A CONSTRUCTION OF SUBSHIFTS AND
A CLASS OF SEMIGROUPS

TOSHIHIRO HAMACHI AND WOLFGANG KRIEGER

ABSTRACT. Subshifts with property (A) are constructed from a class of directed graphs. As special cases the Markov-Dyck shifts are shown to have property (A). The $\mathcal{R}$-graph semigroups, that are associated to topologically transitive subshifts with Property (A), are characterized.

1. INTRODUCTION

Let Σ be a finite alphabet, and let S be the shift on the shift space $\Sigma^{\mathbb{Z}}$,

$$S((x_i)_{i \in \mathbb{Z}}) = (x_{i+1})_{i \in \mathbb{Z}}, \qquad (x_i)_{i \in \mathbb{Z}} \in \Sigma^{\mathbb{Z}}.$$

An S-invariant closed subset X of $\Sigma^{\mathbb{Z}}$ is called a subshift. For an introduction to the theory of subshifts see [Ki] or [LM]. In [Kr2] a Property (A) of subshifts was introduced that is an invariant of topological conjugacy. Also in [Kr2] a semigroup was constructed that is invariantly attached to a subshift with property (A). Prototypes of subshifts with Property (A) are the Dyck shifts [Kr1]. To recall the construction of the Dyck shifts, let $N > 1$, and let $\alpha^-(n), \alpha^+(n), 0 \leq n < N$, be the generators of the Dyck inverse monoid (the polycyclic monoid [NP]) $\mathcal{D}_N$, that satisfy the relations

$$\alpha^-(n)\alpha^+(m) = \begin{cases} 1, & \text{if } n = m, \\ 0, & \text{if } n \neq m. \end{cases}$$

The Dyck shifts are defined as the subshifts

$$D_N \subset (\{\alpha^-(n) : 0 \leq n < N\} \cup \{\alpha^+(n) : 0 \leq n < N\})^{\mathbb{Z}}$$

with the admissible words $(\sigma_i)_{1 \leq i \leq I}, I \in \mathbb{N}$, of $D_N, N > 1$, given by the condition

$$\tag{1} \prod_{1 \leq i \leq I} \sigma_i \neq 0.$$

The Dyck inverse monoid $\mathcal{D}_N$ is associated to the Dyck shift D_N.

We recall from [Kr4] the notion of a partitioned directed graph and of an $\mathcal{R}$-graph. Let there be given a finite directed graph with vertex set $\mathfrak{P}$ and edge set $\mathcal{E}$. Assume also given a partition

$$\mathcal{E} = \mathcal{E}^- \cup \mathcal{E}^+.$$

With s and t denoting the source and the target vertex of a directed edge we set

$$\mathcal{E}^-(\mathfrak{q}, \mathfrak{r}) = \{e^- \in \mathcal{E}^- : s(e^-) = \mathfrak{q}, \ t(e^-) = \mathfrak{r}\},$$
$$\mathcal{E}^+(\mathfrak{q}, \mathfrak{r}) = \{e^- \in \mathcal{E}^+ : s(e^+) = \mathfrak{r}, \ t(e^+) = \mathfrak{q}\}, \qquad \mathfrak{q}, \mathfrak{r} \in \mathfrak{P}.$$

We assume that $\mathcal{E}^-(\mathfrak{q}, \mathfrak{r}) \neq \emptyset$ if and only if $\mathcal{E}^+(\mathfrak{q}, \mathfrak{r}) \neq \emptyset, \mathfrak{q}, \mathfrak{r} \in \mathfrak{P}$, and we assume that the directed graph $(\mathfrak{P}, \mathcal{E}^-)$ is strongly connected, or, equivalently, that the directed graph $(\mathfrak{P}, \mathcal{E}^+)$ is strongly connected. We call the structure $(\mathfrak{P}, \mathcal{E}^-, \mathcal{E}^+)$ a partitioned directed graph. Let there further be given relations

$$\mathcal{R}(\mathfrak{q}, \mathfrak{r}) \subset \mathcal{E}^-(\mathfrak{q}, \mathfrak{r}) \times \mathcal{E}^+(\mathfrak{q}, \mathfrak{r}), \qquad \mathfrak{q}, \mathfrak{r} \in \mathfrak{P},$$

and set
$$\mathcal{R} = \bigcup_{\mathfrak{q}, \mathfrak{r} \in \mathfrak{P}} \mathcal{R}(\mathfrak{q}, \mathfrak{r}).$$

We call he resulting structure, for which we use the notation $\mathcal{G}_{\mathcal{R}}(\mathfrak{P}, \mathcal{E}^-, \mathcal{E}^+)$. an $\mathcal{R}$-graph.

We also recall the construction of a semigroup (with zero) $\mathcal{S}_{\mathcal{R}}(\mathfrak{P}, \mathcal{E}^-, \mathcal{E}^+)$ from an $\mathcal{R}$-graph $\mathcal{G}_{\mathcal{R}}(\mathfrak{P}, \mathcal{E}^-, \mathcal{E}^+)$ as described in [Kr4]. The semigroup $\mathcal{S}_{\mathcal{R}}(\mathfrak{P}, \mathcal{E}^-, \mathcal{E}^+)$ contains idempotents $\mathbf{1}_{\mathfrak{p}}, \mathfrak{p} \in \mathfrak{P}$, and has $\mathcal{E}$ as a generating set. Besides $\mathbf{1}_{\mathfrak{p}}^2 = \mathbf{1}_{\mathfrak{p}}, \mathfrak{p} \in \mathfrak{P}$, the defining relations are:

$$f^- g^+ = \mathbf{1}_{\mathfrak{q}}, \quad f^- \in \mathcal{E}^-(\mathfrak{q}, \mathfrak{r}), g^+ \in \mathcal{E}^+(\mathfrak{q}, \mathfrak{r}), (f^-, g^+) \in \mathcal{R}(\mathfrak{q}, \mathfrak{r}), \quad \mathfrak{q}, \mathfrak{r} \in \mathfrak{P},$$

and

$$\mathbf{1}_{\mathfrak{q}} e^- = e^- \mathbf{1}_{\mathfrak{r}} = e^-, \quad e^- \in \mathcal{E}^-(\mathfrak{q}, \mathfrak{r}),$$
$$\mathbf{1}_{\mathfrak{r}} e^+ = e^+ \mathbf{1}_{\mathfrak{q}} = e^+, \quad e^+ \in \mathcal{E}^+(\mathfrak{q}, \mathfrak{r}), \quad \mathfrak{q}, \mathfrak{r} \in \mathfrak{P},$$

$$f^- g^+ = \begin{cases} \mathbf{1}_{\mathfrak{q}}, & \text{if } (f^-, g^+) \in \mathcal{R}(\mathfrak{q}, \mathfrak{r}), \\ 0, & \text{if } (f^-, g^+) \notin \mathcal{R}(\mathfrak{q}, \mathfrak{r}), \end{cases} \quad f^- \in \mathcal{E}^-(\mathfrak{q}, \mathfrak{r}), g^+ \in \mathcal{E}^+(\mathfrak{q}, \mathfrak{r}), \mathfrak{q}, \mathfrak{r} \in \mathfrak{P},$$

and

$$\mathbf{1}_{\mathfrak{q}} \mathbf{1}_{\mathfrak{r}} = 0, \quad \mathfrak{q}, \mathfrak{r} \in \mathfrak{P}, \mathfrak{q} \neq \mathfrak{r}.$$

We call $\mathcal{S}_{\mathcal{R}}(\mathfrak{P}, \mathcal{E}^-, \mathcal{E}^+)$ an $\mathcal{R}$-graph semigroup.

Special cases are the graph inverse semigroups of finite directed graphs $\mathcal{G} = (\mathfrak{P}, \mathcal{E}_\circ)$ ([AH],[L, Section 10.7.]). With the edge set $\mathcal{E}_\circ^- = \{e_\circ^- : e_\circ \in \mathcal{E}_\circ\}$ of a copy of $(\mathfrak{P}, \mathcal{E}_\circ)$, and with the edge set $\mathcal{E}^+ = \{e_\circ^- : e_\circ \in \mathcal{E}_\circ\}$ of the reversal of $(\mathfrak{P}, \mathcal{E}_\circ)$, the graph inverse semigroup $\mathcal{S}_{\mathcal{G}}$ of $(\mathfrak{P}, \mathcal{E}_\circ)$ is the $\mathcal{R}$-graph semigroup of the partitioned graph $(\mathfrak{P}, \mathcal{E}_\circ^-, \mathcal{E}_\circ^+)$ with the relations

$$\mathcal{R}(\mathfrak{q}, \mathfrak{r}) = \{(e_\circ^-, e_\circ^+) : e_\circ \in \mathcal{E}_\circ, s(e_\circ) = \mathfrak{q}, t(e_\circ) = \mathfrak{r}\}, \quad \mathfrak{q}, \mathfrak{r} \in \mathfrak{P}.$$

For the $\mathcal{R}$-graph semigroups with underlying graph a one-vertex graph see also [HK, Section 4].

In [HI] a criterion was given for the existence of an embedding of an irreducible subshift of finite type into a Dyck shift and this result was extended in [HIK] to a larger class of target shifts with Property (A). These target shifts were constructed by a method that presents the subshifts by means of a suitably structured irreducible finite labeled directed graph with labels taken from the inverse semigroup of an irreducible finite directed graph, in which every vertex has at least two incoming edges. This method was extended in [Kr4] by the use of $\mathcal{R}$-graph semigroups. Following [HIK, Kr4] we describe this construction. Given an $\mathcal{R}$-graph semigroup $\mathcal{S}_{\mathcal{R}}(\mathfrak{P}, \mathcal{E}^-, \mathcal{E}^+)$, denote by $\mathcal{S}_{\mathcal{R}}^-(\mathfrak{P}, \mathcal{E}^-)(\mathcal{S}_{\mathcal{R}}^+(\mathfrak{P}, \mathcal{E}^+))$ the subset of $\mathcal{S}_{\mathcal{R}}(\mathfrak{P}, \mathcal{E}^-, \mathcal{E}^+)$ that contains the non-zero elements of the subsemigroup of $\mathcal{S}_{\mathcal{R}}(\mathfrak{P}, \mathcal{E}^-, \mathcal{E}^+)$ that is generated by $\mathcal{E}^-(\mathcal{E}^+)$. Let there be given a finite strongly connected labeled directed graph with vertex set $\mathcal{V}$ and edge set Σ, and a labeling map λ that assign to every edge $\sigma \in \Sigma$ a label

(G 1) $$\lambda(\sigma) \in S_{\mathcal{R}}^-(\mathfrak{P}, \mathcal{E}^-) \cup \{\mathbf{1}_{\mathfrak{p}}, \mathfrak{p} \in \mathfrak{P}\} \cup S_{\mathcal{R}}^+(\mathfrak{P}, \mathcal{E}^+).$$

The label map λ extends to finite paths $(\sigma_i)_{1 \leq i \leq I}$ in the graph $(\mathcal{V}, \Sigma)$ by

$$\lambda((\sigma_i)_{1 \leq i \leq I}) = \prod_{1 \leq i \leq I} \lambda(\sigma_i).$$

Denoting for $\mathfrak{p} \in \mathfrak{P}$ by $\mathcal{V}(\mathfrak{p})$ the set of $V \in \mathcal{V}$ such that there is a cycle $(\sigma_i)_{1 \leq i \leq I}, I \in \mathbb{N}$, in the graph $(\mathcal{V}, \Sigma)$ from V to V such that

$$\lambda((\sigma_i)_{1 \leq i \leq I}) = \mathbf{1}_{\mathfrak{p}},$$

we require the following conditions (G 2 - 5) to be satisfied:

(G 2) $$\mathcal{V}(\mathfrak{p}) \neq \emptyset, \quad \mathfrak{p} \in \mathfrak{P},$$

(G 3) $$\{\mathcal{V}(\mathfrak{p}) : \mathfrak{p} \in \mathfrak{P}\} \text{ is a partition of } \mathcal{V},$$

(G 4) For $V \in \mathcal{V}(\mathfrak{p}), \mathfrak{p} \in \mathfrak{P}$, and for all edges σ that leave $V, \mathbf{1}_{\mathfrak{p}}\lambda(\sigma) \neq 0$, and for all edges σ that enter $V, \lambda(\sigma)\mathbf{1}_{\mathfrak{p}} \neq 0$,

(G 5) For $f \in \mathcal{S}_{\mathcal{R}}(\mathfrak{P}, \mathcal{E}^-, \mathcal{E}^+), \mathfrak{q}, \mathfrak{r} \in \mathfrak{P}$, such that $\mathbf{1}_{\mathfrak{q}} f \mathbf{1}_{\mathfrak{r}} \neq 0$, and for $U \in \mathcal{V}(\mathfrak{q}), W \in \mathcal{V}(\mathfrak{r})$, there exists a path b in the labeled directed graph $(\mathcal{V}, \Sigma, \lambda)$ from U to W such that $\lambda(b) = f$.

A finite labeled directed graph $(\mathcal{V}, \Sigma, \lambda)$, that satisfies conditions (G 1 - 5), gives rise to a subshift $X(\mathfrak{P}, \Sigma, \lambda)$ that has as its language of admissible words the set of finite paths b in the graph $(\mathcal{V}, \Sigma, \lambda)$ such that $\lambda(b) \neq 0$. We call the subshift $X(\mathfrak{P}, \Sigma, \lambda)$ an $\mathcal{S}_{\mathcal{R}}(\mathfrak{P}, \mathcal{E}^-, \mathcal{E}^+)$-presentation. Using the identity map $\mathrm{id}_{\mathcal{E}^- \cup \mathcal{E}^+}$ on $\mathcal{E}^- \cup \mathcal{E}^+ \subset \mathcal{S}_{\mathcal{R}}(\mathfrak{P}, \mathcal{E}^-, \mathcal{E}^+)$ as label map, one obtains particular cases of $\mathcal{S}_{\mathcal{R}}(\mathfrak{P}, \mathcal{E}^-, \mathcal{E}^+)$-presentations that we denote by $X(\mathcal{E}^- \cup \mathcal{E}^+, \mathfrak{P}, \mathrm{id}_{\mathcal{E}^- \cup \mathcal{E}^+})$. In the case of the inverse semigroups $\mathcal{S}_{\mathcal{G}}$ of strongly connected finite directed graphs $\mathcal{G}$ these $\mathcal{S}_{\mathcal{G}}$-presentations are Markov-Dyck shifts [M]. Also the Markov-Motzkin shifts [KM] of strongly connected finite directed graphs $\mathcal{G}$ are $\mathcal{S}_{\mathcal{G}}$-presentations.

