(p_1, p_3)R_{12}^\circ(p_1, p_2)
\]

\[
p_i = (a_i, b_i)
\]

This is indeed satisfied because of the brane construction after uplift to M-theory.
Surface defect as transfer matrix
We consider **D3-brane insertion** in the previous Type IIB brane configuration which preserves 2 supercharges:

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>N D5</td>
<td>×</td>
<td>×</td>
<td>×</td>
<td>×</td>
<td>×</td>
<td>×</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>NS5</td>
<td>×</td>
<td>×</td>
<td>×</td>
<td>×</td>
<td>×</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>NS5’</td>
<td>×</td>
<td>×</td>
<td>×</td>
<td>×</td>
<td></td>
<td>×</td>
<td>×</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>D3</td>
<td>×</td>
<td>×</td>
<td></td>
<td>×</td>
<td></td>
<td></td>
<td></td>
<td>×</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Depending on the NS5-brane worldvolume direction, we have, for example:

These can be regarded as **symmetric** and **antisymmetric** representation of SU(N).
“L-operator”

We denote the projection of D3-branes in $x^4$-$x^6$ surface by the dashed arrow:

![](image)

Similar to the line from NS5, the dashed line associates the fugacity $c$, and the representation space $W_R$ of rep. $R$.

By using this and zigzag path, we now have the following three types of intersection:

$$
\tilde{R}_{ij} = \begin{array}{c}
\uparrow \\
\downarrow
\end{array} \quad \tilde{L}_{ij} = \begin{array}{c}
\uparrow \\
\downarrow
\end{array}_c \quad \tilde{\mathcal{R}}_{ij} = \begin{array}{c}
\uparrow \\
\downarrow
\end{array}_{cl}.
$$

$$L(c, (a_j, b_j)) : W_i \otimes V_j \rightarrow V_j \otimes W_i$$
“Yang-Baxter equations”

Accordingly, three types of Yang-Baxter like equations can be considered:

**One dashed line:**

\[ \begin{array}{c}
 1 \quad 2 \\
\end{array}
\begin{array}{c}
\overline{1} \quad \overline{2}
\end{array}
\begin{array}{c}
= \\
\end{array}
\begin{array}{c}
\overline{1} \quad \overline{2}
\end{array}
\begin{array}{c}
1 \quad 2 \\
\end{array}
\iff \hat{L}_1 \hat{L}_2 \hat{R}_{12} = \hat{R}_{12} \hat{L}_2 \hat{L}_1

**Two dashed lines:**

\[ \begin{array}{c}
1 \quad 2 \\
\end{array}
\begin{array}{c}
\overline{1} \quad \overline{2}
\end{array}
\begin{array}{c}
= \\
\end{array}
\begin{array}{c}
\overline{1} \quad \overline{2}
\end{array}
\begin{array}{c}
1 \quad 2 \\
\end{array}
\iff \hat{R}_{12} \hat{L}_1 \hat{L}_2 = \hat{L}_2 \hat{L}_1 \hat{R}_{12},

**Three dashed lines:**

\[ \begin{array}{c}
1 \quad 2 \\
\end{array}
\begin{array}{c}
\overline{1} \quad \overline{2} \\
\overline{3} \quad \overline{3}
\end{array}
\begin{array}{c}
= \\
\end{array}
\begin{array}{c}
\overline{1} \quad \overline{2} \\
\overline{3} \quad \overline{3}
\end{array}
\begin{array}{c}
1 \quad 2 \\
\end{array}
\iff \hat{R}_{12} \hat{R}_{13} \hat{R}_{23} = \hat{R}_{23} \hat{R}_{13} \hat{R}_{12},

Again these are indeed satisfied because of the brane construction after uplift to M-theory.
Let us focus on N=2 (SU(2)) case and the representation of surface defect is the fundamental one. **It is known that there is an operator in integrable model**, so-called **L-operator** $L: W_i \otimes V_j \rightarrow V_j \otimes W_i$

This can be represented by 2x2 matrix whose entries are operators acting on function in $V_i$.

**Yang-Baxter equations:**

**One dashed line:**

$$= \iff \tilde{L}_1 \tilde{L}_2 \tilde{R}_{12} = \tilde{R}_{12} \tilde{L}_2 \tilde{L}_1$$

[Derkachov-Spiridonov]

**Two dashed lines:**

$$= \iff \tilde{R}_{12} \tilde{L}_1 \tilde{L}_2 = \tilde{L}_2 \tilde{L}_1 \tilde{R}_{12},$$

[Sklyanin]

**Three dashed lines:**

$$= \iff \tilde{R}_{12} \tilde{R}_{13} \tilde{R}_{23} = \tilde{R}_{23} \tilde{R}_{13} \tilde{R}_{12},$$

[Baxter]
We propose that the transfer matrix from these \( L \)-operators is identified with the index of the surface defect where Sklyanin found that the following operator solves the RLL relation:

\[
L = \sum_{a=0}^{3} w_a \sigma_a \otimes S_a
\]

where \( S_a \) satisfies the so-called Sklyanin algebra and is represented by a difference operator.

