Nonlocal interactions: Dislocations and beyond

Titles and Abstracts

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The large-N behaviour of N-particle minimisers of the interaction energy

José Alfredo Cañizo

We consider the N-particle minimisers of the interaction energy with attractive-repulsive potentials and study their asymptotic behaviour as N tends to infinity. We show that a different behaviour takes place for H-stable and non H-stable potentials: particles tend to fill the space in the first case, while they stay in a bounded region in the second case, approaching a minimiser of the continuum interaction energy as N grows. We will also comment on some open problems and conjectures based on numerical simulations. This talk is based on works with José A. Carrillo and Francesco Patacchini.

Splitting schemes & segregation in reaction-(cross-)diffusion systems

José Antonio Carrillo de la Plata

One of the most fascinating phenomena observed in reaction-diffusion systems is the emergence of segregated solutions, population densities with disjoint supports. We analyse such a reaction cross-diffusion system. In order to prove existence of weak solutions for a wide class of initial data without restriction about their supports or their positivity, we propose a variational splitting scheme combining ODEs with methods from optimal transport. In addition, this approach allows us to prove conservation of segregation for initially segregated data even in the presence of vacuum.

The isoperimetric inequality and the Wulff problem on lattices

Marco Cicalese

We review the quantitative isoperimetric inequality. Then we discuss a possible application of quantitative estimates to study rate of convergences of asymptotic minimisers of sequences of energies. As a result, we show how to simply derive sharp asymptotic estimates for the maximal fluctuation of Wulff shapes on lattices.

Existence of minimizers for Griffith's fracture model

Sergio Conti

Variational models of geometrically linear brittle fracture are formulated in terms of deformation fields which jump over a closed set of finite $(n-1)$-dimensional measure and whose symmetrized gradient is square integrable on the rest of the domain. By general theory, the relaxation of this model has minimizers in the space SBD^p; existence of classical minimizers is far less clear. I shall discuss recent progress in this direction, also in relation to approximation results for SBD^p functions and to regularity results for elasticity-type elliptic problems with p growth. This talk is based on joint work with Antonin Chambolle, Matteo Focardi and Flaviana Iurlano.

Variational analysis for dipoles of topological singularities in two dimensions.

Lucia De Luca

We present two continuous models for the study of topological singularities in 2D: the core-radius approach and the Ginzburg-Landau theory. It is well known that - at zero temperature and under natural energy bounds - the energies associated to these models tend to concentrate, as the length scale parameter ε goes to zero, around a finite number of points, the so-called vortices. We focus on low energy regimes that prevent the formation of vortices in the limit as ε tends to zero, but that are compatible (for positive ε) with configurations of short (in terms of ε) dipoles, and more in general with short clusters of vortices having zero average. By using a Γ-convergence approach, we provide a quantitative analysis of the energy induced by such configurations on a continuous range of length scales. Joint work with M. Ponsiglione (Rome).

Mathematical Analysis of Novel Advanced Materials: Epitaxy and Quantum Dots

Irene Fonseca

Quantum dots are man-made nanocrystals of semiconducting materials. Their formation and assembly patterns play a central role in nanotechnology, and in particular in the optoelectronic properties of semiconductors. Changing the dots' size and shape gives rise to many applications that permeate our daily lives, such as the new Samsung QLED TV monitor that uses quantum dots to turn "light into perfect color"!

Quantum dots are obtained via the deposition of a crystalline overlayer (epitaxial film) on a crystalline substrate. When the thickness of the film reaches a critical value, the profile of the film becomes corrugated and islands (quantum dots) form. As the creation of quantum dots evolves with time, materials defects appear. Their modelling is of great interest in materials science since material properties, including rigidity and conductivity, can be strongly influenced by the presence of defects such as dislocations.

Patterns for line dislocations

Adriana Garroni

I will present the upscaling of a 2d non local variational model for line defects in crystals lying on a single slip plane. Here dislocations are described by a regularized phase field, the plastic slips. We show that, in a scaling regime where the total length of the dislocations is large, the phase field model reduces to a simpler model of the strain-gradient type. The limiting model contains a term describing the three-dimensional elastic energy and a strain-gradient term describing the energy of the geometrically necessary dislocations, characterized by the tangential gradient of the slip. The energy density appearing in the strain-gradient term is determined by the solution of a cell problem, which depends on the line tension energy of dislocations. In some physically relevant cases we can show that this upscaling procedure requires the formation of microstructure, i.e. network of dislocations.