Given finite sets $\mathcal{E}^-$ and $\mathcal{E}^+$ and a relation $\mathcal{R} \subset \mathcal{E}^- \times \mathcal{E}^+$, we set

$$\mathcal{E}^-(\mathcal{R}) = \{e^- \in \mathcal{E}^- : \{e^-\} \times \mathcal{E}^+ \subset \mathcal{R}\}, \quad \mathcal{E}^+(\mathcal{R}) = \{e^+ \in \mathcal{E}^+ : \mathcal{E}^- \times \{e^+\} \subset \mathcal{R}\}.$$

For a partitioned directed graph $(\mathfrak{P}, \mathcal{E}^-, \mathcal{E}^+)$ denote by $\mathfrak{P}^{(1)}$ the set of vertices in $\mathfrak{P}$ that have a single predecessor vertex in $\mathcal{E}^-$, or, equivalently, that have a single successor vertex in $\mathcal{E}^+$. For $\mathfrak{p} \in \mathfrak{P}^{(1)}$ the predecessor vertex of $\mathfrak{p}$ in $\mathcal{E}^-$, which is identical to the successor vertex of $\mathfrak{p}$ in $\mathcal{E}^+$, will be denoted by $\eta(\mathfrak{p})$. For an $\mathcal{R}$-graph $\mathcal{G}_{\mathcal{R}}(\mathfrak{P}, \mathcal{E}^-, \mathcal{E}^+)$ we set

$$\mathcal{E}_{\mathcal{R}}^- = \bigcup_{\mathfrak{p} \in \mathfrak{P}^{(1)}} \mathcal{E}^-(\mathcal{R}(\eta(\mathfrak{p}), \mathfrak{p})), \quad \mathcal{E}_{\mathcal{R}}^+ = \bigcup_{\mathfrak{p} \in \mathfrak{P}^{(1)}} \mathcal{E}^+(\mathcal{R}(\eta(\mathfrak{p}), \mathfrak{p})),$$

and

$$\mathfrak{P}_{\mathcal{R}}^{(1)} = \{\mathfrak{p} \in \mathfrak{P}^{(1)} : \mathcal{R}(\eta(\mathfrak{p}), \mathfrak{p}) = \mathcal{E}^-(\eta(\mathfrak{p}), \mathfrak{p}) \times \mathcal{E}^+(\eta(\mathfrak{p}), \mathfrak{p})\}.$$

We formulate three conditions (I), (II) and (III) on $\mathcal{R}$-graphs $\mathcal{G}_{\mathcal{R}}(\mathfrak{P}, \mathcal{E}^-, \mathcal{E}^+)$. Condition (II) comes in two parts (II$-$) and (II$+$) that are symmetric to one another:

(I) For $\mathfrak{p} \in \mathfrak{P}^{(1)} \setminus \mathfrak{P}_{\mathcal{R}}^{(1)}$, $\mathcal{E}^-(\mathcal{R}(\eta(\mathfrak{p}), \mathfrak{p})) = \emptyset$, or $\mathcal{E}^+(\mathcal{R}(\eta(\mathfrak{p}), \mathfrak{p})) = \emptyset$.

(II$-$) There is no cycle in the directed graph $(\mathfrak{P}, \mathcal{E}^-)$ that contains only edges in $\mathcal{E}_{\mathcal{R}}^-$ and that contains at least one edge $e^- \in \mathcal{E}_{\mathcal{R}}^-$ such that $t(e^-) \in \mathfrak{P}^{(1)} \setminus \mathfrak{P}_{\mathcal{R}}^{(1)}$.

(II$+$) There is no cycle in the directed graph $(\mathfrak{P}, \mathcal{E}^+)$ that contains only edges in $\mathcal{E}_{\mathcal{R}}^+$ and that contains at least one edge $e^+ \in \mathcal{E}_{\mathcal{R}}^+$ such that $s(e^+) \in \mathfrak{P}^{(1)} \setminus \mathfrak{P}_{\mathcal{R}}^{(1)}$.

(III) For $\mathfrak{q}, \mathfrak{r} \in \mathfrak{P}^{(1)}$, $\mathfrak{q} \neq \mathfrak{r}$, there does not simultaneously exist a path f^- in $\mathcal{E}_{\mathcal{R}}^-$ from $\mathfrak{q}$ to $\mathfrak{r}$ and a path f^+ in $\mathcal{E}_{\mathcal{R}}^+$ from $\mathfrak{q}$ to $\mathfrak{r}$, such that there is at least one edge e^- in f^- such that $t(e^-) \in \mathfrak{P}^{(1)} \setminus \mathfrak{P}_{\mathcal{R}}^{(1)}$, or there is at least one edge e^+ in f^+ such that $s(e^+) \in \mathfrak{P}^{(1)} \setminus \mathfrak{P}_{\mathcal{R}}^{(1)}$.

We show in Section 2 that an $\mathcal{S}_{\mathcal{R}}(\mathfrak{P}, \mathcal{E}^-, \mathcal{E}^+)$-presentation has Property (A) if and only if the $\mathcal{R}$-graph $\mathcal{G}_{\mathcal{R}}(\mathfrak{P}, \mathcal{E}^-, \mathcal{E}^+)$ satisfies Conditions (I), (II) and (III). In

particular the $\mathcal{S}_{\mathcal{R}}(\mathfrak{P}, \mathcal{E}^-, \mathcal{E}^+)$-presentations $X(\mathcal{E}^- \cup \mathcal{E}^+, \mathrm{id}_{\mathcal{S}_{\mathcal{R}(\mathfrak{P}, \mathcal{E}^-, \mathcal{E}^+)}})$ have Property (A) if and only if the $\mathcal{R}$-graph $\mathcal{G}_{\mathcal{R}}(\mathfrak{P}, \mathcal{E}^-, \mathcal{E}^+)$ satisfies Conditions (I), (II) and (III). This implies that Markov-Dyck shifts of strongly connected finite directed graphs have Property (A). Also the Markov-Motzkin shifts of strongly connected finite directed graphs have Property (A).

In Section 3 we describe how one can obtain from an $\mathcal{R}$-graph $\mathcal{G}_{\mathcal{R}}(\mathfrak{P}, \mathcal{E}^-, \mathcal{E}^+)$, that satisfies conditions (I), (II) and (III), an $\mathcal{R}$-graph $\mathcal{G}_{\widehat{\mathcal{R}}}(\mathfrak{P}, \widehat{\mathcal{E}}^-, \widehat{\mathcal{E}}^+)$ such that the $\mathcal{R}$-graph semigroup $\mathcal{S}_{\widehat{\mathcal{R}}}(\mathfrak{P}, \widehat{\mathcal{E}}^-, \widehat{\mathcal{E}}^+)$ is associated to any $\mathcal{S}_{\mathcal{R}}(\mathfrak{P}, \mathcal{E}^-, \mathcal{E}^+)$-presentation. The procedure first produces an intermediate directed graph by identifying edges in the directed graph $\mathcal{G}_{\mathcal{R}}(\mathfrak{P}, \mathcal{E}^-, \mathcal{E}^+)$, and then contracts the rooted subtrees of the intermediate directed graph to their roots. Here $\mathcal{S}_{\widehat{\mathcal{R}}}(\mathfrak{P}, \widehat{\mathcal{E}}^-, \widehat{\mathcal{E}}^+)$ is a homomorphic image of $\mathcal{S}_{\mathcal{R}}(\mathfrak{P}, \mathcal{E}^-, \mathcal{E}^+)$.

In Section 4 we look at examples of $\mathcal{R}$-graphs with two or three vertices, that give rise to a subshift whose associated semigroup is the semigroup of an $\mathcal{R}$-graph with one vertex.

In Section 5 we consider the Markov-Dyck shifts of directed graphs with three vertices whose associated semigroup is the graph inverse semigroup of a directed graph with two or three vertices. Given finite sets $\mathcal{E}^-$ and $\mathcal{E}^+$ and a relation $\mathcal{R} \subset \mathcal{E}^- \times \mathcal{E}^+$, we set

$$\Omega_{\mathcal{R}}^+(e^-) = \{e^+ \in \mathcal{E}^+ : (e^-, e^+) \in \mathcal{R}\}, \quad e^- \in \mathcal{E}^-,$$
$$\Omega_{\mathcal{R}}^-(e^+) = \{e^- \in \mathcal{E}^- : (e^-, e^+) \in \mathcal{R}\}, \quad e^+ \in \mathcal{E}^+.$$

In Section 6 we prove that an $\mathcal{R}$-graph semigroup $\mathcal{S}_{\mathcal{R}}(\mathfrak{P}, \mathcal{E}^-, \mathcal{E}^-)$ is associated to a topologically transitive subshift with Property (A) if and only if the $\mathcal{R}$-graph $\mathcal{G}_{\mathcal{R}}(\mathfrak{P}, \mathcal{E}^-, \mathcal{E}^-)$ satisfies the following conditions $(a), (b), (c)$ and (d). Condition (a) and (b) come in two symmetric parts parts, $(a-)$ and $(a+)$, and also Condition (b) comes in two symmetric parts parts, $(b-)$ and $(b+)$:

$(a-)$ $\quad \Omega_{\mathcal{R}(\mathfrak{q}, \mathfrak{r})}^+(e^-) \neq \Omega^+(\tilde{e}^-), \qquad e^-, \tilde{e}^- \in \mathcal{E}^-(\mathfrak{q}, \mathfrak{r}), e^- \neq \tilde{e}^-, \qquad \mathfrak{q}, \mathfrak{r} \in \mathfrak{P}.$

$(a+)$ $\quad \Omega_{\mathcal{R}(\mathfrak{q}, \mathfrak{r})}^-(e^+) \neq \Omega^-(\tilde{e}^+), \qquad e^+, \tilde{e}^+ \in \mathcal{E}^+(\mathfrak{r}, \mathfrak{q}), e^+ \neq \tilde{e}^+, \qquad \mathfrak{q}, \mathfrak{r} \in \mathfrak{P}.$

$(b-)$ $\quad$ There is no non-empty cycle in $\mathcal{E}_{\mathcal{R}}^-$.

$(b+)$ $\quad$ There is no non-empty cycle in $\mathcal{E}_{\mathcal{R}}^+$.

(c) $\quad$ For $\mathfrak{p} \in \mathfrak{P}^{(1)}$ such that $\eta(\mathfrak{p}) \neq \mathfrak{p}$, $\mathcal{E}^-(\mathcal{R}(\eta(\mathfrak{p}), \mathfrak{p})) = \emptyset$, or $\mathcal{E}^-(\mathcal{R}(\eta(\mathfrak{p}), \mathfrak{p})) = \emptyset$.

(d) $\quad$ For $\mathfrak{q}, \mathfrak{r} \in \mathfrak{P}^{(1)}$, $\mathfrak{q} \neq \mathfrak{r}$, there do not simultaneously exist a path in $\mathcal{E}_{\mathcal{R}}^-$ from $\mathfrak{q}$ to $\mathfrak{r}$ and a path in $\mathcal{E}_{\mathcal{R}}^+$ from $\mathfrak{q}$ to $\mathfrak{r}$.

2. $\mathcal{S}_{\mathcal{R}}(\mathfrak{P}, \mathcal{E}^-, \mathcal{E}^+)$-PRESENTATIONS

For a semigroup (with zero) $\mathcal{S}$, and for $F \in \mathcal{S}$ we set

$$\Gamma(F) = \{(G^-, G^+) \in \mathcal{S} \times \mathcal{S} : G^- F G^+ \neq 0\},$$

and we call $\Gamma(F)$ the context of F.

Lemma 2.1. *Let $\mathcal{G}_{\mathcal{R}}(\mathfrak{P}, \mathcal{E}^-, \mathcal{E}^+)$ be an $\mathcal{R}$-graph such that*

$$\mathfrak{P}^{(1)} \setminus \mathfrak{P}_{\mathcal{R}}^{(1)} \neq \emptyset.$$

Let $\mathfrak{p} \in \mathfrak{P}_{\mathcal{R}}^{(1)}$, and let $M \in \mathbb{N}$ be the maximal length of a path in $\mathcal{E}^-$ that ends at $\mathfrak{p}$ and that enters only vertices in $\mathfrak{P}_{\mathcal{R}}^{(1)}$. For an $m_\circ \in [1, M]$, let h^+ be a path in $\mathcal{E}^+$ of length $m_\circ$ that begins at $\mathfrak{p}$, and let h^- be a path in $\mathcal{E}^-$ of length $m_\circ$ that ends at

$\mathfrak{p}$. *Also let f^- be a path in $\mathcal{E}^-$ that ends at $\mathfrak{p}$ and f^+ be a path in $\mathcal{E}^+$ that begins at $\mathfrak{p}$. Then the context of $f^- f^+$ is equal to the context of $f^- h^+ h^- f^+$.*

Proof. The lemma reduces to the case that the paths f^- and f^+ are empty. What is to be proved is that the context of $h^+ h^-$ is equal to the context of $\mathbf{1}_{\mathfrak{p}}$. Let $\mathfrak{p}_m, 1 \leq m \leq M$, be the vertices that are entered by a path in $\mathcal{E}^+$ of length M that begins at $\mathfrak{p}$. The proof is based on the observation that for a path g^- in $\mathcal{E}^-$ of length $m_\circ$ that begins at $\mathfrak{p}$, and a path g^+ in $\mathcal{E}^+$ of length $m_\circ$ that ends at $\mathfrak{p}$, one has

$$(2.1) \qquad g^- h^+ h^- g^+ = \mathbf{1}_{p_{m_\circ}} = g^- g^+.$$

For a path

$$g^- = (e_m^-)_{1 \leq m \leq m(-)}, \quad m(-) \in \mathbb{N},$$

in $\mathcal{E}^-$ that ends at $\mathfrak{p}$, and a path

$$g^+ = (e_m^+)_{1 \leq m \leq m(+)}, \quad m(+) \in \mathbb{N},$$

in $\mathcal{E}^+$ that begins at $\mathfrak{p}$, one has, in case that $m(-), m(+) > M$, that

$$g^- g^+ = \Big(\prod_{1 \leq m \leq m(-)-M} e_m^- \Big) \Big(\prod_{1 \leq m \leq m(+)} e_m^- \Big),$$

and by (2.1)

$$g^- h^+ h^- g^+ =$$
$$\Big(\prod_{1 \leq m \leq m(-)-M} e_m^- \Big) \Big(\prod_{1 \leq m \leq m(+)} e_m^- \Big) h^+ h^- \Big(\prod_{1 \leq m \leq m(-)-M} e_m^- \Big) \Big(\prod_{1 \leq m \leq m(+)} e_m^- \Big) =$$

$$\Big(\prod_{1 \leq m \leq m(+)} e_m^- \Big) \mathbf{1}_{\mathfrak{p}_{m_\circ}} \Big(\prod_{1 \leq m \leq m(-)-M} e_m^- \Big) = \Big(\prod_{1 \leq m \leq m(-)-M} e_m^- \Big) \Big(\prod_{1 \leq m \leq m(+)} e_m^- \Big).$$

Also one finds in the case that $m(-) \leq M$ or $m(+) \leq M$, that the pair (g^-, g^+) is in the context of $h^+ h^-$ and also in the context of $\mathbf{1}_{\mathfrak{p}}$. $\square$

Lemma 2.2. *Let $\mathcal{G}_{\mathcal{R}}(\mathfrak{P}, \mathcal{E}^-, \mathcal{E}^+)$ be an $\mathcal{R}$-graph that satisfies conditions (I), (II) and (III), such that*

$$(2.2) \qquad \mathfrak{P}^{(1)} \setminus \mathfrak{P}_{\mathcal{R}}^{(1)} \neq \emptyset,$$

and let $\mathfrak{q}, \mathfrak{r} \in \mathfrak{P}^{(1)}, \mathfrak{q} \neq \mathfrak{r}$. Then at most one of the following cases (A−), (A+) and (B) can occur

    (A+) There exists a path in $\mathcal{E}_{\mathcal{R}}^+$ from $\mathfrak{q}$ to $\mathfrak{r}$.
    (A−) There exists a path in $\mathcal{E}_{\mathcal{R}}^-$ from $\mathfrak{q}$ to $\mathfrak{r}$.
    (B) There exists a vertex $\mathfrak{p} \in \mathfrak{P}$ together with a path

$$g^+ = (e_{i_+}^+)_{1 \leq i_+ \leq I_+}, \quad I_+ \in \mathbb{N},$$

in $\mathcal{E}_{\mathcal{R}}^+$ from $\mathfrak{q}$ to $\mathfrak{p}$, and a path

$$g^- = (e_{i_+}^-)_{1 \leq i_+ \leq I_+}, \quad I_+ \in \mathbb{N},$$

in $\mathcal{E}_{\mathcal{R}}^-$ from $\mathfrak{p}$ to $\mathfrak{r}$, such that

$$(2.3) \qquad s(e_{I_+}^+) \neq t(e_1^-).$$

    In case (B) the vertex $\mathfrak{p}$ is uniquely determined by the vertices $\mathfrak{q}$ and $\mathfrak{r}$.

