THIRD SCOTTISH PDE COLLOQUIUM

JUNE 1-2, 2015
UNIVERSITY OF STRATHCLYDE, GLASGOW

- PROGRAMME AND ABSTRACTS -

This booklet gathers the abstracts of the talks presented at the third Scottish Partial Differential Equations Colloquium. This meeting started as a space in which academics whose research interests are related to Partial Differential Equations could gather to discuss research, and possible interactions. This is the first time the Scottish PDE Colloquium takes place at the University of Strathclyde, Glasgow, as in the two previous occasions it took place in ICMS, Edinburgh. We hope this will be the start of a rotating character of this meeting, and look forward to future incarnations of it to be held elsewhere around Scotland.

In this occasion, 16 speakers from all around Scotland (and England) have agreed to give invited talks. Also, academics and PhD students from various Scottish universities have registered to participate in it. So, we expect that this will constitute a very valuable research and networking activity for the PDE-related community in Scotland.

As organisers, we sincerely thank the following institutions for their financial support:

- Edinburgh Mathematical Society;
- Glasgow Mathematical Journal Trust;
- Glasgow City Marketing Bureau;
- University of Strathclyde.

Finally, we want to thanks all the participants for coming to Glasgow, and making this meeting possible.

Lucia Scardia & Gabriel Barrenechea
Glasgow, June 1-2, 2015

PROGRAMME

MONDAY 1 JUNE

- 09:45-10:20 Registration
- 10:20-10:30 Welcome

Chair: Lucia Scardia

- 10:30-11:30 **Geoffrey Burton**
  Variational problems involving rearrangements of functions
- 11:30-12:00 Coffee break

Chair: József Farkas

- 12:00-12:30 **Serena Dipierro**
  Boundary properties of nonlocal minimal surfaces
- 12:30-13:00 **Michael Grinfeld**
  Aspects of saturating flux diffusion
- 13:00-14:00 Lunch

Chair: Nigel Mottram

- 14:00-14:30 **James Maddison**
  On dynamically active divergent fluxes in geophysical flows
- 14:30-15:00 **Chuong Van Tran**
  A physicist's approach to Navier–Stokes regularity
- 15:00-15:30 **Peter Stewart**
  Discrete-to-Continuum models for foam fracture
- 15:30-16:00 Coffee break

Chair: Robin Knops

- 16:00-16:30 **Luigi Vergori**
  Nonlinear Elastodynamics of Materials with Strong Ellipticity Condition: Carroll-Type Solutions
- 16:30-17:00 **Matthias Langer**
  Schrödinger operators with δ-potentials supported on hypersurfaces
- 17:00-18:30 Wine reception
- 19:00 Social Dinner

TUESDAY 2 JUNE

Chair: Gabriel Barrenechea

- 09:30-10:30 **Endre Süli**
  Numerical approximation of non-divergence-form PDEs

- 10:30-11:00 Coffee break

Chair: Dugald Duncan

- 11:00-11:30 **Mariya Ptashnyk**
  Gene Regulatory Network and Functional Analysis: Hopf Bifurcation causes by molecular movement
- 11:30-12:00 **Lehel Banjai**
  A Trefftz space-time discontinuous Galerkin method for the second-order wave equation
- 12:00-12:30 **Iain Stewart**
  Pressure-driven Flow in Smectic A Liquid Crystals
- 12:30-13:00 **Heiko Gimperlein**
  Finite and boundary elements for interface problems in nonlinear elasticity: adaptivity and *hp*–stabilization

- 13:00-14:00 Lunch

Chair: John MacKenzie

- 14:00-14:30 **Hiroko Kamei**
  The Lattice of Partial Synchronies of a Coupled Cell Network
- 14:30-15:00 **Lyonell Boulton**
  Numerical approximation of Poincaré - Friedrichs constants
- 15:00-15:30 **József Farkas**
  Modelling structured populations: from partial differential equation to delay formulation

- 15:30-16:00 Coffee break

ABSTRACTS

- *Geoffrey Burton (University of Bath): Variational problems involving rearrangements of functions*

The rearrangements of a real function f_0 are the functions f such that, for every real α, the sets $\{x \mid f(x) \geq \alpha\}$ and $\{x \mid f(x) \geq \alpha\}$ have equal measure. In classical analysis, integrals are frequently extremised by functions with special properties such as symmetry and monotonicity, comparison of a function with a symmetric or monotonic rearrangement of itself often playing a role in the proof.