A theory of Discrete Dislocation Dynamics in three dimensions

Thomas Hudson

We present a mathematical framework within which Discrete Dislocation Dynamics in three dimensions is well-posed. By considering smooth distributions of slip, we derive a regularised energy for curved dislocations, and rigorously derive the Peach-Koehler force on the dislocation network via an inner variation. We propose a dissipative evolution law which is cast as a generalised gradient flow, and using a discrete-in-time approximation scheme, existence and regularity results are obtained for the evolution, up until the first time at which an infinite density of dislocation lines forms.

A new class of fractional Sobolev spaces

Giovanni Leoni

We present a new class of fractional Sobolev spaces which are related to the trace spaces of homogeneous Sobolev spaces in infinite strip-like domains

Dislocations and Kirchhoff ellipses
Joan Mateu

In this talk I will present some results on the characterization of the minimisers for a given nonlocal energy. These minimisers coincide with the family of Kirchhoff ellipses, which are rotating vortex patches for the 2D-Euler equation. This result is a joint work with Carrillo, Mora, Rondi, Scardia and Verdera.

Homogenization of parabolic equations
Matteo Novaga

I consider the homogenization of a semilinear parabolic equation with vanishing viscosity and with an oscillating potential. According to the rate between the frequency of oscillations in the potential and the vanishing viscosity factor, the limit evolution exhibits different regimes, and it turns out that in the strong diffusion regime the effective operator is discontinuous in the gradient entry. I discuss the main properties of the solutions to the effective problem, and show uniqueness for some classes of initial data.

Convergence of gradient flows: tilted is better
Mark Peletier

It is a very common challenge to pass to the limit in a sequence of evolution equations $\dot{x}_\epsilon = f_\epsilon(x_\epsilon)$, and over the years I have proved many theorems of this kind. In particular, the theory of gradient flows provides various useful handles for proving such convergences. In this talk I want to report on a slightly surprising observation: sometimes it is better not to study the original sequence $\dot{x}_\epsilon = f_\epsilon(x_\epsilon)$, but instead embed the sequence into a larger class of sequences, and study the convergence of this whole class instead. The resulting limiting gradient-flow structures may be different from those obtained by more classical methods, and in some cases they better represent the modelling aspects of the limits. This is work together with Alexander Mielke and Alberto Montefusco.

Lattice energies and crystallization for different pairwise interaction potentials

Mircea Petrache

Consider the following fundamental crystallization question: We consider N points $x_1, .., x_N$ which form a ground state for the energy E_f given as the sum of pairwise energies $f(|x_i - x_j|)$. What principles force the configuration to form a triangular lattice in 2-dimensions asymptotically for very large N? The two possible settings are the one in which we fix the particle density as a constraint for the minimization, and the one in which we fix the scale via the shape of f, by looking at one-well potentials f. I will recall some important known results and present some new results and counterexamples, concerning the relevant and useful properties of the potential f which allow to treat the minimization of E_f amongst lattice-like, and sometimes amongst more general, configurations. This is based on joint work with L. Betermin.

Γ-convergence of the Heitmann-Radin sticky disc energy to the crystalline perimeter

Marcello Ponsiglione

We will introduce and analyse the Heitmann-Radin sticky discs functional, in the limit of a diverging number of discs. For configurations whose energy scales like the perimeter, we prove a compactness result which shows the emergence of polycrystalline structures: The empirical measure converges to a set of finite perimeter, while a microscopic variable, representing the orientation of the underlying lattice, converges to a locally constant function. Whenever the limit configuration is a single crystal, i.e., it has constant orientation, we show that the Γ-limit is the anisotropic perimeter, corresponding to the Finsler metric determined by the orientation of the single crystal. Finally, we will discuss some related tessellation problems, some partial results concerning polycrystals and some open problems. The results are in collaboration with Lucia De Luca and Matteo Novaga.