Proof. That (A−) and (A+) cannot occur simultaneously is Condition (III).

We prove that (B) cannot occur simultaneously with (A+). For this we note first, that, as a consequence of (2.2), there cannot simultaneously exist a path

$$f^- = (e_{i_-}^-)_{1 \le i_- \le I_-}, \quad I_- \in \mathbb{N},$$

in $\mathcal{E}_\mathcal{R}^-$ from $\mathfrak{q}$ to $\mathfrak{r}$ such that

$$t(e_{i_-}^-) \in \mathfrak{P}_\mathcal{R}^{(1)}, \quad 1 \le i_- \le I_-,$$

and a path

$$f^+ = (e_{i_+}^+)_{1 \le i_+ \le I_+}, \quad I_+ \in \mathbb{N},$$

in $\mathcal{E}_\mathcal{R}^+$ from $\mathfrak{q}$ to $\mathfrak{r}$, such that

$$s(e_{i_+}^+) \in \mathfrak{P}_\mathcal{R}^{(1)}, \quad 1 \le i_+ \le I_+.$$

For, in this case one could choose edges $\widetilde{e}_{i_-}^+ \in \mathcal{E}_\mathcal{R}^+(s(e_{i_-}^-), t(e_{i_-}^-)), I_- \ge i_- \ge 1$, to obtain a path

$$\widetilde{f}^+ = (\widetilde{e}_{i_+}^+)_{I_- \ge i_- \ge 1},$$

in $\mathcal{E}_\mathcal{R}^+$ from $\mathfrak{r}$ to $\mathfrak{q}$, in this way producing a cycle in $\mathcal{E}_\mathcal{R}^+$, all of whose vertices are in $\mathfrak{P}_\mathcal{R}^{(1)}$, contradicting (2.3) by the irreducibility assumption on the partitioned graph $(\mathfrak{P}, \mathcal{E}^-, \mathcal{E}^+)$.

As a consequence of Condition (II) one can consider the path

$$h^+ = (e_{i_+}^+)_{1 \le i_+ \le H_+},$$

of maximal length H_+ in $\mathcal{E}_\mathcal{R}^+(\mathcal{E}^-)$ that starts at $\mathfrak{q}$ and also the path

$$h^- = (e_{i_-}^+)_{1 \le i_- \le H_-},$$

of maximal length H_- in $\mathcal{E}_\mathcal{R}^-(\mathcal{E}^+)$ that ends at $\mathfrak{r}$. The vertex set

$$\{t(e_{i_+}^+) : 1 \le i_+ \le H_+\}$$

is uniquely determined by $\mathfrak{q}$ and the vertex set

$$\{s(e_{i_-}^-) : 1 \le i_- \le H_-\}$$

is uniquely determined by $\mathfrak{r}$. The case (B) occurs if and only if these two vertex sets intersect. We assume that this is the case and we observe that there is then a unique vertex $\mathfrak{p}$ in the intersection such that, with the indices $i_+^\circ \in [1, H_+]$ and $i_-^\circ \in [1, H_-]$ given by

$$\mathfrak{p} = t(e_{i_+^\circ}^+) = s(e_{i_-^\circ}^+),$$

one has that

$$(2.4) \qquad\qquad\qquad s(e_{i_+^\circ}^+) \ne t(e_{i_-^\circ}^+).$$

Assume now that there is a path

$$f^+ = (e_{i_+}^+)_{1 \le i_+ \le I_+}, \quad I_+ \in \mathbb{N},$$

in $\mathcal{E}_\mathcal{R}^+(\mathcal{E}^-)$ from $\mathfrak{q}$ to $\mathfrak{r}$. $I_+ > i_+^\circ$ would violate Condition (III), since the paths

$$(e_{i_-}^+)_{i_+^\circ \le i_+ \le I_+},$$

and

$$(e_{i_-}^-)_{i_-^\circ \le i_- \le H_-},$$

would both start at $\mathfrak{p}$ and end at $\mathfrak{r}$. $I_+ = i_+^\circ$ would violate Condition (II−), and $I_+ < i^\circ$ is impossible by (2.4). It follows that (B) cannot occur simultaneously with (A+). The proof that (B) cannot occur simultaneously with (A−) is symmetric. $\square$

We introduce notation and terminology for subshifts. Given a subshift $X \subset \Sigma^{\mathbb{Z}}$ we set

$$x_{[i,k]} = (x_j)_{i \leq j \leq k}, \quad x \in X, i, k \in \mathbb{Z}, i \leq k,$$

and

$$X_{[i,k]} = \{x_{[i,k]} : x \in X\}, i, k \in \mathbb{Z}, i \leq k.$$

π_p denotes the period of a periodic point p of X.

We set

$$\Gamma(a) = \bigcup_{n,m \in \mathbb{N}} \{(b,c) \in X_{[i-n,i]} \times X_{[k,k+m]} : (b,a,c) \in X_{[i-n,k+m]}\},$$

$$a \in X_{[i,k]}, i, k \in \mathbb{Z}, i \leq k.$$

and call $\Gamma(a)$ the context of a. We set

$$\Gamma_n^+(a) = \{b \in X_{(k,k+n]} : (a,b) \in X_{[i,k+n]}\},$$

$$n \in \mathbb{N}, a \in X_{[i,k]}, i, k \in \mathbb{Z}, i \leq k.$$

Γ^- has the symmetric meaning. We set

$$\omega^+(a) = \bigcup_{n \in \mathbb{N}} \bigcap_{n \in \mathbb{N}} \bigcap_{c \in \Gamma_n^-(a)} \{b \in X_{(k,k+n]} : (c,a,b) \in X_{[i-n,k+m]}\},$$

$$a \in X_{[i,k]}, i, k \in \mathbb{Z}, i \leq k.$$

ω^- has the symmetric meaning.

Given a subshift $X \subset \Sigma^{\mathbb{Z}}$ we define a subshift of finite type (more precicely, an n-step Markov shift) $A_n(X)$ by

$$A_n(X) = \bigcap_{i \in \mathbb{Z}} (\{x \in X : x_i \in \omega^+(x_{[i-n,i)})\} \cap (\{x \in X : x_i \in \omega^-(x_{(i,i+n]})\}) \quad n \in \mathbb{N},$$

and we set

$$A(X) = \bigcup_{n \in \mathbb{N}} A_n(X).$$

We denote the set of periodic points in $A(X)$ by $P(A(X))$. The subshifts $X \subset \Sigma^{\mathbb{Z}}$: that we consider in this paper are such that $P(A(X))$ is dense in X. We introduce a preorder relation $\gtrsim$ into the set $P(A(X))$ where for $q, r \in P(A(X))$ means that there exists a point in $A(X)$ that is left asymptotic to the orbit of q and right asymptotic to the orbit of r. The equivalence relation on $P(A(X))$ that results from the preorder relation $\gtrsim$ we denote by $\approx$.

We recall from [Kr2] the definition of Property (A). For $n \in \mathbb{N}$ a subshift $X \subset \Sigma^{\mathbb{Z}}$, has property $(a, n, H), H \in \mathbb{N}$, if for $h, \widetilde{h} \geq 3H$ and for $I_-, I_+, \widetilde{I}_-, \widetilde{I}_+ \in \mathbb{Z}$, such that

$$I_+ - I_-, \widetilde{I}_+ - \widetilde{I}_- \geq 3H,$$

and for

$$a \in A_n(X)_{(I_-,I_+]}, \quad \widetilde{a} \in A_n(X)_{(\widetilde{I}_-,\widetilde{I}_+]},$$

such that

$$a_{(I_-,I_-+H]} = \tilde{a}_{(\widetilde{I}_-,\widetilde{I}_-+H]}, \quad a_{(I_+-H,I_+]} = \tilde{a}_{(\widetilde{I}_+-H,\widetilde{I}_+]},$$

one has that a and $\tilde{a}$ have the same context. A subshift $X \subset \Sigma^{\mathbb{Z}}$ has property (A) if there are $H_n, n \in \mathbb{N}$, such that X has the properties $(a, n, H_n), n \in \mathbb{N}$.

Theorem 2.3. *For a $\mathcal{R}$-graph $\mathcal{G}_{\mathcal{R}}(\mathfrak{P}, \mathcal{E}^-, \mathcal{E}^+)$, $\mathcal{S}_{\mathcal{R}}(\mathfrak{P}, \mathcal{E}^-, \mathcal{E}^+)$-presentations have property (A) if and only if $\mathcal{G}_{\mathcal{R}}(\mathfrak{P}, \mathcal{E}^-, \mathcal{E}^+)$ satisfies Conditions (I), (II) and (III).*

Proof. Assume that the $\mathcal{R}$-graph $\mathcal{G}_\mathcal{R}(\mathcal{P}, \mathcal{E}^-, \mathcal{E}^+)$ does not satisfy Condition (I). Then let

$$\mathfrak{p} \in \mathfrak{P}^{(1)} \setminus \mathfrak{P}_\mathcal{R}^{(1)}, \tag{2.5}$$

and let

$$e^- \in \mathcal{E}^-(\mathfrak{p}), \quad e^+ \in \mathcal{E}^+(\mathfrak{p}). \tag{2.6}$$

Choose vertices $U \in \mathcal{V}(\mathfrak{p}), W \in \mathcal{V}(\eta(\mathfrak{p}))$ and choose a cycle b in the directed graph $(\mathcal{V}, \Sigma)$ from U to U such that $\lambda(b) = \mathbf{1}_\mathfrak{p}$. It is

$$b^{2m} \in \mathcal{L}(A_n(X(\Sigma, \lambda))), \quad n \geq \ell(b), \qquad m \in \mathbb{N}. \tag{2.7}$$

Also choose a path b^- in the directed graph $(\mathcal{V}, \Sigma)$ from U to W such that $\lambda(b^-) = e^-$ and also a path b^+ from U to W such that $\lambda(b^+) = e^+$. By (2.6) and (2.7)

$$b^m b^- b^+ b^m \in \mathcal{L}(A_n(X(\Sigma, \lambda))), \quad n \geq \ell(b), \ell(b^-), \ell(b^+), \qquad m \in \mathbb{N}. \tag{2.8}$$

By (2.5) there are

$$\widetilde{e}^- \in \mathcal{E}^-(\eta(\mathfrak{p}), \mathfrak{p}), \quad \widetilde{e}^+ \in \mathcal{E}^+(\eta(\mathfrak{p}), \mathfrak{p}),$$

such that

$$\widetilde{e}^- \widetilde{e}^+ = 0. \tag{2.9}$$

Choose a path $\widetilde{b}^-$ in the directed graph $(\mathcal{V}, \Sigma)$ from U to W such that $\lambda(\widetilde{b}^-) = \widetilde{e}^-$ and a path $\widetilde{b}^+$ from V to U such that $\lambda(\widetilde{b}^+) = \widetilde{e}^+$. By (2.6) and (2.9)

$$\lambda(\widetilde{b}^- b^m b^- b_+ b^m \widetilde{b}^+) = \mathbf{1}_{\eta(\mathfrak{p})}, \quad \lambda(\widetilde{b}^- b^{2m} \widetilde{b}^+) = 0, \qquad m \in \mathbb{N},$$

which means that

$$\Gamma(b^m b_- b_+ b^m) \neq \Gamma(b^{2m}), \qquad m \in \mathbb{N},$$

and in view of (2.6) and (2.8) it follows that the subshift $X(\mathfrak{P}, \Sigma, \lambda)$ does not have Property (A).

Assume that the $\mathcal{R}$-graph $\mathcal{G}_\mathcal{R}(\mathcal{P}, \mathcal{E}^-, \mathcal{E}^+)$ does not satisfy Condition (II−), and let $(e_k^-)_{k \in \mathbb{Z}/K\mathbb{Z}}, K \in \mathbb{N}$, be a cycle in $\mathcal{E}^-$, where

$$e_k^- \in \mathcal{E}_\mathcal{R}^-, \quad k \in \mathbb{Z}/K\mathbb{Z}. \tag{2.10}$$

Set

$$\mathfrak{p}_k = t(e_k^-), \quad k \in \mathbb{Z}/K\mathbb{Z}.$$

We note that due to the irreducibility assumption on the partitioned directed graph $(\mathfrak{P}, \mathcal{E}^-, \mathcal{E}^+)$, one has that $\mathfrak{P} = \{\mathfrak{p}_k : k \in \mathbb{Z}/K\mathbb{Z}\}$. Also let

$$\mathfrak{p}_0 \in \mathfrak{P}^{(1)} \setminus \mathfrak{P}_\mathcal{R}^{(1)}. \tag{2.11}$$

Choose vertices

$$V_k \in \mathcal{V}(\mathfrak{p}_k), \quad k \in \mathbb{Z}/K\mathbb{Z}.$$

Choose a cycle b in the directed graph $(\mathcal{V}, \Sigma)$ from V_0 to V_0 such that $\lambda(b) = \mathbf{1}_{\mathfrak{p}_0}$. One has

$$b^{2m} \in \mathcal{L}(A_n(X(\Sigma, \lambda))), \quad n \geq \ell(b) \qquad m \in \mathbb{N}. \tag{2.12}$$

Choose paths b_k in the directed graph $(\mathcal{V}, \Sigma)$ from $V_{k-1(\text{mod } K)}$ to $V_k, k \in \mathbb{Z}/K\mathbb{Z}$, such that

$$\lambda(b_k) = e_k^-, \quad k \in \mathbb{Z}/K\mathbb{Z},$$

One has by (2.10) that

$$b^m ((b_k)_{0 \leq k < K}) b^m \in \mathcal{L}(A_n(X(\Sigma, \lambda))), \quad n \geq \ell(b_k), k \in \mathbb{Z}/K\mathbb{Z}, \tag{2.13}$$
$$n \geq \ell(b) \qquad m \in \mathbb{N}.$$

By (2.11) there are

$$\widetilde{e}^- \in \mathcal{E}^-(\mathcal{R}(\mathfrak{p}_{-1(\mathrm{mod}\,K)}, \mathfrak{p}_0)), \quad \widetilde{e}^+ \in \mathcal{E}^+(\mathcal{R}(\mathfrak{p}_{-1(\mathrm{mod}\,K)}, \mathfrak{p}_0)),$$

such that

$$(2.14) \qquad\qquad\qquad\qquad \widetilde{e}^-\widetilde{e}^+ = 0.$$

Choose a path $\widetilde{b}^-$ in the directed graph $(\mathcal{V}, \Sigma)$ from $V_{-1(\mathrm{mod}\,K)}$ to V_0 such that

$$\lambda(\widetilde{b}^-) = \widetilde{e}^-,$$

and choose also a path $\widetilde{b}^+$ from V_0 to $V_{-1(\mathrm{mod}\,K)}$ such that

$$\lambda(\widetilde{b}^+) = \widetilde{e}^+.$$

Then by (2.8) and (2.14)

$$\lambda(\widetilde{b}_-b^m((b_k)_{0\le k<K})b^m\widetilde{b}_+) = \mathbf{1}_{\mathfrak{p}_{-1(\mathrm{mod}\,K)}}, \quad \lambda(\widetilde{b}_-b^{2m}\widetilde{b}_+) = 0, \qquad m \in \mathbb{N},$$

which means that

$$\Gamma(b^m((b_k)_{0\le k<K})b^m) \ne \Gamma(b^{2m}), \qquad m \in \mathbb{N},$$

and in view of (2.12) and (2.14) it follows that the subshift $X(\mathfrak{P}, \Sigma, \lambda)$ does not have Property (A).