The focus of this talk will be extreme rearrangements that are not constructed explicitly but arise as solutions to variational problems having the set of rearrangements of a given f_0 as a constraint set. This theory is worked out most thoroughly for functions on bounded domains in $\mathbb{R}^N$ and weakly continuous integral functionals, where existence theorems, first variation conditions and minimax theorems have been established.

A key ingredient is the study of the set of rearrangements of f_0 as a set in L^p including J.V. Ryff's remarkable discovery that its closure in the weak topology of L^p is convex. The methods of convex analysis are of much greater utility than might be thought for this apparently non-convex variational problem.

Whereas in the context of bounded domains the extreme points of the weak closure of the rearrangements are precisely the rearrangements themselves, in the context of unbounded domains there can be more extreme points. This phenomenon is related to non-existence of extrema in certain situations.

The motivating example is that of vortices in a planar ideal fluid, where the (scalar) vorticity evolves within the set of rearrangements of its initial state (a phenomenon related to Kelvin's circulation theorem) and kinetic energy is a conserved convex functional. The general theory leads to existence theorems for vortices of extreme energy on an "equivortical surface" in function space and the first variation condition yields the criterion for the vortex to be steady, as anticipated by V.I Arnol′d and T.B. Benjamin.

- *Endre Süli (University of Oxford): Numerical approximation of non-divergence-form PDEs*

Non-divergence form partial differential equations with discontinuous coefficients do not generally posses a weak formulation, thus presenting an obstacle to their numerical solution by classical finite element methods. Such equations arise in many applications from areas such as probability and stochastic processes. These equations also arise as linearizations to fully nonlinear PDEs, as obtained for instance from the use of iterative solution algorithms. In such cases, it can rarely be expected that the coefficients of the operator be smooth or even continuous. For example, in applications to Hamilton–Jacobi–Bellman equations, the coefficients will usually be merely essentially bounded. In contrast to the study of divergence form equations, it is usually not possible to define a notion of weak solution when the coefficients are non-smooth. In the case of continuous but possibly non-differentiable coefficients, the Calderon–Zygmund theory of strong solutions establishes the well-posedness of the problem in sufficiently smooth domains. However, without additional hypotheses, well-posedness is generally lost in the case of discontinuous coefficients. The aim of the lecture is to survey recent developments concerning the numerical approximation of such problems by finite element methods. The talk is based on joint work with Iain Smears (University of Oxford).

- *Lehel Banjai (Heriot-Watt University): A Trefftz space-time discontinuous Galerkin method for the second-order wave equation*

  In recent years there has been much interest in using non-polynomial basis functions to discretize wave propagation problems in the frequency domain. A prominent example is the use of plane wave bases to solve the Helmholtz equation at high frequencies. The motivation behind this approach is to reduce the number of degrees of freedom per wavelength required to obtain accurate results. Thus obtained Trefftz methods have proved very successful in practice, hence it is a natural question to ask whether they can be extended and whether they can be equally successful in the time-domain.

  In this talk we present a new space-time interior penalty discontinuous Galerkin method for the second order wave equation, that allows the use of Trefftz bases. We present a stability and convergence analysis and illustrate the effectiveness of the method by numerical experiments.

  (This is work in collaboration with Emmanuil Georgoulis and Oluwaseun F. Lijoka.)

