INVERSE SEMIGROUPS
TOPOLOGICAL GROUPOIDS
$C^*$-ALGEBRAS
AND NON-COMMUTATIVE STONE
DUALITY

Mark V Lawson
Heriot-Watt University
and the
Maxwell Institute for Mathematical Sciences
Edinburgh, Scotland
UK
Talk given at ICMS, 23rd February 2012
The work described in this talk was developed in collaboration with Daniel Lenz (Jena).
Philosophische Bemerkung

Before beginning a Hunt, it is wise to ask someone what you are looking for before you begin looking for it.

Winnie-the-Pooh
1. Background

Inverse semigroups, topological groupoids and $C^*$-algebras are somehow related. The question is how and why? And perhaps — why should I care?

Three key references for the general theory are:


My interest in this area stems mainly from three papers by Johannes Kellendonk:


He showed how to associate an inverse semigroup with any tiling, the *tiling semigroup*. In the case of aperiodic tilings, this semigroup can be viewed as the *inverse semigroup of partial translational symmetries* of the tiling.
Further motivation and insight came from the following:

- The work of Ehresmann on algebraic models of pseudogroups.


This suggested that Higman-Thompson-like groups and topos theory were somehow also connected.
Basic question

What is the relationship between inverse semi-groups and topological groupoids as used in $C^*$-algebras?

Short answer

This is contained in the preprint with Lenz: *Pseudogroups and their étale groupoids*, arXiv:1107.5511v2.
Ghosts of departed quantities

A *frame* is a complete infinitely distributive lattice.

The theory of frames can be viewed as an approach to spaces in which open sets, and not points, are taken as basic.

This is the stance taken in Peter Johnstone’s book *Stone spaces*, CUP, 1986:
The Big Idea

Inverse semigroup theory can be regarded as non-commutative frame theory.

This provides the setting for understanding the connection between inverse semigroups and topological groupoids.
Peter Johnstone writes

It was Ehresmann ... and his student Bénabou ... who first took the decisive step in regarding complete Heyting algebras as ‘generalized topological spaces’.

So, Ehresmann was right all along.
Some definitions

• We use the term boolean algebra to mean generalized boolean algebra. If a boolean algebra has a top element we shall say that it is unital.

• A semigroup is a set equipped with an everywhere defined associative binary operation. Monoids are semigroups with an identity. Semigroups with zero are called semigroups with zero.

• A groupoid $G$ is a (for us, small) category with every arrow invertible. The set of identities (or objects) of $G$ is denoted by $G_0$. The ‘o’ stands for ‘objects’.
Inverse semigroups

A semigroup $S$ is said to be inverse if for each $s \in S$ there exists a unique $s^{-1} \in S$ such that

$$s = ss^{-1}s \text{ and } s^{-1} = s^{-1}ss^{-1}.$$ 

Observe that $s^{-1}s$ and $ss^{-1}$ are idempotents, and that $(s^{-1})^{-1} = s$ and $(st)^{-1} = t^{-1}s^{-1}$.

It can be proved that idempotents commute (Munn and Penrose).

Set of idempotents of $S$, denoted by $E(S)$, equipped with an order $e \leq f$ iff $e = ef = fe$ which makes $E(S)$ a meet semilattice.

The set of all partial bijections $I(X)$ on the set $X$ is an inverse semigroup, called the symmetric inverse monoid.
Order and compatibility

An inverse semigroup $S$ is equipped with two important relations:

- $s \leq t$ is defined if and only if $s = te$ for some idempotent $e$. Despite appearances ambidextrous. Called the natural partial order. Compatible with multiplication.

- $s \sim t$ if and only if $st^{-1}$ and $s^{-1}t$ both idempotents. Compatibility relation. Not in general an equivalence relation. Controls when pairs of elements are eligible to have a join.
2. Non-commutative Stone duality

Our theory is founded on two ideas and two theorems.
Idea 1

**Theorem** [Marshall H. Stone]  *The category of Boolean algebras is dual to the category of Boolean spaces — that is, Hausdorff topological spaces with a basis of compact-open sets.*

This theorem links algebra and order, in the guise of boolean algebras, with topology.

We shall generalize this theorem by replacing *Boolean algebras* by *Boolean inverse semigroups* and *Boolean spaces* by *Boolean groupoids.*
An inverse semigroup $S$ is said to be *Boolean* if it satisfies the following three conditions:

1. Each pair of compatible elements has a join.

2. Products distribute over joins where they exist.

3. The semilattice of idempotents of $S$ is a Boolean algebra.

An inverse semigroup satisfying just (1) and (2) is said to be *distributive*.

**Example** Symmetric inverse monoids are Boolean.
A groupoid $G$ is said to be *Boolean* if it satisfies the following

1. $G$ is a topological groupoid.

2. $G$ is étale — this means that the domain map is a local homeomorphism.

3. The space of identities $G_o$ is Boolean.

Our non-commutative generalization of Stone duality can now be stated.

**Theorem 1** [Lawson, Lenz] *The category of Boolean inverse semigroups is dual to the category of Boolean spaces.*
Proof

There are two proofs.

The first is a direct generalization of classical Stone duality and uses ultrafilters and local bisections.

The second works within non-commutative frame theory.
This is all well and good, but where are the Boolean inverse semigroups?
Idea 2

Let \( A \subseteq s \downarrow \) be a finite subset of the elements beneath \( s \). We say that \( A \) is a Lenz cover of \( s \) or more simply that \( A \) covers \( s \) and write \( s \to A \) if for each \( 0 \neq t \leq s \) there exists \( a \in A \) such that \( a \) and \( t \) have a non-zero lower bound.