The proof, that the subshift $X(\mathfrak{P}, \Sigma, \lambda)$ does not have Property (A), if the $\mathcal{R}$-graph $\mathcal{G}_\mathcal{R}(\mathfrak{P}, \mathcal{E}^-, \mathcal{E}^+)$ does not satisfy Condition (II+), is symmetric.

Assume that the $\mathcal{R}$-graph $\mathcal{G}_\mathcal{R}(\mathfrak{P}, \mathcal{E}^-, \mathcal{E}^+)$ does not satisfy Condition (III), and let there be given $\mathfrak{q}, \mathfrak{r} \in \mathfrak{P}^{(1)}, \mathfrak{q} \ne \mathfrak{r}$, and a path $(e^-_{k_-})_{1\le k_-\le K_-}, K_- \in \mathbb{N}$, in $\mathcal{E}^-$ from $\mathfrak{q}$ to $\mathfrak{r}$, where

$$(2.15) \qquad\qquad\qquad e^-_{k_-} \in \mathcal{E}^-_\mathcal{R}, \quad 1 \le k_- \le K_-,$$

and assume for a $k^\circ_- \in [1, K_-]$ that

$$(2.16) \qquad\qquad\qquad \mathfrak{p}_{k^\circ_-} \in \mathfrak{P}^{(1)} \setminus \mathfrak{P}^{(1)}_\mathcal{R}.$$

Let there also be given a path $(e^+_{k_+})_{1\le k_+\le K_+}, K_+ \in \mathbb{N}$, in $\mathcal{E}^+$ from $\mathfrak{q}$ to $\mathfrak{r}$, where

$$(2.17) \qquad\qquad\qquad e^+_{k_+} \in \mathcal{E}^+_\mathcal{R}, \quad 1 \le k_+ \le K_+.$$

Set

$$\mathfrak{p}^-_0 = \mathfrak{p}^+_0 = \mathfrak{q}, \quad \mathfrak{p}^-_{K_-} = \mathfrak{p}^+_{K_+} = \mathfrak{r},$$

and

$$\mathfrak{p}^-_{k_-} = t(e^-_{k_-}), \quad 1 \le k_- < K_-,$$
$$\mathfrak{p}^+_{k_+} = t(e^+_{k_+}), \quad 1 \le k_+ < K_+.$$

We choose $\widetilde{e}^+_{k_-} \in \mathcal{E}^+(\mathfrak{p}^-_{k_--1}, \mathfrak{p}^-_{k_-}), 1 \le k_- \le K_-$, such that

$$e^-_{k_-}\widetilde{e}^+_{k_-} = \mathbf{1}_{k_--1}, \quad 1 \le k_- \le K_-,$$

and $\widetilde{e}^-_{k_+} \in \mathcal{E}^-(\mathfrak{p}^+_{k_+}, \mathfrak{p}^+_{k_+-1}), 1 \le k_+ \le K_+$. such that

$$\widetilde{e}^-_{k_+}e^+_{k_+} = \mathbf{1}_{k_+}, \quad 1 \le k_+ \le K_+.$$

By (2.16) there are $e^- \in \mathcal{E}^-(\mathfrak{p}^-_{k^\circ_--1}, \mathfrak{p}^-_{k^\circ_-}), e^+ \in \mathcal{E}^+(\mathfrak{p}^-_{k^\circ_--1}, \mathfrak{p}^-_{k^\circ_-})$ such that

$$(2.18) \qquad\qquad\qquad e^-e^+ = 0.$$

Choose vertices

$$V_\mathfrak{q} \in \mathcal{V}(\mathfrak{q}), \quad V_\mathfrak{r} \in \mathcal{V}(\mathfrak{r}),$$

and vertices

$$V_{\mathfrak{p}^-_{k_-}} \in \mathcal{V}(\mathfrak{p}^-_{k_-}), \quad 1 \le k_- < K_-,$$

and
$$V_{\mathfrak{p}_{k_+}^+} \in \mathcal{V}(\mathfrak{p}_{k_+}^+), \quad 1 \le k_+ < K_+.$$

Choose a cycle $b_{\mathfrak{q}}$ in the directed graph $(\mathcal{V}, \Sigma)$ from $V_{\mathfrak{q}}$ to $V_{\mathfrak{q}}$ such that $\lambda(b_{\mathfrak{q}}) = \mathbf{1}_{\mathfrak{q}}$, and also a cycle $b_{\mathfrak{q}}$ from $V_{\mathfrak{q}}$ to $V_{\mathfrak{q}}$ such that $\lambda(b_{\mathfrak{q}}) = \mathbf{1}_{\mathfrak{q}}$, Also choose paths $b_{k_-}^-, 1 \le k_- \le K_-$, in the directed graph $(\mathcal{V}, \Sigma)$ from $V_{\mathfrak{p}_{k_- -1}^-}$ to $V_{\mathfrak{p}_{k_-}^-}$ such that

$$\lambda(b_{k_-}^-) = e_{k_-}^-, \quad 1 \le k_- \le K_-,$$

and also paths $b_{k_+}^+, 1 \le k_+ \le K_+$, from $V_{\mathfrak{p}_{k_+ -1}^+}$ to $V_{\mathfrak{p}_{k_+}^+}$ such that

$$\lambda(b_{k_+}^+) = e_{k_+}^+, \quad 1 \le k_+ \le K_+$$

As a consequence of (2.15) and (2.17) one has

$$(2.19) \qquad b_{\mathfrak{q}}^m((b_{k_-}^-)_{1 \le k_- \le K_-})b_{\mathfrak{r}}^m \in \mathcal{L}(A_n(X(\Sigma, \lambda))), \quad n \ge \ell(b_{k_-}^-), 1 \le k_- \le K_-$$

$$(2.20) \qquad b_{\mathfrak{q}}^m((b_{k_+}^+)_{1 \le k_+ \le K_+})b_{\mathfrak{r}}^m \in \mathcal{L}(A_n(X(\Sigma, \lambda))), \quad n \ge \ell(b_{k_+}^+), 1 \le k_+ \le K_+$$

$$n \ge \ell(b_{\mathfrak{q}}), \ell(b_{\mathfrak{r}}), \quad m \in \mathbb{N}.$$

Choose paths $\widetilde{b}_{k_-}^+, 1 \le k_- \le K_-$, in the directed graph $(\mathcal{V}, \Sigma)$ from $V_{\mathfrak{p}_{k_-}}$ to $V_{\mathfrak{p}_{k_- -1}}$ such that

$$\lambda(\widetilde{b}_{k_-}^+) = \widetilde{e}_{k_-}^+, \quad 1 \le k_- \le K_-,$$

and also paths $\widetilde{b}_{k_+}^-, 1 \le k_+ \le K_+$, from $V_{\mathfrak{p}_{k_+}^+}$ to $V_{\mathfrak{p}_{k_+ -1}^+}$ such that

$$\lambda(\widetilde{b}_{k_+}^-) = \widetilde{e}_{k_+}^-, \quad 1 \le k_+ \le K_+.$$

One has

$$\lambda(b^-((b_{k_-}^-)_{k_{k_-}^\circ < k_- \le K_-})((\widetilde{b}_{k_+}^-)_{K_+ \ge k_+ \ge 1})$$

$$b_{\mathfrak{q}}^m((b_{k_-}^-)_{1 \le k_- \le K_-})b_{\mathfrak{r}}^m$$

$$((\widetilde{b}_{k_-}^+)_{K_- \ge k_- \ge k_-^\circ}), b^+) =$$

$$e^-(\prod_{k_{k_-}^\circ < k_- \le K_-} e_{k_-}^-))(\prod_{K_+ \ge k_+ \ge 1} \widetilde{e}_{k_+}^-))(\prod_{1 \le k_- \le k_-^\circ} e_{k_-}^-)), \quad m \in \mathbb{N}.$$

By (2.18)

$$\lambda(b^-((b_{k_-}^-)_{k_{k_-}^\circ < k_- \le K_-})((\widetilde{b}_{k_+}^-)_{K_+ \ge k_+ \ge 1})$$

$$b_{\mathfrak{q}}^m(b_{k_+}^+)_{1 \le k_+ \le K_+} b_{\mathfrak{r}}^m$$

$$((\widetilde{b}_{k_-}^+)_{K_- \ge k_- \ge k_-^\circ})b^+) = 0, \quad m \in \mathbb{N}.$$

One sees from this that

$$\Gamma(b_{\mathfrak{q}}^m((b_{k_-}^-)_{1 \le k_- \le K_-})b_{\mathfrak{r}}^m) \ne \Gamma(b_{\mathfrak{q}}^m(b_{k_+}^+)_{1 \le k_+ \le K_+}b_{\mathfrak{r}}^m), \quad m \in \mathbb{N},$$

and in view of (2.19) and (2.20) it follows that the subshift $X(\mathfrak{P}, \Sigma, \lambda)$ does not have Property (A).

The proof, that the subshift $X(\mathfrak{P}, \Sigma, \lambda)$ does not have Property (A), if, with a $k_+^\circ \in [1, K_+]$, the assumption (12) is replaced by

$$\mathfrak{P}_{k_+^\circ}^+ \in \mathfrak{P}^{(1)} \setminus \mathfrak{P}_{\mathcal{R}}^{(1)},$$

is symmetric.

For the converse consider an $\mathcal{R}$-graph $\mathcal{G}_{\mathcal{R}}(\mathfrak{P}, \mathcal{E}^-, \mathcal{E}^+)$ that satisfies conditions (I) and (III) and that has no cycle in $\mathcal{E}_{\mathcal{R}}^+$ nor in $\mathcal{E}_{\mathcal{R}}^-$. For a given $\mathcal{S}_{\mathcal{R}}(\mathcal{P}, \mathcal{E}^-, \mathcal{E}^+)$-presentation $X(\mathfrak{P}, \Sigma, \lambda)$ we determine for $n \in \mathbb{N}$ and $Q_-, Q^+ \in \mathbb{Z}, Q_+ - Q_- > 2n$, the blocks

$$(\sigma_q)_{Q_- < q \leq Q_+} \in X(\Sigma, \lambda)_{(Q_-, Q_+]}$$

such that

(2.19)
$$(\sigma_q)_{Q_- + n < q \leq Q_+} \in \omega^+((\sigma_q)_{Q_- < q \leq Q_- + n}),$$

and

(2.20)
$$(\sigma_q)_{Q_- < q \leq Q_+ - n} \in \omega^-((\sigma_q)_{Q_- - n < q \leq Q_+}).$$

One has $\mathfrak{p}(-), p(+) \in \mathfrak{P}$, $J_-(-), J_+(-), J_-(+), J_+(+) \in \mathbb{N}$, and

$$e_{j_+(-)}^+(-) \in \mathcal{E}^+, \quad J_+(-) \geq j_+(-) > 0,$$

$$e_{j_-(-)}^-(-) \in \mathcal{E}^-, \quad 0 < j_-(-) \leq J_-(-),$$

$$e_{j_+(+)}^+(+) \in \mathcal{E}^+, \quad J_+(+) \geq j_+(+) > 0,$$

$$e_{j_-(+)}^-(+) \in \mathcal{E}^-, \quad 0 < j_-(+) \leq J_-(+),$$

such that

$$\lambda((\sigma_q)_{Q_- < q \leq Q_- + n}) = (\prod_{J_+(-) \geq j_+(-) > 0} e_{j_+(-)}^+)\mathbf{1}_{\mathfrak{p}(-)}(\prod_{0 < j_-(-) \leq J_-(-)} e_{j_-(-)}^-),$$

$$\lambda((\sigma_q)_{Q_+ - n < q \leq Q_+}) = (\prod_{J_+(+) \geq j_+(+) > 0} e_{j_+(+)}^+)\mathbf{1}_{\mathfrak{p}(+)}(\prod_{0 < j_-(+) \leq J_-(+)} e_{j_-(+)}^-),$$

and one also has $\mathfrak{p} \in \mathfrak{P}$, $J_-, J_+ \in \mathbb{N}$, and

$$e_{j_+}^+ \in \mathcal{E}^+, \quad J_+ \geq j_+ > 0,$$

$$e_{j_-}^- \in \mathcal{E}^-, \quad 0 < j_- \leq J_-,$$

such that

$$\lambda((\sigma_q)_{Q_+ < q \leq Q_+}) = (\prod_{J_+(-) \geq j_+(-) > 0} e_{j_+(-)}^+)\mathbf{1}_{\mathfrak{p}(-)}$$

$$(\prod_{J_+ \geq j_+ > 0} e_{j_+}^+)\mathbf{1}_{\mathfrak{p}}(\prod_{0 < j_- \leq J_-} e_{j_-}^-)$$

$$\mathbf{1}_{\mathfrak{p}(+)}(\prod_{0 < j_-(+) \leq J_-(+)} e_{j_-(+)}^-).$$

As a consequence of (2.19)

$$\mathfrak{p}(-) \in \mathfrak{P}^{(1)},$$

and

$$e_{j_+}^+ \in \mathcal{E}_{\mathcal{R}}^+, \quad J_+ \geq j_+ > 0,$$

and a s a consequence of (2.20)

$$\mathfrak{p}(+) \in \mathfrak{P}^{(1)},$$

and

$$e_{j_-}^- \in \mathcal{E}_{\mathcal{R}}^-, \quad 0 < j_- \leq J_-,$$

There are the following cases (A), (B) and (C)

    (A) $J_- = 0, J_+ > 0$

    (B)$J_- > 0, J_+ = 0$

    (C), $J_- = J_+ = 0$ or $J_- > 0, J_+ = 0$.

Given that the $\mathcal{R}$-graph $\mathcal{G}_{\mathcal{R}}(\mathfrak{P}, \mathcal{E}^-, \mathcal{E}^+)$ satisfies Conditions (I), (II) and (III), we see from Lemma (2.2) that $\mathfrak{p}(-)$ and $\mathfrak{p}(+)$ determine which one of these cases

occurs, and from Lemma (2.1) one sees that the context of $(\sigma_q)_{Q_- < q \leq Q_+}$ is always determined by the pair

$$((\prod_{J_+(-) \geq j_+(-) > 0} e^+_{j_+(-)}) \mathbf{1}_{\mathfrak{p}(-)}, \mathbf{1}_{\mathfrak{p}(+)} (\prod_{0 < j_-(+) \leq J_-(+)} e^-_{j_-(+)})),$$

which means that this context is determined by the segments $(\sigma_q)_{Q_- < q \leq Q_- + n}$ and $(\sigma_q)_{Q_- - n < q \leq Q_+}$. We have shown that the subshift $X(\mathfrak{P}, \Sigma, \lambda)$ has Property (a, n, n).