- *Lyonell Boulton (Heriot-Watt University): Numerical approximation of Poincaré - Friedrichs constants*

  The amplitude of a regular function which vanishes on the boundary of a compact region is controlled (integral square) by a constant times the gradient. The smallest possible value of the constant, allowing this to hold true for all functions in the suitable Sobolev space, is non-zero and it is often called the Poincaré - Friedrichs constant. A similar statement is still valid, if we replace the gradient by the curl operator on fields with zero divergence subject to suitable boundary conditions.

  Unfortunately, we cannot estimate numerically guaranteed upper bounds for Poincaré - Friedrichs constants by means of a direct application of the classical Galerkin method. The latter might lead, for example, to variational collapse in the case of the curl operator. In this talk I will examine two methods for overcoming this difficulty on regions with very rough boundary. The context of this research has been motivated by recent work by Vejchodsky. I will report on very recent results by myself produced in collaboration with Banjai and Hobiny.

- *Serena Dipierro (University of Edinburgh): Boundary properties of nonlocal minimal surfaces*

  In this talk we recall the notion of nonlocal minimal surfaces and we discuss their qualitative and quantitative boundary behavior. In particular, we present some optimal examples in which the surfaces stick at the boundary. This is a joint work with O. Savin and E. Valdinoci.

- *József Farkas (University of Stirling): Modelling structured populations: from partial differential equation to delay formulation*

  We consider a structured population model with a distributed recruitment process. The model is formulated as a first order hyperbolic partial integro-differential equation. It is well-posed on the biologically relevant state space of Lebesgue integrable functions. We also formulate a delayed integral equation for the distributed birth rate of the population. We illustrate the connection between the partial integro-differential and the delayed integral equation formulation of the model utilising a recent spectral theoretic result. Furthermore, we prove that any (non-negative) solution of the integral equation determines a unique solution of the partial differential equation, but not necessarily vice versa.

  Joint work with A. Calsina (Universitat Autonoma de Barcelona)

- *Heiko Gimperlein (Heriot-Watt University): Finite and boundary elements for interface problems in nonlinear elasticity: adaptivity and hp–stabilization*

  Adaptive finite element / boundary element procedures provide an efficient and extensively investigated tool for the numerical solution of uniformly elliptic transmission or contact problems. However, models of strongly nonlinear materials often lead to nonelliptic partial differential equations, where the standard Hilbert space analysis is no longer appropriate to analyze the numerical approximations by coupled finite and boundary element methods.

  As a model problem, we consider a p-Laplacian-variant of the Lamé equation in a bounded domain Ω coupled to the homogeneous Lamé equation in $\mathbb{R}^n \setminus \overline{\Omega}$. The exterior problem is reduced to an integral equation on $\partial\Omega$, and an equivalent coupled boundary/domain variational inequality is solved.

  This talk reviews basic convergence results and adaptive mesh refinements, then analyzes a mixed stabilized *hp*–adaptive boundary element method for the integral equation.

  The stabilization technique, originally due to Barbosa–Hughes for finite elements, circumvents the need to satisfy an inf-sup condition for the mixed problem. It thereby allows discretisations with convenient meshes and polynomial degrees. We consider the stabilization of integral equations and obtain a priori convergence rates in terms of the mesh size and polynomial degree for a Tresca frictional contact problem. In addition, a residual based a posteriori error estimate is derived, which gives rise to *hp*–adaptive mesh refinements. We comment on the theoretical challenges for more realistic frictional contact laws, such as Coulomb friction.

- *Michael Grinfeld (University of Strathclyde): Aspects of saturating flux diffusion.*

  In the 1990s, Rosenau argued for a mechanism of diffusion with saturating flux. We will discuss the following aspects of such equations: the curious exchange of stability and continuity of stationary states in a model of solid-solid phase transitions; long-time behaviour of solutions to Burgers' type equation with saturating flux, and travelling waves.