A general multiscale theory in imaging

Luca Rondi

We extend the hierarchical decomposition of an image as a sum of constituents of different scales, introduced by Tadmor, Nezzar and Vese in 2004, to a nonlinear setting. We develop a general framework for multiscale decompositions which is applicable to a wide range of imaging and nonlinear inverse problems, for example we settle an open problem on the decomposition of Tadmor, Nezzar and Vese for arbitrary L^2 functions. We apply our theory to image registration, in the framework of the so-called Large Deformation Diffeomorphic Metric Mapping

(LDDMM). Image registration is an important problem in medical imaging: one seeks an optimal diffeomorphism between two given images, in order to align images obtained at different times or with different instrumentation by transforming one to the other. We construct analogous hierarchical expansions for such diffeomorphisms, with the sum replaced by composition of maps.

This is a joint work with Klas Modin and Adrian Nachman.

Regularity theory for free boundary problems with dynamic boundary conditions

Xavier Ros-Oton

We discuss the regularity of solutions and free boundaries in (parabolic) obstacle problems with dynamic boundary conditions. This is equivalent to the obstacle problem for the operator $\partial_t + (-\Delta)^{1/2}$.

Despite significant recent advances in the regularity theory for solutions and free boundaries in obstacle problems with nonlocal operators, there was a lack of understanding of parabolic problems with *critical* scaling, such as the obstacle problem for $\partial_t + (-\Delta)^{1/2}$. No regularity result for free boundaries was known for any parabolic problem with such scaling.

In this talk, we present some new results in this context, which allow us to understand for the first time the fine behavior of solutions, as well as the regularity of free boundaries. This is a joint work with A. Figalli and J. Serra.

Mean-Field limits for Coulomb dynamics

Sylvia Serfaty

We consider a system of N particles evolving according to the gradient flow of their Coulomb or Riesz interaction, or a similar conservative flow. By Riesz interaction, we mean inverse power s of the distance with s between d-2 and d where d denotes the dimension. After reviewing results for energy minimizers, we state a convergence result as N tends to infinity to the expected limiting evolution equation.

Are atomistic equilibrium distributions ordered?

Florial Theil

It is an open problem whether ordered crystalline structures can be explained as equilibrium configurations of atomistic energies at low temperatures. We demonstrate that in two dimensions configurations exhibit orientiational order if dislocations are tightly bound. This is a consequence of rigidity estimates. Therefore the central question is whether order also holds if no constraints are imposed on disclocations. First results in this direction are shown for the Ariza-Ortiz' model for discrete crystal elasticity and discrete dislocations in crystals.

Chaotic orbits for nonlocal equations and applications to atom dislocation dynamics in crystals

Enrico Valdinoci

We consider a nonlocal equation driven by a perturbed periodic potential. We construct multibump solutions that connect one integer point to another one in a prescribed way. In particular, heteroclinic, homoclinic and chaotic trajectories are constructed. This result regarding symbolic dynamics in a fractional framework is part of a study of the Peierls-Nabarro model for crystal dislocations. The associated evolution equation can be studied in the mesoscopic and macroscopic limit. Namely, the dislocation function has the tendency to concentrate at single points of the crystal, where the size of the slip coincides with the natural periodicity of the medium. These dislocation points evolve according to the external stress and an interior potential, which can be either repulsive or attractive, depending on the relative orientations of the dislocations. For opposite orientations, collisions occur, after which the system relaxes exponentially fast.

Nonlinear Elliptic and Parabolic Equations involving Fractional Operators on Bounded Domains

Juan Luis Vazquez

The talk presents work on the existence and behaviour of solutions of nonlinear fractional elliptic and parabolic equations, mainly when posed in bounded domains. Attention is given to functional aspects, to the boundary behaviour, and to the long-time asymptotics.

Two classical theorems on vortex patches

Joan Verdera

The vorticity form of the planar Euler equation says that vorticity is constant along particle trajectories. A vortex patch is a weak solution of the vorticity equation with initial condition the characteristic function of a domain D_0. Thus at time t vorticity is the characteristic function of a domain D_t. Simulations show that the evolution of D_t is extremely complicated. In spite of this general fact there are some special domains, called V-states, whose evolution is just rotation around the center of mass with constant angular velocity. Ellipses are examples of V-states. I will discuss Burbea's proof of existence of other V-states and then I will discuss the smoothness of their boundary (joint work with Hmidi and Mateu). For general vortex patches, if the initial condition is the characteristic function of a domain with boundary of class $C^{1+\gamma}$, then the boundary of D_t conserves the regularity for all times (Chemin's theorem). I will mention a similar result for the aggregation equation in higher dimensions (joint work with Bertozzi, Garnett and Laurent).