To conclude the proof, observe that in the case that the $\mathcal{R}$- graph $\mathcal{G}_{\mathcal{R}}(\mathfrak{P}, \mathcal{E}^-, \mathcal{E}^+)$ is such that $\mathfrak{P} = \mathfrak{P}^{(1)}_{\mathcal{R}}$, the subshift $X(\mathfrak{P}, \Sigma, \lambda)$ is the edge shift of the directed graph $(\mathcal{V}, \Sigma)$, and as a topological Markov shift it has Property (A). □

3. The $\mathcal{R}$-graph semigroup associated to an $\mathcal{S}_{\mathcal{R}}(\mathcal{P}, \mathcal{E}^-, \mathcal{E}^+)$-presentation

Following the terminology that was introduced in [HI] we say for an $\mathcal{R}$-graph and an $\mathcal{S}_{\mathcal{R}}(\mathfrak{P}, \mathcal{E}^+, \mathcal{E}^-)$-presentation $X(\mathfrak{P}, \Sigma, \lambda)$ that a periodic point in $A(X(\Sigma, \lambda))$ is neutral if there exist $I \in \mathbb{Z}$ and $\mathfrak{p} \in \mathfrak{P}$ such that $\lambda(p_{[I, I + \pi_p)}) = \mathbf{1}_{\mathfrak{p}}$. and we say that a periodic point in $A(X(\Sigma, \lambda))$ has negative (positive) multiplier if there exist $I \in \mathbb{Z}$ and $\mathfrak{p} \in \mathfrak{P}$ such that $\lambda(p_{[I, I + \pi_p)}) \in \mathcal{S}^-_{\mathcal{R}}(\mathfrak{P}, \mathcal{E}^+, \mathcal{E}^-)(\mathcal{S}^+_{\mathcal{R}}(\mathfrak{P}, \mathcal{E}^+, \mathcal{E}^-))$.

Lemma 3.1. *Let $\mathcal{G}_{\mathcal{R}}(\mathfrak{P}, \mathcal{E}^-, \mathcal{E}^+)$ be an $\mathcal{R}$-graph, that satisfies Conditions (II) and such that*

$$(3.1) \qquad \mathfrak{P} \setminus \mathfrak{P}^{(1)}_{\mathcal{R}} \neq \emptyset,$$

and let $X(\mathfrak{P}, \Sigma, \lambda)$ be an $\mathcal{S}_{\mathcal{R}}(\mathfrak{P}, \mathcal{E}^-, \mathcal{E}^+)$-presentation. Then a periodic point of $X(\mathfrak{P}, \Sigma, \lambda)$ is in $A(X(\mathfrak{P}, \Sigma, \lambda))$ if and only if it is neutral.

Proof. Let

$$(3.2) \qquad p \in A(X(\mathfrak{P}, \Sigma, \lambda)).$$

and let $I \in \mathbb{Z}$ be such that

$$\lambda(p_{[I, I + \pi_p)}) \in \mathcal{S}^+_{\mathcal{R}}(\mathfrak{P}, \mathcal{E}^-, \mathcal{E}^+) \cup \{\mathbf{1}_{\mathfrak{p}} : \mathfrak{p} \in \mathfrak{P}\} \cup \mathcal{S}^-_{\mathcal{R}}(\mathfrak{P}, \mathcal{E}^-, \mathcal{E}^+).$$

If here

$$\lambda(p_{[I, I + \pi_p)}) \in \mathcal{S}^+_{\mathcal{R}}(\mathfrak{P}, \mathcal{E}^-, \mathcal{E}^+),$$

then it follows from (3.2) that $\lambda(p_{[I, I + \pi_p)})$ is given by a cycle in the directed graph $\mathcal{E}^-$ that goes from $\mathfrak{p}$ to in $\mathfrak{p}$, all of whose edges are in $\mathcal{E}^-_{\mathcal{R}}$, contradicting Condition (II-) and (3.1). For the case that

$$\lambda(p_{[I, I + \pi_p)}) \in \mathcal{S}^+_{\mathcal{R}}(\mathfrak{P}, \mathcal{E}^-, \mathcal{E}^+),$$

one has the symmetric argument.

For the converse, note that $\lambda(p_{[I, I + \pi_p)}) = \mathbf{1}_{\mathfrak{p}}$, implies $p \in A_{\pi_p}(X(\Sigma, \lambda))$. □

Given finite sets $\mathcal{E}^-$ and $\mathcal{E}^-$ and a relation $\mathcal{R} \subset \mathcal{E}^- \times \mathcal{E}^+$, we introduce an equivalence relation $\sim (\mathcal{R}, -)$ into $\mathcal{E}^-$, where $e^- \sim (\mathcal{R}, -)\widetilde{e}^-$ if and only if $\Omega^+_{\mathcal{R}}(e^-) = \Omega^+_{\mathcal{R}}(\widetilde{e}^-), e^-, \widetilde{e}^- \in \mathcal{E}^-$. An equivalence relation $\sim (\mathcal{R}, +)$ on $\mathcal{E}^+$ is defined symmetrically.

From an $\mathcal{R}$-graph $\mathcal{G}_{\mathcal{R}}(\mathfrak{P}, \mathcal{E}^-, \mathcal{E}^+)$ we derive an $\mathcal{R}$-graph $\mathcal{G}_{\widehat{\mathcal{R}}}(\mathfrak{P}, \widehat{\mathcal{E}}^-_{\mathcal{R}}, \widehat{\mathcal{E}}^+_{\mathcal{R}})$, by identifying edges that are $(\mathcal{R}, -)$-equivalent, that is, by setting

$$\widehat{\mathcal{E}}^-_{\mathcal{R}}(\mathfrak{q}, \mathfrak{r}) = [\mathcal{E}^-(\mathfrak{q}, \mathfrak{r})]_{\sim (\mathcal{R}, -)}, \quad \mathfrak{q}, \mathfrak{r} \in \mathfrak{P},$$

$$\widehat{\mathcal{E}}^-_{\mathcal{R}} = \bigcup_{\mathfrak{q}, \mathfrak{r} \in \mathfrak{P}} \widehat{\mathcal{E}}^-_{\mathcal{R}}(\mathfrak{q}, \mathfrak{r}),$$

and by identifying edges that are $\sim (\mathcal{R},+)$-equivalent, that is, by setting

$$\widehat{\mathcal{E}}_{\mathcal{R}}^+(\mathfrak{q},\mathfrak{r}) = [\mathcal{E}^+(\mathfrak{q},\mathfrak{r})]_{\sim(\mathcal{R},+)}, \quad \mathfrak{q},\mathfrak{r} \in \mathfrak{P},$$

$$\widehat{\mathcal{E}}_{\mathcal{R}}^+ = \bigcup_{\mathfrak{q},\mathfrak{r}\in\mathfrak{P}} \widehat{\mathcal{E}}_{\mathcal{R}}^+(\mathfrak{q},\mathfrak{r}),$$

setting

$$s([e^-]_{\sim(\mathcal{R},-)}) = s(e^-), \quad t([e^-]_{\sim(\mathcal{R},-)}) = t(e^-), \qquad e^- \in \mathcal{E}^-,$$

$$s([e^+]_{\sim(\mathcal{R},+)}) = s(e^+), \quad t([e^-]_{\sim(\mathcal{R},+)}) = t(e^-), \qquad e^+ \in \mathcal{E}^+,$$

$$\widehat{\mathcal{R}} = \{([e^-]_{\sim(\mathcal{R},-)}, [e^+]_{\sim(\mathcal{R},+)}) : (e^-, e^+) \in \mathcal{R}\}.$$

We note that $\mathfrak{P}_{\mathcal{R}}^{(1)}$ is the set of vertices in $\mathfrak{P}$ that have a single incoming edge in $\widehat{\mathcal{E}}_{\mathcal{R}}^+$, or, equivalently, that have a single outgoing edge in $\widehat{\mathcal{E}}_{\mathcal{R}}^+$, and we set

$$\widehat{\mathcal{E}}_{\mathcal{R}}^-(1) = \{\widehat{e}^- \in \widehat{\mathcal{E}}_{\mathcal{R}}^- : t(\widehat{e}^-) \in \mathfrak{P}_{\mathcal{R}}^{(1)}\}, \quad \widehat{\mathcal{E}}_{\mathcal{R}}^+(1) = \{\widehat{e}^+ \in \widehat{\mathcal{E}}_{\mathcal{R}}^+ : s(\widehat{e}^+) \in \mathfrak{P}_{\mathcal{R}}^{(1)}\}.$$

For an $\mathcal{R}$-graph $\mathcal{G}_{\mathcal{R}}(\mathfrak{P}, \mathcal{E}^+, \mathcal{E}^-)$ such that $\mathfrak{P} \setminus \mathfrak{P}_{\mathcal{R}}^{(1)} \neq \emptyset$ we denote for $\mathfrak{p} \in \mathfrak{P} \setminus \mathfrak{P}_{\mathcal{R}}^{(1)}$ by $\mathfrak{P}_{\mathfrak{p}}$ the set of target vertices of the paths (including the empty path) in $\widehat{\mathcal{E}}_{\mathcal{R}}^-(1)$, that have $\mathfrak{p}$ as source vertex (Symmetrically, $\mathfrak{P}_{\mathfrak{p}}$ can be defined as the set of source vertices of the paths (including the empty path) in $\widehat{\mathcal{E}}_{\mathcal{R}}^+(1)$, that have $\mathfrak{p}$ as target vertex). $\{\mathfrak{P}_{\mathfrak{p}} : \mathfrak{p} \in \mathfrak{P} \setminus \mathfrak{P}_{\mathcal{R}}^{(1)}\}$ is a partition of $\mathfrak{P}$. We note that the set $\mathfrak{P} \setminus \mathfrak{P}_{\mathcal{R}}^{(1)}$ is the set of roots of the subtrees (degenerate and non-degenerate) of the (possibly degenerate) directed graphs $(\mathfrak{P}, \widehat{\mathcal{E}}_{\mathcal{R}}^+(1))$ and $(\mathfrak{P}, \widehat{\mathcal{E}}_{\mathcal{R}}^-(1))$, that are reversals of one another, and that are the union of their sub-trees (degenerate and non-degenerate).

Theorem 3.2. *Let $\mathcal{G}_{\mathcal{R}}(\mathfrak{P}, \mathcal{E}^-, \mathcal{E}^+)$ be an $\mathcal{R}$-graph, that satisfies conditions (I). (II) and (III) and let*

$$\mathfrak{P} \setminus \mathfrak{P}_{\mathcal{R}}^{(1)} \neq \emptyset.$$

Let $X(\mathfrak{P}, \Sigma, \lambda)$ be an $\mathcal{S}_{\mathcal{R}}(\mathfrak{P}, \mathcal{E}^-, \mathcal{E}^+)$-presentation, let q, r be neutral periodic points of $X(\mathfrak{P}, \Sigma, \lambda)$, and let $\mathfrak{q}, \mathfrak{r} \in \mathfrak{P}$, and $I(q), I(r) \in \mathbb{Z}$ be such that

$$p_{(I(q),I(q)+\pi(q)]} = \mathbf{1}_{\mathfrak{q}}, \quad p_{(I(r),I(r)+\pi(r)]} = \mathbf{1}_{\mathfrak{r}}.$$

Then it is $q \approx r$ if and only if $\mathfrak{q}$ and $\mathfrak{r}$ are in the same element of the partition $\{\mathfrak{P}_{\mathfrak{p}} : \mathfrak{p} \in \mathfrak{P} \setminus \mathfrak{P}_{\mathcal{R}}^{(1)}\}$.

Proof. Assume that $p \approx q$. Let $N \in \mathbb{N}$ and

(3.3) $$x^{(q,r)}, x^{(r,q)} \in A_N(X(\Sigma, \lambda)),$$

and let

$$I_-(q,r), I_+(q,r), \quad I_-(r,q), I_+(r,q) \in \mathbb{Z},$$
$$I_-(q,r) < I_+(q,r), \quad I_-(r,q) < I_+(r,q),$$

be such that

$$x^{(q,r)}_{(-\infty,I_-(q,r)]} = q_{(-\infty,I(q)]}, \quad x^{(q,r)}_{(I_+(q,r),\infty)} = r_{(I(r),\infty)},$$

$$x^{(r,q)}_{(-\infty,I_-(r,q)]} = r_{(-\infty,I(r)]}, \quad x^{(r,q)}_{(I_+(r,q),\infty)} = q_{(I(q),\infty)}.$$

By (3.3)

(3.4) $$(x^{(q,r)}_{(-\infty,I_+(q,r)]}, r_{(I(r),I(r)+N\pi(r)]}, x^{(r,q)}_{(I_-(q,r),\infty)}) \in X(\Sigma, \lambda),$$

and

(3.5) $$(x^{(r,q)}_{(-\infty,I_+(r,q)]}, q_{(I(q),I(q)+N\pi(q)]}, x^{(q,r)}_{(I_-(r,q),\infty)}) \in X(\Sigma, \lambda).$$

As a consequence of (3.3) there are also

$$\mathfrak{p}(q,r) \in \mathfrak{P}, \quad J_-(q,r), J_+(q,r) \in \mathbb{N},$$

and

$$e^+_{j_+(q,r)}(q,r) \in \mathcal{E}^+_\mathcal{R}, \quad J_+(q,r) \geq j_+(q,r) > 0,$$

$$e^-_{j_-(q,r)}(q,r) \in \mathcal{E}^-_\mathcal{R}, \quad 0 < j_-(q,r) \leq J_-(q,r),$$

such that

$$(3.6) \quad \lambda(x^{(q,r)}_{(I_-(q.r),I_+(q,r)]}) =$$

$$\mathbf{1}_\mathfrak{q}(\prod_{J_+(q,r) \geq j_+(q,r) > 0} e^+_{j_+}(q,r)) \mathbf{1}_{\mathfrak{p}^{(q,r)}} (\prod_{0 < j_-(q,r) \leq J_-(q,r)} e^-_{j_-(q,r)}(q,r)) \mathbf{1}_\mathfrak{r},$$

and there are also

$$\mathfrak{p}(r,q) \in \mathfrak{P}, \quad J_-(r,q), J_+(r,q) \in \mathbb{N},$$

and

$$e^+_{j_+(r,q)}(q,r) \in \mathcal{E}^+_\mathcal{R}, \quad J_+(r,q) \geq j_+(r,q) > 0,$$

$$e^-_{j_-(r,q)}(q,r) \in \mathcal{E}^-_\mathcal{R}, \quad 0 < j_-(r,q) \leq J_-(r,q),$$

such that

$$(3.7) \quad \lambda(x^{(r,q)}_{(J_-(r,q),J_+(r,q)]}) =$$

$$\mathbf{1}_\mathfrak{r}(\prod_{J_+(r,q) \geq j_+(r,q) > 0} e^+_{j_+}(r,q)) \mathbf{1}_{\mathfrak{p}^{(r,q)}} (\prod_{0 < j_-(r,q) \leq J_-(r,q)} e^-_{j_-(r,q)}(r,q)) \mathbf{1}_\mathfrak{q}.$$

By (3.4), and in case that $J_+(q,r) \geq J_+(r,q)$

$$\mathbf{1}_\mathfrak{q}(\prod_{J_+(q,r) \geq j_+(q,r) > 0} e^+_{j_+}(q,r)) \mathbf{1}_{\mathfrak{p}^{(q,r)}} (\prod_{0 < j_-(q,r) \leq J_-(q,r)} e^-_{j_-(q,r)}(q,r)) \mathbf{1}_\mathfrak{r}$$

$$\mathbf{1}_\mathfrak{r}(\prod_{J_+(r,q) \geq j_+(r,q) > 0} e^+_{j_+}(r,q)) \mathbf{1}_{\mathfrak{p}^{(r,q)}} (\prod_{0 < j_-(r,q) \leq J_-(r,q)} e^-_{j_-(r,q)}(r,q)) \mathbf{1}_\mathfrak{q} =$$

$$\mathbf{1}_\mathfrak{q}(\prod_{J_+(q,r) \geq j_+(q,r) > 0} e^+_{j_+}(q,r)) \mathbf{1}_{\mathfrak{p}^{(q,r)}}$$

$$(\prod_{0 < j_-(q,r) \leq J_-(q,r) - J_+(r,q)} e^-_{j_-(q,r)}(q,r)) \mathbf{1}_{\mathfrak{p}^{(r,q)}} (\prod_{0 < j_-(r,q) \leq J_-(r,q)} e^-_{j_-(r,q)}(r,q)) \mathbf{1}_\mathfrak{q} \neq 0.$$

From this it follows by Condition (I) that

$$(\widehat{e}^+_{j_+}(q,r))_{J_+(q,r) \geq j_+(q,r) > 0}$$

is a path in $\widehat{\mathcal{E}}^+(1)$ from $\mathfrak{q}$ to $\mathfrak{p}^{(q,r)}$, and

$$((\widehat{e}^-)_{0 < j_-(q,r) \leq J_-(q,r) - J_+(r,q)}, (\widehat{e}^-)_{0 < j_-(r,q) \leq J_-(r,q)}))$$

is a path in $\widehat{\mathcal{E}}^-(1)$ from $\mathfrak{p}^{(q,r)}$ to $\mathfrak{r}$ that passes through $\mathfrak{p}^{(r,q)}$ and one sees that $\mathfrak{p}^{(q,r)}, \mathfrak{p}^{(r,q)}$ and $\mathfrak{q}$ are in the same element of the partition $\{\mathfrak{P}_\mathfrak{p} : \mathfrak{p} \in \mathfrak{P} \setminus \mathfrak{P}^{(1)}_\mathcal{R}\}$. By the same argument for the case that $J_+(q,r) \leq J_+(q,r)$, and by the symmetric argument that uses (3.7), one sees that in fact $\mathfrak{p}^{(q,r)}, \mathfrak{p}^{(r,q)}$ and $\mathfrak{q}$ and $\mathfrak{r}$ are in the same element of the partition $\{\mathfrak{P}_\mathfrak{p} : \mathfrak{p} \in \mathfrak{P} \setminus \mathfrak{P}^{(1)}_\mathcal{R}\}$.