- *Hiroko Kamei (University of Dundee): The Lattice of Partial Synchronies of a Coupled Cell Network*

  In many areas of science, interacting dynamical systems can be represented as a network. We are interested in partial synchronisation where a fully synchronised network breaks into clusters such that all individual dynamical systems within one cluster are perfectly synchronised. We consider partial synchronisation of a network using the coupled cell network formalism which represents interacting (coupled) individual systems (cells) as a directed graph, and studies how the network structure constrains its dynamics. In this formalism, partial synchronies of a network are given as a special type of invariant subspaces of the adjacency matrix, which represents the network structure. We represent all possible partial synchronies of a given network as a complete lattice, which can be used to classify networks with similar synchrony clusters, and identify specific dynamics. We also show that lattice structures have a direct link to the eigenvalues and eigenvectors of an adjacency matrix and can be used to analyse synchrony-breaking bifurcation of a given network.

- *Matthias Langer (University of Strathclyde): Schrödinger operators with δ-potentials supported on hypersurfaces*

Schrödinger operators with potentials supported on sets with zero measure have been studied in quantum physics for a long time. They can be used to approximate more complicated potentials, like point interactions approximating short-range potentials. In this talk I will discuss ways of rigorously defining Schrödinger operators with potentials that are supported on hypersurfaces, like δ and δ'-potentials. Moreover, I will present results about their spectral properties and associated wave and scattering operators.

- *James Maddison (University of Edinburgh): On dynamically active divergent fluxes in geophysical flows*

It is typical, in the study of geostrophic turbulence, for small-scale fields to influence the large-scale through the divergence of eddy fluxes. As such the dynamics is independent of any non-divergent flux components. In practice these components may be large in magnitude, and this obscures their interpretation. This is a particular issue in the development and study of ocean eddy parameterisations: in principle, only the (typically much smaller) divergent components need be parameterised. A candidate scheme may suggest parameterised fluxes which differ very significantly from those diagnosed in an eddying model, but may nevertheless perform well if it yields accurate flux divergences.

Here the definition of divergent fluxes is considered via a study of the structure of the relevant dynamical equations. For the two-dimensional Navier-Stokes equations any rotational eddy vorticity fluxes can be shown to project onto the pressure gradient. This suggests an intuitive definition for dynamically active divergent fluxes. More general approaches are considered, and in particular the application to the quasi-geostrophic equations is described.

- *Mariya Ptashnyk (University of Dundee): Gene Regulatory Network and Functional Analysis: Hopf Bifurcation causes by molecular movement*

Gene regulatory networks, i.e. DNA segments in a cell which interact with each other indirectly through their RNA and protein products, lie at the heart of many important intracellular signal transduction processes. We consider a mathematical model of a canonical gene regulatory network consisting of a single negative feedback loop between a protein and its mRNA (e.g. the Hes1 transcription factor system). The model consists of two partial differential equations describing the spatio-temporal interactions between the protein and its mRNA. Such intracellular negative feedback systems are known to exhibit oscillatory behaviour. Applying linearised stability analysis, we study the stability of a (spatially inhomogeneous) steady state of the system of two reaction-diffusion equations. Our results show that the diffusion coefficient of the protein/mRNA acts as a bifurcation parameter and gives rise to a Hopf bifurcation. The main difficulty of the analysis is that the steady state of the model is not constant. We show the existence of a Hopf bifurcation by considering a limit problem associated with the original model equations. The method of collective compactness is applied to relate the spectrum of the limit operator to the spectrum of the original operator. To show the stability of periodic solutions and to determine the type of Hopf bifurcation, we use a weakly nonlinear analysis and normal form theory. Our result has implications for transcription factors such as p53, NF-κB and heat shock proteins which are involved in regulating important cellular processes such as inflammation, meiosis, apoptosis and the heat shock response, and are linked to diseases such as arthritis and cancer.