We propose that the **transfer matrix from these \( L \)-operators is identified with the index of the surface defect**

\[
\text{Tr}(L(c, (a_n, b_n)) \ldots L(c, (a_1, b_1))) = \sum_{s_1 = \{\pm 1\}} \ldots \sum_{s_n = \{\pm 1\}} \prod_{i=1}^{n} l(z_i^{s_i}, z_{i+1}^{s_{i+1}}; \frac{b_i}{c}, \frac{a_{i+1}}{c}) \prod_{j=1}^{n} \Delta^{s_j/2}_j.
\]

where

\[
(\Delta^{s_1/2}_i f)(z_i) = f(q^{\pm 1/2} z_i) \quad l(z_i, z_{i+1}; \frac{b_i}{c}, \frac{a_{i+1}}{c}) = \frac{1}{\theta(z_i^2)} \theta(\sqrt{pq} \frac{c^2}{b_i a_{i+1}} \frac{z_{i+1}}{z_i}) \theta(\sqrt{pq} \frac{b_i}{a_{i+1}} z_i z_{i+1}).
\]
Generalizations

- We can consider the surface defects supported on $S^1 \times S^1$. The $S^1$ can be one of two circles in $S^3$ ($|\xi_1|^2 + |\xi_2|^2 = 1$). Having two choices corresponds to modular double of Sklyanin algebra. ($p \leftrightarrow q$)

- General $N$ and general representations have been studied in [Yagi, ’17]
Surface defect in class $S$ theory
Now we come back to class S theory. Actually some theories in class S are a special case of brane tiling models.

Let us consider the following tiling:
M5 on cylinder

The previous brane tiling is produced by the brane system like:

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>N D5</strong></td>
<td>×</td>
<td>×</td>
<td>×</td>
<td>×</td>
<td>×</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>NS5</strong></td>
<td>×</td>
<td>×</td>
<td>×</td>
<td>×</td>
<td>×</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

T-dual

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>N D4</strong></td>
<td>×</td>
<td>×</td>
<td>×</td>
<td>×</td>
<td></td>
<td>×</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>NS5</strong></td>
<td>×</td>
<td>×</td>
<td>×</td>
<td>×</td>
<td>×</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

N M5-branes on
A surface defect

We add to this a D3 brane to describe a surface defect.

Depending on the directions, we have two types of defects:

1. | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
---|---|---|---|---|---|---|---|---|---|---|
N D5 | x | x | x | x | x | x |   |   |   |   |
NS5  | x | x | x | x | x | x |   |   |   |   |
D3   | x | x |   |   | x |   |   |   |   |   |

2. | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
---|---|---|---|---|---|---|---|---|---|---|
N D5 | x | x | x | x | x | x |   |   |   |   |
NS5  | x | x | x | x | x | x |   |   |   |   |
D3’  | x | x |   |   | x | x |   |   |   |   |
The first case:

From the proposal from integrable lattice model, the inclusion of the surface defect corresponds just to the trace of single L-operator:

\[
\text{Tr} \left( \tilde{L}^\diamond (d, (c, b)) \right) \propto \sum_{s=\pm 1} \frac{1}{\theta(z^{2s})} \theta \left( \sqrt{pq \frac{d^2}{bc}} \right) \theta \left( \sqrt{pq \frac{b}{c} z^{2s}} \right) \Delta^{s/2}.
\]

The difference operator acts on the flavor or gauge fugacity \( z \)

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>N D4</td>
<td>×</td>
<td>×</td>
<td>×</td>
<td>×</td>
<td>×</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>NS5</td>
<td>×</td>
<td>×</td>
<td>×</td>
<td>×</td>
<td>×</td>
<td>×</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>D2</td>
<td>×</td>
<td>×</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

N M5-branes on

[Gaiotto-Rastelli-Razamat]
The second case:

This corresponds to the trace of multiple L-operators:

\[ \prod \sum_{i} g_i(z_i, a_i, b, c) \Delta^{s_i/2} \]

\[
\begin{array}{cccccccccc}
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
\textbf{N D4} & \times & \times & \times & \times & \times & \times & & & \\
\textbf{NS5} & \times & \times & \times & \times & \times & \times & & & \\
\textbf{D4} & \times & \times & & \times & \times & \times & & & \\
\end{array}
\]
Conclusion

We propose that the index of the half-BPS surface defect from D3-branes in brane tiling model can be identified with the transfer matrix of L-operators in the corresponding integrable model.

This gives the unified picture of the indices of the M2- and M5-branes defects in class S theories.

Outlook

- M2-M5’ defect from Baxter’s R matrix
- another T-dual gives line defects in class S
- 3d reduction