For the proof of the converse let $p \in \mathfrak{P} \setminus \mathfrak{P}^{(1)}_\mathcal{R}$ and let $\mathfrak{q}, \mathfrak{r} \in \mathfrak{P}_\mathfrak{p}$. There is a path $(\widehat{e}^+_{j_+(q)})_{J_+(q) \geq j_+(q) > 0}$ in $\widehat{\mathcal{E}}^+(1)$ from $\mathfrak{q}$ to $\mathfrak{p}$, a path $(\widehat{e}^-_{j_-}(r))_{0 < j_-(r) \leq J_-(r)}$ in $\widehat{\mathcal{E}}^-(1)$ from $\mathfrak{p}$ to $\mathfrak{r}$, a path $(\widehat{e}^+_{j_+}(r))_{J_+(r) \geq j_+(r) > 0}$ in $\widehat{\mathcal{E}}^+(1)$ from $\mathfrak{r}$ to $\mathfrak{p}$, and a path $(\widehat{e}^-_{j_-}(q))_{0 < j_-(r) \leq J_-(q)}$ in $\widehat{\mathcal{E}}^-(1)$ from $\mathfrak{p}$ to $\mathfrak{q}$. Choose a vertex $V(p) \in \mathcal{V}(\mathfrak{p})$ and choose

$$e^+_{j_+(q)} \in \widehat{e}^+_{j_+(q)}, \quad J_+(q) \geq j_+(q) > 0$$

and choose a path $b^+(q)$ in the directed graph $(\mathcal{V}, \Sigma)$ from the target vertex of $q_{(-\infty, I(q)]}$ to $V(p)$, such that

$$\lambda(b^+(q)) = \prod_{J_+(q) \geq j_+(q) > 0} e^+_{j_+(q)},$$

and choose

$$e^+_{j_-(r)} \in \widehat{e}^+_{j_-(r)}, \quad 0 < j_-(r) \leq J_-(r)$$

and also a path $b^-(r)$ from $V(p)$ to the source vertex of $q_{(I(r), \infty)}$, such that

$$\lambda(b^-(r)) = \prod_{0 < j_-(r) \leq J_-(r)} e^-_{j_-}(r),$$

choose

$$e^+_{j_+(r)} \in \widehat{e}^+_{j_+(r)}, \quad J_+(q) \geq j_+(r) > 0$$

and choose a path $b^+(q)$ in the directed graph $(\mathcal{V}, \Sigma)$ from the target vertex of $q_{(-\infty, I(q)]}$ to $V(p)$, such that

$$\lambda(b^+(r)) = \prod_{J_+(r) \geq j_+(r) > 0} e^+_{j_+(r)},$$

and choose

$$e^-_{j_-(q)} \in \widehat{e^-_{j_-(q)}}, \quad 0 < j_-(q) \leq J_-(q)$$

and also a path $b^-(q)$ from $V(p)$ to the source vertex of $q_{(I(q), \infty)}$, such that

$$\lambda(b^-(q)) = \prod_{0 < j_-(q) \leq J_-(q)} e^-_{j_-}(q),$$

Then

$$(q_{(-\infty, I(q)]}, b^+(q), b^-(r), r_{(I(r), \infty)}) \in A(X(\Sigma, \lambda)),$$
$$(r_{(-\infty, I(r)]}, b^+(r), b^-(q), q_{(I(q), \infty)}) \in A(X(\Sigma, \lambda)). \quad \square$$

We recall at this point the construction of the associated semigroup. For a property (A) subshift $X \subset \Sigma^{\mathbb{Z}}$ we denote by $Y(X)$ the set of points in X that are left asymptotic to a point in $P(A(X))$ and also right-asymptotic to a point in $P(A(X))$. Let $y, \tilde{y} \in Y(X)$, let y be left asymptotic to $q \in P(A(X))$ and right asymptotic to $r \in P(A(X))$, and let $\tilde{y}$ be left asymptotic to $\tilde{q} \in P(A(X))$ and right asymptotic to $\tilde{r} \in P(A(X))$. Given that X has the properties $(a, n, H_n), n \in \mathbb{N}$, we say that y and $\tilde{y}$ are equivalent, $y \approx \tilde{y}$, if $q \approx \tilde{q}$ and $r \approx \tilde{r}$, and if for $n \in \mathbb{N}$ such that $q, r, \tilde{q}, \tilde{r} \in P(A_n(X))$ and for $I, J, \tilde{I}, \tilde{J} \in \mathbb{Z}, I < J, \tilde{I} < \tilde{J}$, such that

$$y_{(-\infty, I]} = q_{(-\infty, 0]}, \quad y_{(J, \infty)} = r_{(0, \infty)},$$
$$\tilde{y}_{(-\infty, \tilde{I}]} = \tilde{q}_{(-\infty, 0]}, \quad \tilde{y}_{(\tilde{J}, \infty)} = \tilde{r}_{(0, \infty)},$$

one has for $h \geq 3H_n$ and for

$$a \in X_{(I-h, J+h]}, \quad \tilde{a} \in X_{(\tilde{I}-h, \tilde{J}+h]},$$

such that

$$a_{(I-H_n, J+H_n]} = y_{(I-H_n, J+H_n]}, \quad \tilde{a}_{(\tilde{I}-H_n, \tilde{J}+H_n]} = \tilde{y}_{(\tilde{I}-H_n, \tilde{J}+H_n]},$$
$$a_{(I-h, I-h+H_n)} = \tilde{a}_{(\tilde{I}-h, \tilde{I}-h+H_n)},$$
$$a_{(J+h-H_n, J+h]} = \tilde{a}_{(\tilde{J}+h-H_n, \tilde{J}+h]},$$

and such that

$$a_{(I-h, I]} \in A_n(X)_{(I-h, I]}, \quad \tilde{a}_{(\tilde{J}-h, \tilde{I}]} \in A_n(X)_{(\tilde{J}-h, \tilde{I}]},$$

$$a_{(J,J+h]} \in A_n(X)_{(J,J+h]}, \quad \tilde{a}_{(\tilde{J},\tilde{J}+h]} \in A_n(X)_{(\tilde{J},\tilde{J}+h]},$$

that a and $\tilde{a}$ have the same context. To give $[Y(X)]_\approx$ the structure of a semigroup, let $u, v \in Y(X)$, let u be right asymptotic to $q \in P(A(X))$ and let v be left asymptotic to $r \in P(A(X))$. If here $q \gtrsim r$, then $[u]_\approx [v]_\approx$ is set equal to $[y]_\approx$ where y is any point in Y such that there are $n \in \mathbb{N}, I, J, \hat{I}, \hat{J} \in \mathbb{Z}, I < J, \hat{I} < \hat{J}$, such that $q, r \in A_n(X)$, and such that

$$u_{(I,\infty)} = q_{(I,\infty)}, \quad v_{(-\infty,J]} = r_{(-\infty,J]},$$

$$y_{(-\infty,\hat{I}+H_n]} = u_{(-\infty,I+H_n]}, \quad y_{(\hat{J}-H_n,\infty)} = v_{(J-H_n,\infty)},$$

and

$$y_{(\hat{I},\hat{J}]} \in A_n(X)_{(\hat{I},\hat{J}]},$$

provided that such a point y exists. If such a point y does not exist, $[u]_\approx [v]_\approx$ is set equal to zero. Also, in the case that one does not have $q \gtrsim r, [u]_\approx [v]_\approx$ is set equal to zero.

From an $\mathcal{R}$-graph $\mathcal{G}_\mathcal{R}(\mathfrak{P}, \mathcal{E}^+, \mathcal{E}^-)$ such that $\mathfrak{P} \setminus \mathfrak{P}_\mathcal{R}(1) \neq \emptyset$, setting

$$\widetilde{\mathfrak{P}} = \mathfrak{P} \setminus \mathfrak{P}_\mathcal{R}(1), \quad \widetilde{\mathcal{E}}^- = \widehat{\mathcal{E}}^- \setminus \widehat{\mathcal{E}}_\mathcal{R}^-(1), \quad \widetilde{\mathcal{E}}^+ = \widehat{\mathcal{E}}^+ \setminus \widehat{\mathcal{E}}_\mathcal{R}^+(1),$$

and

$$\widetilde{\mathcal{R}} = \widehat{\mathcal{R}} \upharpoonright (\widehat{\mathcal{E}}^- \setminus \widehat{\mathcal{E}}_\mathcal{R}^-(1)) \times (\widehat{\mathcal{E}}^- \setminus \widehat{\mathcal{E}}_\mathcal{R}^-(1)),$$

one obtains a $\mathcal{R}$-graph $\mathcal{G}_{\widetilde{\mathcal{R}}}(\widetilde{\mathfrak{P}}, \widetilde{\mathcal{E}}^-, \widetilde{\mathcal{E}}^+)$ with source and target maps

$$\widetilde{s} : \widetilde{\mathcal{E}}^- \cup \widetilde{\mathcal{E}}^+ \to \widetilde{\mathfrak{P}}, \quad \widetilde{t} : \widetilde{\mathcal{E}}^- \cup \widetilde{\mathcal{E}}^+ \to \widetilde{\mathfrak{P}},$$

given by

$$\widetilde{s}(\widehat{e^-}) = p, \qquad \widehat{e^-} \in \widetilde{\mathcal{E}}^-, p \in \widetilde{\mathfrak{P}}, s(\widehat{e^-}) \in \mathfrak{P}_p,$$
$$\widetilde{s}(\widehat{e^+}) = p, \qquad \widehat{e^+} \in \widetilde{\mathcal{E}}^+, p \in \widetilde{\mathfrak{P}}, s(\widehat{e^+}) \in \mathfrak{P}_p,$$
$$\widetilde{t}(\widehat{e^-}) = p, \qquad \widehat{e^-} \in \widetilde{\mathcal{E}}^-, p \in \widetilde{\mathfrak{P}}, t(\widehat{e^-}) \in \mathfrak{P}_p,$$
$$\widetilde{t}(\widehat{e^+}) = p, \qquad \widehat{e^+} \in \widetilde{\mathcal{E}}^+, p \in \widetilde{\mathfrak{P}}, t(\widehat{e^+}) \in \mathfrak{P}_p.$$

Theorem 3.3. *Let $\mathcal{G}_\mathcal{R}(\mathfrak{P}, \mathcal{E}^+, \mathcal{E}^-)$ be an $\mathcal{R}$-graph, such that*

$$\mathfrak{P} \setminus \mathfrak{P}_\mathcal{R}^{(1)} \neq \emptyset.$$

that satisfies conditions (I), (II) and (III), and let $X(\mathfrak{P}, \Sigma, \lambda)$ be an $\mathcal{S}_\mathcal{R}(\mathfrak{P}, \mathcal{E}^-, \mathcal{E}^+)$-presentation. Then the semigroup $\mathcal{S}_{\widetilde{\mathcal{R}}}(\widetilde{\mathfrak{P}}, \widetilde{\mathcal{E}}^-, \widetilde{\mathcal{E}}^+)$ is associated to $X(\mathfrak{P}, \Sigma, \lambda)$, and $X(\mathfrak{P}, \Sigma, \lambda)$ has an $\mathcal{S}_{\widetilde{\mathcal{R}}}(\widetilde{\mathfrak{P}}, \widetilde{\mathcal{E}}^-, \widetilde{\mathcal{E}}^+)$-presentation.