- *Iain Stewart (University of Strathclyde): Pressure-driven Flow in Smectic A Liquid Crystals*

Following a brief description of liquid crystals, the steady flow of a smectic A liquid crystal sample is examined. The sample is initially aligned in a classical 'bookshelf' geometry confined between parallel plates and is then subjected to a lateral pressure gradient which is perpendicular to the initial local smectic layer arrangement. The usual molecular alignment of liquid crystals, modelled by the director **n** (a unit vector) is allowed to separate from the local smectic layer unit normal vector **a** in order to model boundary layer effects. The nonlinear dynamic equations are derived via the continuum theory developed in [4] which allows for discrepancies between these two unit vectors. The governing equations can be linearised and solved exactly to reveal two characteristic length scales that can be identified in terms of the material parameters and reflect the boundary layer behaviour of the velocity and the orientations of the director and smectic layer normal. This resolves one of the long-standing major dilemmas that arose from the theoretical and experimental work of de Gennes [1, 2]; earlier linearised theories could not detect the presence of multiple boundary layer effects that were evident in experimental much work (e.g., Elston [3, 5]). The asymptotic properties of the nonlinear equations are investigated to find that these length scales apparently manifest themselves in various aspects of the solutions to the nonlinear steady state equations, especially in the separation between the orientations of the director and smectic layer normal. Non-Newtonian plug-like flow occurs and the solutions for the director profile and smectic layer normal share features identified elsewhere in static liquid crystal configurations. Comparisons with numerical solutions of the nonlinear equations will also be made. This is a joint work with M. Vynnycky (KTH, Stockolm), S. McKee (Strathclyde), and M.F. Tomé (Sao Paulo, Brazil).

References

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[5] H. G. Walton, I. W. Stewart, and M. J. Towler, *Flow past finite obstacles in smectic liquid crystals: permeative flow induced S_A to S_C phase transition*, Liq. Cryst., 20 (1996), pp. 665–668.

- *Peter Stewart (University of Glasgow): Discrete-to-Continuum models for foam fracture*

We examine the fracture of a quasi two-dimensional aqueous foam under an applied driving pressure. Experiments indicate two distinct mechanisms of failure in this system, analogous to those observed in a crystalline solid: a slow ductile mode when the driving pressure is applied slowly, where the void propagates as bubbles interchange neighbours through the T1 process, and a rapid brittle mode for faster application of pressures, where the void advances by successive rupture of liquid films driven by Rayleigh–Taylor instability.

Starting with PDE continuum models for each of the two phases in the foam (liquid and gas), we form a reduced model for the dynamics of fracture by constructing a discrete network model of ODEs and algebraic constraints using a rational asymptotic reduction in the limit of low liquid fraction. Simulations of this network model allow detailed insight into the mechanics of the fracturing medium and the role of its microstructure in determining the direction of propagation.

In order to further quantify the global mechanisms of foam fracture we form a new one-dimensional continuum PDE model from this ODE network model using discrete-to-continuum methods, comprising a coupled system of four nonlinear telegraph equations. This PDE model enables analytical prediction of the pressure distribution through the foam as well as detailed insight into the interplay between the speed of crack propagation (driven by film breakage) and the speed of shear wave propagation ahead of the crack tip (driven by post-rupture rearrangement). We demonstrate how the presence of the shear wave can significantly enhance the crack speed and play a key role in localising brittle fracture into a single line of breakages.

(This is work in collaboration with Sascha Hilgenfeldt.)

- *Chuong Van Tran (University of St. Andrews): A physicist's approach to Navier–Stokes regularity*

  This talk derives some new regularity criteria for solutions of the Navier–Stokes equations evolving from sufficiently regular initial velocity fields. A semi-analytic approach is used to argue that viscous effects are strong enough to regularise the Navier–Stokes dynamics.

- *Luigi Vergori (University of Glasgow): Nonlinear Elastodynamics of Materials with Strong Ellipticity Condition: Carroll-Type Solutions*

  Classes of deformations in nonlinear elastodynamics with origin in pioneering work of Carroll are investigated for an isotropic elastic solid subject to body forces corresponding to a nonlinear substrate potential. Exact solutions are obtained which, *inter alia*, are descriptive of the propagation of compact waves and motions with oscillatory spatial dependence. It is shown that a description of slowly modulated waves leads to a novel class of generalized nonlinear Schrödinger equations.