This condition says that \( s \) is eligible to be the join of \( A \).

The set of covers of the elements of an inverse semigroup forms what is called the tight coverage of the inverse semigroup.

A homomorphism \( \theta: S \to T \) to a distributive inverse semigroup is said to be tight if

\[
a \to \{a_1, \ldots, a_m\} \Rightarrow \theta(a) = \bigvee_{i=1}^{m} \theta(a_i).
\]

Thus tight homomorphisms map potential joins to actual joins.
A completion theorem

**Theorem 2** [Lawson, Lenz] *For each inverse semigroup $S$ there is a distributive inverse semigroup $D(S)$ and a tight homomorphism $\delta: S \rightarrow D(S)$ which is universal for tight maps to distributive inverse semigroups.*
Pre-Boolean inverse semigroups

An inverse semigroup \( S \) is said to be \textit{pre-Boolean} or satisfy the \textit{compactness condition} if \( D(S) \) is Boolean.

If \( S \) is pre-Boolean we say that \( D(S) \) is its \textit{Boolean completion}.

\textbf{Really important examples} Under suitable assumptions, graph inverse semigroups and tiling semigroups are pre-Boolean.
Intuitions

• An inverse semigroup equipped with its tight coverage should be regarded as a presentation of the distributive (or Boolean) inverse semigroup that is its completion.

• The elements of the coverage should be thought of as relations of the form

\[ a = a_1 \lor \ldots \lor a_n. \]

• A Boolean inverse semigroup is a little like a semiring only the addition is only partially defined. (A sort of hemisemiring?).

• There is a more general notion of coverage of which tight ones are special cases.
The passage from inverse semigroup plus coverage to Boolean inverse semigroup is a \textit{local-to-global} construction.
**Example 1**

The finite symmetric inverse monoid $I(X)$ or the symmetric inverse monoid $I^f(X)$ of partial bijections with finite domain.

These are already boolean inverse monoids.

The associated groupoid is the set $X \times X$ with the usual groupoid multiplication equipped with the discrete topology.

When $|X| = n$, then $I(X)$ is related to the $C^*$-algebra $M_n(\mathbb{C})$. When $|X| = \aleph_0$, then $I^f(X)$ is related to the $C^*$-algebra of bounded operators.
Example 2

The polycyclic monoid $P_n$, where $n \geq 2$, is defined as a monoid with zero generated by the variables $a_1, \ldots, a_n, a_1^{-1}, \ldots, a_n^{-1}$ subject to the relations

$$a_i^{-1}a_i = 1 \text{ and } a_i^{-1}a_j = 0, i \neq j.$$  

Every non-zero element of $P_n$ is of the form $yx^{-1}$ where $x$ and $y$ are elements of the free monoid on $\{a_1, \ldots, a_n\}$.

The product of two elements $yx^{-1}$ and $vu^{-1}$ is zero unless $x$ and $v$ are prefix-comparable in which case

$$yx^{-1} \cdot vu^{-1} = \begin{cases} yzu^{-1} & \text{ if } v = xz \text{ for some } z \\ y(uz)^{-1} & \text{ if } x = vz \text{ for some } z \end{cases}$$
The polycyclic monoid $P_n$ is an inverse monoid not a Boolean inverse monoid.

It is a combinatorial inverse semigroup meaning that there are no non-trivial groups lurking inside it.

Its tight coverage is equivalent to the single relation

$$1 = \bigvee_{i=1}^{n} a_i a_i^{-1}.$$
**Theorem** The boolean completion of $P_n$ is called (here) the *Cuntz inverse monoid* $CI_n$.

1. This monoid is congruence-free.

2. Its group of units is the Thompson group $V_{n,1}$.

3. Its associated groupoid is the groupoid also associated with the Cuntz $C^*$-algebra $C_n$. 
Worth noting

O. Bratteli, P. E. T. Jorgensen, *Iterated function systems and permutation representations of the Cuntz algebra*, Memoirs of the A.M.S. No. 663, (1999) is, in fact, a study of tight maps from $P_n$ to $I(X)$.

This theory leads to the construction of interesting groups. The groups are obtained by gluing together partial symmetries into global symmetries.

Something similar can be done for the Cuntz-Krieger $C^*$-algebras. In the case of finite directed graphs, this gives rise to groups that should be analogues of the Higman-Thompson groups.

What can we say about the groups of units of Boolean inverse monoids?
3. Vague thoughts, vaguely expressed

1. The theory of Boolean inverse semigroups is intimately connected to Cantor-type spaces. Lying behind the different manifestations of the theory there should be Cantor structures, whatever that might mean.

2. Connections: inverse semigroups, topological groupoids, $C^*$-algebras, topos theory, groups related to the Higman-Thompson groups, linear logic and reversible computation, MV-algebras, symbolic dynamics.

3. Why are inverse playing such a prominent role in $C^*$-algebra theory? Perhaps because the Elliot-programme to classify $C^*$-algebras via their $K_0$-groups requires many projections, and inverse semigroups supply those projections.
Concluding remark

Kellendonk was inspired by Shechtman’s work on quasi-crystals.

Shechtman’s work has led to much research into the theory of aperiodic tilings and he received the 2011 Nobel prize for chemistry.

It raises general questions about the nature of symmetry and how it can be formalized mathematically.

I believe that this is one of the reasons that inverse semigroups arise in this field — they extend the classical notion of group and enable us to deal with more exotic notions of symmetry.