Proof. There is a homomorphism Ψ of $\mathcal{S}_\mathcal{R}(\mathfrak{P}, \mathcal{E}^-, \mathcal{E}^+)$ onto $\mathcal{S}_{\widetilde{\mathcal{R}}}(\widetilde{\mathfrak{P}}, \widetilde{\mathcal{E}}^-, \widetilde{\mathcal{E}}^+)$ that is given by setting

$$\Phi(\mathbf{1}_{\mathfrak{p}'}) = \mathbf{1}_\mathfrak{p}, \qquad \mathfrak{p}' \in \mathfrak{P}, \quad \mathfrak{p} \in \widetilde{\mathfrak{P}},$$
$$\Psi(e^-) = \mathbf{1}_\mathfrak{p}, \qquad e^- \in \mathcal{E}^-, \{s(e^-), t(e^-)\} \subset \mathfrak{P}(\mathfrak{p}), \quad \mathfrak{p} \in \widetilde{\mathfrak{P}},$$
$$\Psi(e^+) = \mathbf{1}_\mathfrak{p}, \qquad e^+ \in \mathcal{E}^+, \{s(e^+), t(e^+)\} \subset \mathfrak{P}(\mathfrak{p}), \quad \mathfrak{p} \in \widetilde{\mathfrak{P}},$$
$$\Psi(e^-) = \widehat{e^-}, \qquad e^- \in \mathcal{E}^-, \quad \widehat{e^-} \in \widetilde{\mathcal{E}}^-,$$
$$\Psi(e^+) = \widehat{e^+}, \qquad e^+ \in \mathcal{E}^+, \quad \widehat{e^+} \in \widetilde{\mathcal{E}}^+,$$

An application of Theorem 2.3 and of Lemma 3.1 and Theorem 3.2 yields that there is an isomorphism ψ of $[Y_{X(\Sigma,\lambda)}]_\approx$ onto $\mathcal{S}_{\widetilde{\mathcal{R}}}(\widetilde{\mathfrak{P}}, \widetilde{\mathcal{E}}^-, \widetilde{\mathcal{E}}^+)$ such that for a $y \in Y_{X(\Sigma,\lambda)}$ that is left asymptotic to a periodic point $q \in A(X(\Sigma, \lambda))$ and right asymptotic to a periodic point $r \in A(X(\Sigma, \lambda))$, and such that more precisely, with $I, J \in \mathbb{Z}, I < J$, such that

$$y_{(-\infty,I]} = q_{(-\infty,I]}, \quad \lambda(q_{[I,I+\pi_q]}) = \mathbf{1}_\mathfrak{q},$$

$$y_{(J,\infty)} = r_{(J,\infty)}, \quad \lambda(r_{[J,J+\pi_q]}) = \mathbf{1}_{\mathfrak{r}},$$

one has

$$\psi([y]_{\approx}) = \Psi(\lambda(y_{[I,J)})).$$

An $\mathcal{S}_{\widetilde{\mathcal{R}}}(\widetilde{\mathfrak{P}}, \widetilde{\mathcal{E}}^-, \widetilde{\mathcal{E}}^+)$-presentation of $X(\Sigma, \lambda)$ is given by $X(\mathfrak{P}, \Sigma, \Psi \circ \lambda)$. $\qquad\square$

4. Examples I

We introduce notation for relations. Given finite non-empty sets $\mathcal{E}^-$ and $\mathcal{E}^-$ and a relation $\mathcal{R} \subset \mathcal{E}^- \times \mathcal{E}^+$, we set

$$D^-(\mathcal{R}) = \gcd \{\operatorname{card}([e^-]_{\sim(\mathcal{R},-)}) : e^- \in \mathcal{E}^-\},$$

$$D^+(\mathcal{R}) = \gcd \{\operatorname{card}([e^+]_{\sim(\mathcal{R},+)}) : e^+ \in \mathcal{E}^-\}.$$

We denote by $\rho^{\triangle}(\mathcal{E}^-, \mathcal{E}^+)$ the set of relations $\mathcal{R} \subset \mathcal{E}^- \times \mathcal{E}^+$ such that

$$\gcd\{\operatorname{card}([e^-]_{\sim(\mathcal{R},-)}), \operatorname{card}([\widetilde{e}^-]_{\sim(\mathcal{R},-)})\} = D^-(\mathcal{R}),$$
$$e^-, \widetilde{e}^- \in \mathcal{E}^-, [e^-]_{\sim(\mathcal{R},-)} \neq [\widetilde{e}^-]_{\sim(\mathcal{R},-)},$$

$$\gcd\{\operatorname{card}([e^+]_{\sim(\mathcal{R},+)}), \operatorname{card}([\widetilde{e}^+]_{\sim(\mathcal{R},+)})\} = D^+(\mathcal{R}),$$
$$e^+, \widetilde{e}^+ \in \mathcal{E}^+, [e^+]_{\sim(\mathcal{R},+)} \neq [\widetilde{e}^+]_{\sim(\mathcal{R},+)},$$

and by $\rho^{\bigcirc}(\mathcal{E}^-, \mathcal{E}^+)$ the set of relations $\mathcal{R} \subset \mathcal{E}^- \times \mathcal{E}^+$ such that

$$\mathcal{E}^-(\mathcal{R}) = \mathcal{E}^+(\mathcal{R}) = \emptyset,$$

and we set

$$\rho^{\triangledown}(\mathcal{E}^-, \mathcal{E}^+) = \{\mathcal{R} \in \rho^{\triangle}(\mathcal{E}^-, \mathcal{E}^+) : D^-(\mathcal{R}) = D^+(\mathcal{R}) = 1\},$$

and

$$\rho^{\bigcirc\triangledown}(\mathcal{E}^-, \mathcal{E}^+) = \rho^{\bigcirc}(\mathcal{E}^-, \mathcal{E}^+) \cap \rho^{\triangle}(\mathcal{E}^-, \mathcal{E}^+).$$

Given disjoint finite sets $\mathcal{E}^-$ and $\mathcal{E}^+$, subsets $\mathcal{E}_{\circ}^- \subset \mathcal{E}^-, \mathcal{E}_{\circ}^+ \subset \mathcal{E}^+$, and a relation $\mathcal{R} \subset \mathcal{E}^- \times \mathcal{E}^+$, we speak of a subrelation $\mathcal{R} \cap (\mathcal{E}_{\circ}^- \times \mathcal{E}_{\circ}^+)$ of $\mathcal{R}$, provided that

$$\bigcup_{e^- \in \mathcal{E}_{\circ}^-} \Omega_{\mathcal{R}}^+(e^-) \subset \mathcal{E}_{\circ}^+, \quad \bigcup_{e^+ \in \mathcal{E}_{\circ}^+} \Omega_{\mathcal{R}}^-(e^+) \subset \mathcal{E}_{\circ}^-.$$

We say that a relation is irreducible if it does not possess a proper subrelation. For a subrelation

$$\mathcal{R}_{\circ} = \mathcal{R} \cap (\mathcal{E}_{\circ}^- \times \mathcal{E}_{\circ}^+)$$

of the relation $\mathcal{R} \subset \mathcal{E}^- \times \mathcal{E}^+$ we define its complementary relation $\mathcal{R} \ominus \mathcal{R}_{\circ}$ by

$$\mathcal{R} \ominus \mathcal{R}_{\circ} = \mathcal{R} \cap ((\mathcal{E}^- \setminus \mathcal{E}_{\circ}^-) \times (\mathcal{E}^+ \setminus \mathcal{E}_{\circ}^+)).$$

Given a finite index set $\mathcal{I}$ and disjoint finite sets $\mathcal{E}_i^-, \mathcal{E}_i^+, i \in \mathcal{I}$, together with relations $\mathcal{R}_i \subset \mathcal{E}_i^- \times \mathcal{E}_i^+, i \in \mathcal{I}$, we say that the relation

$$\bigoplus_{i \in \mathcal{I}} \mathcal{R}_i = \bigcup_{i \in \mathcal{I}} \mathcal{R}_i \subset (\bigcup_{i \in \mathcal{I}} \mathcal{E}_i^-) \times (\bigcup_{i \in \mathcal{I}} \mathcal{E}_i^+)$$

is the Kronecker sum of the relations $\mathcal{R}_i, i \in \mathcal{I}$. Every relation is the Kronecker sum of its irreducible subrelations, which are uniquely determined by it.

For finite sets $\mathcal{E}^-, \mathcal{E}^+$ and $\widetilde{\mathcal{E}}^-, \widetilde{\mathcal{E}}^+$ and relations $\mathcal{R} \subset \mathcal{E}^- \times \mathcal{E}^+$ and $\widetilde{\mathcal{R}} \subset \widetilde{\mathcal{E}}^- \times \widetilde{\mathcal{E}}^+$ their Kronecker product

$$\mathcal{R} \otimes \widetilde{\mathcal{R}} \subset (\mathcal{E}^- \times \widetilde{\mathcal{E}}^-) \times (\mathcal{E}^+ \times \widetilde{\mathcal{E}}^+)$$

is given by

$$\mathcal{R} \otimes \widetilde{\mathcal{R}} = \{((e^-, \widetilde{e}^-), (e^+, \widetilde{e}^+)) : (e^-, e^+) \in \mathcal{R}, (\widetilde{e}^-, \widetilde{e}^+) \in \widetilde{\mathcal{R}}\}.$$

For finite sets $\mathcal{E}^-, \mathcal{E}^+$ and a relation $\mathcal{R} \subset \mathcal{E}^- \times \mathcal{E}^+$ we define for $n \in \mathbb{N}$ a relation $\mathcal{R}^{\langle n \rangle} \subset (\mathcal{E}^-)^{\mathbb{Z}/n\mathbb{Z}} \times (\mathcal{E}^+)^{\mathbb{Z}/n\mathbb{Z}}$ by setting

$$\mathcal{R}^{\langle n \rangle} =$$

$$\bigcup_{k \in \mathbb{Z}/n\mathbb{Z}} \bigcap_{i \in \mathbb{Z}/n\mathbb{Z}} \{((e_i^-)_{i \in \mathbb{Z}/n\mathbb{Z}}, (e_i^+)_{i \in \mathbb{Z}/n\mathbb{Z}}) \in (\mathcal{E}^-)^{\mathbb{Z}/n\mathbb{Z}} \times (\mathcal{E}^+)^{\mathbb{Z}/n\mathbb{Z}} : (e_i^-, e_{i+k}^+) \in \mathcal{R}\}.$$

For finite sets $\mathcal{E}^-, \mathcal{E}^+$ and $\widetilde{\mathcal{E}}^-, \widetilde{\mathcal{E}}^+$ we say that a relation $\mathcal{R} \subset \mathcal{E}^- \times \mathcal{E}^+$ is isomorphic to a relation $\widetilde{\mathcal{R}} \subset \widetilde{\mathcal{E}}^- \times \widetilde{\mathcal{E}}^+$ if there exist bijections

$$\psi^- : \mathcal{E}^- \to \widetilde{\mathcal{E}}^-, \quad \psi^+ : \mathcal{E}^+ \to \widetilde{\mathcal{E}}^+,$$

such that

$$\widetilde{\mathcal{R}} = (\psi^- \times \psi^+)(\mathcal{R}).$$

The isomorphism class of a relation $\mathcal{R}$ determines the isomorphism class of the relation $\mathcal{R}^{\langle n \rangle}, n \in \mathbb{Z}$.

Let Ξ denote a system of representatives of the isomorphism classes irreducible relations on finite sets. We describe the isomorphism type of a relation $\mathcal{R} \subset \mathcal{E}^- \times \mathcal{E}^+$ by a vector $(\mu_{\mathcal{R}_\circ}(\mathcal{R}))_{\mathcal{R}_\circ \in \Xi}$, where $\mu_{\mathcal{R}_\circ}(\mathcal{R})$ is the multiplicity of the irreducible subrelations of $\mathcal{R}$ that are isomorphic to $\mathcal{R}_\circ$.

We note, that for the set of (isomorphism classes of) relations, the (class of the) empty relation acts as the zero element.

For a subshift X denote its set of orbits of length n by $\mathcal{O}_n(X), n \in \mathbb{N}$. To a subshift $X \subset \Sigma^{\mathbb{Z}}$ there are invariantly attached the sets $\mathcal{O}_n^{(-)}(X)$ of $O \in \mathcal{O}_n(X)$ such that for all $O' \in \cup_{n \in \mathbb{N}} \mathcal{O}_n(X)$ there is a point on X that is left asymptotic to O' and right asymptotic to $O, n \in \mathbb{N}$. Invariantly attached sets $\mathcal{O}_n^{(+)}(X), n \in \mathbb{N}$, are defined symmetrically. We set

$$\mathcal{O}_n^-(X) = \mathcal{O}_n^{(-)}(X) \setminus (\mathcal{O}_n^{(-)}(X) \cap \mathcal{O}_n^{(+)}(X)),$$
$$\mathcal{O}_n^+(X) = \mathcal{O}_n^{(+)}(X) \setminus (\mathcal{O}_n^{(-)}(X) \cap \mathcal{O}_n^{(+)}(X)), \quad n \in \mathbb{N},$$

and

$$\mathcal{O}^-(X) = \bigcup_{n \in \mathbb{N}} \mathcal{O}_n^-(X), \quad \mathcal{O}^+(X) = \bigcup_{n \in \mathbb{N}} \mathcal{O}_n^+(X).$$

For $k, l \in \mathbb{N}$ we denote by $\mathcal{O}_{k,l}^-(X)(\mathcal{O}_{k,l}^+(X))$ the set of $O^- \in \mathcal{O}_k^-(X)$ $(O^+ \in \mathcal{O}_l^+(X))$ such that X has a point that is left (right) asymptotic to $O^-(O^+)$ and right(left) asymptotic to a point in $\mathcal{O}_l^+(X)(\mathcal{O}_k^-(X))$. One has the relations

$$\mathcal{R}_{k,l}(X) \subset \mathcal{O}_k^-(X) \times \mathcal{O}_l^+(X)$$

where $O^- \in \mathcal{O}_k^-(X)$ and $O^+ \in \mathcal{O}_l^+(X)$ are related if X has a point that is left asymptotic to O^- and right asymptotic to O^+. (For $\mathcal{R}_{n,n}$ we write $\mathcal{R}_n$.)

One has for $\mathcal{S}_{\mathcal{R}}$-presentations $X(\Sigma, \mathfrak{P}, \lambda)$ of $\mathcal{R}$-graphs the set $\mathcal{O}^-(X(\Sigma, \mathfrak{P}, \lambda))$ $(\mathcal{O}^+(X(\Sigma, \mathfrak{P}, \lambda)))$ coincides with the set of periodic orbits of X that have negative (positive) multiplier precisely if there is no cycle in $\mathcal{E}_{\mathcal{R}}^-(\mathcal{E}_{\mathcal{R}}^+)$. In this case therefore the set of neutral periodic orbits, and the set of orbits with positive multiplier, and, symmetrically, with negative multiplier, are invariantly attached to the $\mathcal{S}_{\mathcal{R}}$-presentations $X(\Sigma, \mathfrak{P}, \lambda)$ (compare [HIK, Section 4]). We denote the number of orbits of length n with a negative multiplier by I_n^-, and the number of neutral orbits of length n by $I_n^0, n \in \mathbb{N}$.

In the way of examples we look now at certain special types of $\mathcal{R}$-graphs with one, two or three vertices, that satisfy conditions $(I), (II)$ and (III). A one-vertex $\mathcal{R}$-graph $\mathcal{G}_{\mathcal{R}}(\{\mathfrak{p}\}, \mathcal{E}^-, \mathcal{E}^+)$ such that

$$\mathcal{R}(\mathfrak{p}, \mathfrak{p}) \in \rho^{\bigcirc}(\mathcal{E}^-, \mathcal{E}^+),$$

we denote by $\mathcal{G}[\mathfrak{p}]$, setting

$$X(\mathcal{G}[\mathfrak{p}]) = X(\mathcal{E}^- \cup \mathcal{E}^+, \{\mathfrak{p}\}, \mathrm{id}_{\mathcal{E}^- \cup \mathcal{E}^+}).$$

The relation $\mathcal{R}(\mathfrak{p}, \mathfrak{p})$ is isomorphic to the relation $\mathcal{R}_1(X(\mathcal{G}[\mathfrak{p}]))$, and its isomorphism class is therefore an invariant of the subshifts $X(\mathcal{G}[\mathfrak{p}])$.

We note that

(4.1) $$I_2^-(X(\mathcal{G}[\mathfrak{p}])) = I_1^-(X(\mathcal{G}[\mathfrak{p}]))(I_1^-(X(\mathcal{G}[\mathfrak{p}])) - 1),$$

(4.2) $$I_2^0(X(\mathcal{G}[\mathfrak{p}])) = \mathrm{card}(\mathcal{R}_1(X(\mathcal{G}[\mathfrak{p}]))).$$

The semigroup that is associated to the subshift $X(\mathcal{G}[\mathfrak{p}])$ is isomorphic to the $\mathcal{R}$-graph semigroup of a one-vertex $\mathcal{R}$-graph with a relation that is isomorphic to $\widehat{\mathcal{R}}(\mathfrak{p}, \mathfrak{p})$.

By $\mathcal{G}[\mathfrak{p}, \mathfrak{q}]$ we denote an $\mathcal{R}$-graph $\mathcal{G}_\mathcal{R}(\{\mathfrak{p}, \mathfrak{q}\}, \mathcal{E}^-, \mathcal{E}^+)$ such that

$$\mathcal{E}^-(\mathfrak{q}, \mathfrak{q}) = \emptyset,$$

and

(4.3) $$\mathcal{E}^-(\mathfrak{p}, \mathfrak{q}) \neq \emptyset, \quad \mathcal{R}(\mathfrak{p}, \mathfrak{q}) = \mathcal{E}^-(\mathfrak{p}, \mathfrak{q}) \times \mathcal{E}^+(\mathfrak{p}, \mathfrak{q}),$$

and such that

$$\mathcal{E}^-(\mathfrak{q}, \mathfrak{p}) \neq \emptyset,$$

and, in case that $\mathcal{E}^-(\mathfrak{p}, \mathfrak{p}) \neq \emptyset$,

(4.4.a) $$\mathcal{R}(\mathfrak{q}, \mathfrak{p}) \in \rho^\triangledown(\mathcal{E}^-(\mathfrak{q}, \mathfrak{p}), \mathcal{E}^+(\mathfrak{q}, \mathfrak{p})),$$

and, in case that $\mathcal{E}^-(\mathfrak{p}, \mathfrak{p}) = \emptyset$,

(4.4.b) $$\mathcal{R}(\mathfrak{q}, \mathfrak{p}) \in \rho^{\bigcirc\triangledown}(\mathcal{E}^-(\mathfrak{q}, \mathfrak{p}), \mathcal{E}^+(\mathfrak{q}, \mathfrak{p}),$$

setting

$$X(\mathcal{G}[\mathfrak{p}, \mathfrak{q}]) = X(\mathcal{E}^- \cup \mathcal{E}^+, \{\mathfrak{p}, \mathfrak{q}\}, \mathrm{id}_{\mathcal{E}^- \cup \mathcal{E}^+}).$$

The relation $\mathcal{R}(\mathfrak{p}, \mathfrak{p})$ is isomorphic to the relation $\mathcal{R}_1(X(\mathcal{G}[\mathfrak{p}, \mathfrak{q}]))$, and its isomorphism class is therefore an invariant of the sushifts $X(\mathcal{G}[\mathfrak{p}, \mathfrak{q}])$. For an $\mathcal{R}$-graph $\mathcal{G}[\mathfrak{p}, \mathfrak{q}]$ denote by $\mathcal{O}_2^-(\mathcal{G}[\mathfrak{p}, \mathfrak{q}])(\mathcal{O}_2^+(\mathcal{G}[\mathfrak{p}, \mathfrak{q}]))$ the set of $O^- \in \mathcal{O}_2^-(X(\mathcal{G}[\mathfrak{p}, \mathfrak{q}]))$ $(O^+ \in \mathcal{O}_2^+(X(\mathcal{G}[\mathfrak{p}, \mathfrak{q}])))$ that contain the points that carry a bi-infinite concatenation of a word in $\mathcal{E}^-(\mathfrak{p}, \mathfrak{q})\mathcal{E}^-(\mathfrak{q}, \mathfrak{p})$ $(\mathcal{E}^+(\mathfrak{q}, \mathfrak{p})\mathcal{E}^+(\mathfrak{p}, \mathfrak{q}))$, and consider the relation

$$\mathcal{Q}(\mathcal{G}[\mathfrak{p}, \mathfrak{q}]) = \mathcal{R}_2(X(\mathcal{G}[\mathfrak{p}, \mathfrak{q}])) \cap (\mathcal{O}_2^-(\mathcal{G}[\mathfrak{p}, \mathfrak{q}]) \times \mathcal{O}_2^+(\mathcal{G}[\mathfrak{p}, \mathfrak{q}]).$$

It is

$$\mu(\mathcal{R}_2(X(\mathcal{G}[\mathfrak{p}, \mathfrak{q}])) \ominus \mathcal{Q}_2(\mathcal{G}[\mathfrak{p}, \mathfrak{q}])) = \tfrac{1}{2}\mu(\mathcal{R}_1(X(\mathcal{G}[\mathfrak{p}, \mathfrak{q}]))^{\langle 2 \rangle}),$$

which implies that the isomorphism class of the relation $\mathcal{Q}_2(\mathcal{G}[\mathfrak{p}, \mathfrak{q}]))$ is an invariant of $X(\mathcal{G}[\mathfrak{p}, \mathfrak{q}])$. By (4.3) and (4.4.a - b)

$$\mathcal{Q}_2(\mathcal{G}[\mathfrak{p}, \mathfrak{q}])) \in \rho^\triangle(\mathcal{E}^-(\mathcal{G}[\mathfrak{p}, \mathfrak{q}]), \mathcal{E}^+(\mathcal{G}[\mathfrak{p}, \mathfrak{q}])),$$

and therefore it follows that

$$\mathrm{card}(\mathcal{E}^-(\mathfrak{p}, \mathfrak{q})) = D^-(\mathcal{Q}_2(\mathcal{G}[\mathfrak{p}, \mathfrak{q}])), \quad \mathrm{card}(\mathcal{E}^+(\mathfrak{p}, \mathfrak{q})) = D^+(\mathcal{Q}_2(\mathcal{G}[\mathfrak{p}, \mathfrak{q}])),$$

and

$$\mu(\mathcal{R}(\mathfrak{q}, \mathfrak{p})) = \frac{\mu(\mathcal{Q}_2(\mathcal{G}[\mathfrak{p}, \mathfrak{q}])}{D^-(\mathcal{Q}_2(\mathcal{G}[\mathfrak{p}, \mathfrak{q}]))D^+(\mathcal{Q}_2(\mathcal{G}[\mathfrak{p}, \mathfrak{q}]))},$$

and it is seen that $\mathrm{card}(\mathcal{E}^-(\mathfrak{p}, \mathfrak{q}))$ and $\mathrm{card}(\mathcal{E}^+(\mathfrak{p}, \mathfrak{q}))$ as well as the isomorphism class of the relation $\mathcal{R}(\mathfrak{q}, \mathfrak{p})$ are invariants of the subshifts $X(G[\mathfrak{p}, \mathfrak{q}])$.

We note that

(4.5) $$I_2^-(X(\mathcal{G}[\mathfrak{p}, \mathfrak{q}])) > I_1^-(X(\mathcal{G}[\mathfrak{p}, \mathfrak{q}]))(I_1^-(X(\mathcal{G}[\mathfrak{p}, \mathfrak{q}])) - 1),$$

$$(4.6) \quad I_2^0(X(\mathcal{G}[\mathfrak{p},\mathfrak{q}])) = \mathrm{card}(\mathcal{R}_1(X(\mathcal{G}[\mathfrak{p},\mathfrak{q}]))) +$$

$$\frac{\mathrm{card}(\mathcal{Q}(X(\mathcal{G}[\mathfrak{p},\mathfrak{q}])))}{D^-(\mathcal{Q}_2(X(\mathcal{G}[\mathfrak{p},\mathfrak{q}])))D^+(\mathcal{Q}_2(X(\mathcal{G}[\mathfrak{p},\mathfrak{q}])))} +$$

$$D^-(\mathcal{Q}(X(\mathcal{G}[\mathfrak{p},\mathfrak{q}])))D^+(\mathcal{Q}_2(X(\mathcal{G}[\mathfrak{p},\mathfrak{q}]))),$$

$$(4.7) \qquad \mathcal{Q}(X(\mathcal{G}[\mathfrak{p},\mathfrak{q}])) \in \rho^\triangle(O_2^-(\mathcal{G}[\mathfrak{p},\mathfrak{q}]), O_2^+(\mathcal{G}[\mathfrak{p},\mathfrak{q}])).$$

The semigroup that is associated to the subshift $X(\mathcal{G}[\mathfrak{p},\mathfrak{q}])$ is isomorphic to the $\mathcal{R}$-graph semigroup of a one-vertex $\mathcal{R}$-graph with a relation that is isomorphic to $\widehat{\mathcal{R}}(\mathfrak{p},\mathfrak{p}) \oplus \widehat{\mathcal{R}}(\mathfrak{q},\mathfrak{p})$.

By $\mathcal{G}_+[\mathfrak{p},\mathfrak{q},\mathfrak{r}]$ we denote an $\mathcal{R}$-graph $\mathcal{G}_\mathcal{R}(\{\mathfrak{p},\mathfrak{q},\mathfrak{r}\}, \mathcal{E}^-, \mathcal{E}^+)$ such that

$$\mathcal{E}^-(\mathfrak{q},\mathfrak{q}) = \emptyset, \quad \mathcal{E}^-(\mathfrak{r},\mathfrak{r}) = \emptyset, \quad \mathcal{E}^-(\mathfrak{p},\mathfrak{r}) = \emptyset, \quad \mathcal{E}^-(\mathfrak{r},\mathfrak{q}) = \emptyset,$$

and

$$(4.8) \qquad \mathcal{E}^-(\mathfrak{p},\mathfrak{q}) \neq \emptyset, \quad \mathcal{R}(\mathfrak{p},\mathfrak{q}) = \mathcal{E}^-(\mathfrak{p},\mathfrak{q}) \times \mathcal{E}^+(\mathfrak{p},\mathfrak{q}),$$

$$(4.9) \qquad \mathcal{E}^-(\mathfrak{q},\mathfrak{r}) \neq \emptyset, \quad \mathcal{R}(\mathfrak{q},\mathfrak{r}) = \mathcal{E}^-(\mathfrak{q},\mathfrak{r}) \times \mathcal{E}^+(\mathfrak{q},\mathfrak{r}),$$

and

$$(4.10) \qquad \mathcal{R}(\mathfrak{q},\mathfrak{p}) \in \rho^\triangledown(\mathcal{E}^-(\mathfrak{q},\mathfrak{p}), \mathcal{E}^+(\mathfrak{q},\mathfrak{p})),$$

$$(4.11) \qquad \mathcal{R}(\mathfrak{r},\mathfrak{p}) \in \rho^\triangledown(\mathcal{E}^-(\mathfrak{r},\mathfrak{p}), \mathcal{E}^+(\mathfrak{r},\mathfrak{p})),$$

setting

$$X(\mathcal{G}_+[\mathfrak{p},\mathfrak{q},\mathfrak{r}]) = X(\mathcal{E}^- \cup \mathcal{E}^+, \{\mathfrak{p},\mathfrak{q},\mathfrak{r}\}, \mathrm{id}_{\mathcal{E}^- \cup \mathcal{E}^+}).$$

The relation $\mathcal{R}(\mathfrak{p},\mathfrak{p})$ is isomorphic to the relation $\mathcal{R}_1(X(\mathcal{G}[\mathfrak{p},\mathfrak{q},\mathfrak{r}])$, and therefore its isomorphism class is an invariant of the subshifts $X(\mathcal{G}_+[\mathfrak{p},\mathfrak{q},\mathfrak{r}])$. For an $\mathcal{R}$-graph $\mathcal{G}_+[\mathfrak{p},\mathfrak{q},\mathfrak{r}]$ denote by $O_2^-(\mathcal{G}_+[\mathfrak{p},\mathfrak{q},\mathfrak{r}])(O_2^+(\mathcal{G}_+[\mathfrak{p},\mathfrak{q},\mathfrak{r}]))$ the set of $O^- \in \mathcal{O}_2^-(X(\mathcal{G}_+[\mathfrak{p},\mathfrak{q},\mathfrak{r}]))$ $(O^+ \in \mathcal{O}_2^+(X(\mathcal{G}_+[\mathfrak{p},\mathfrak{q},\mathfrak{r}])))$ that contain the points that carry a bi-infinite concatenation of a word in $\mathcal{E}^-(\mathfrak{p},\mathfrak{q})\mathcal{E}^-(\mathfrak{q},\mathfrak{p})$ $(\mathcal{E}^+(\mathfrak{q},\mathfrak{p})\mathcal{E}^+(\mathfrak{p},\mathfrak{q}))$, and consider the relation

$$\mathcal{Q}_2(\mathcal{G}_+[\mathfrak{p},\mathfrak{q},\mathfrak{r}]) = \mathcal{R}_2(X(\mathcal{G}_+[\mathfrak{p},\mathfrak{q},\mathfrak{r}])) \cap (O_2^-(\mathcal{G}_+[\mathfrak{p},\mathfrak{q},\mathfrak{r}]) \times O_2^+(\mathcal{G}_+[\mathfrak{p},\mathfrak{q},\mathfrak{r}])).$$

It is

$$\mu(\mathcal{R}_2(X(\mathcal{G}_+[\mathfrak{p},\mathfrak{q},\mathfrak{r}])) \ominus \mathcal{Q}_2(\mathcal{G}_+[\mathfrak{p},\mathfrak{q},\mathfrak{r}]) = \tfrac{1}{2}\mu(\mathcal{R}_1(X(\mathcal{G}_+[\mathfrak{p},\mathfrak{q},\mathfrak{r}]))^{\langle 2 \rangle}),$$

which implies that the isomorphism class of the relation $\mathcal{Q}_2(\mathcal{G}_+[\mathfrak{p},\mathfrak{q},\mathfrak{r}])$ is an invariant of the subshifts $X(\mathcal{G}[\mathfrak{p},\mathfrak{q}])$. By (4.8) and (4.10)

$$\mathcal{Q}_2(\mathcal{G}_+[\mathfrak{p},\mathfrak{q},\mathfrak{r}])) \in \rho^\triangle(\mathcal{E}^-(\mathcal{G}[\mathfrak{p},\mathfrak{q}]), \mathcal{E}^+(\mathcal{G}_+[\mathfrak{p},\mathfrak{q},\mathfrak{r}])).$$

and it follows therefore that

$$\mathrm{card}(\mathcal{E}^-(\mathfrak{p},\mathfrak{q})) = D^-(\mathcal{Q}_2(\mathcal{G}_+[\mathfrak{p},\mathfrak{q},\mathfrak{r}])), \quad \mathrm{card}(\mathcal{E}^+(\mathfrak{p},\mathfrak{q})) = D^+(\mathcal{Q}_2(\mathcal{G}_+[\mathfrak{p},\mathfrak{q},\mathfrak{r}])),$$

and

$$\mu(\mathcal{R}(\mathfrak{q},\mathfrak{p})) = \frac{\mu(\mathcal{Q}_2(\mathcal{G}_+[\mathfrak{p},\mathfrak{q},\mathfrak{r}])}{D^-(\mathcal{Q}_2(\mathcal{G}_+[\mathfrak{p},\mathfrak{q},\mathfrak{r}]))D^+(\mathcal{Q}_2(\mathcal{G}_+[\mathfrak{p},\mathfrak{q},\mathfrak{r}]))},$$

and it is seen that $\mathrm{card}(\mathcal{E}^-(\mathfrak{p},\mathfrak{q}))$ and $\mathrm{card}(\mathcal{E}^-(\mathfrak{p},\mathfrak{q}))$ as well as the isomorphism class of $\mathcal{R}(\mathfrak{q},\mathfrak{p})$ are invariants of the subshifts $X(G_+[\mathfrak{p},\mathfrak{q},\mathfrak{r}])$.

For an $\mathcal{R}$-graph $\mathcal{G}_+[\mathfrak{p},\mathfrak{q},\mathfrak{r}]$ denote also by $O_3^-(\mathcal{G}_+[\mathfrak{p},\mathfrak{q},\mathfrak{r}])(O_3^+(\mathcal{G}[\mathfrak{p},\mathfrak{q},\mathfrak{r}]))$ the set of $O^- \in \mathcal{O}_3^-(X(\mathcal{G}_+[\mathfrak{p},\mathfrak{q},\mathfrak{r}]))$ $(O^+ \in \mathcal{O}_3^+(X(\mathcal{G}_+[\mathfrak{p},\mathfrak{q},\mathfrak{r}])))$ that contain the points that carry a bi-infinite concatenation of a word in $\mathcal{E}^-(\mathfrak{p},\mathfrak{q})\mathcal{E}^-(\mathfrak{q},\mathfrak{r})\mathcal{E}^-(\mathfrak{r},\mathfrak{p})(\mathcal{E}^+(\mathfrak{r},\mathfrak{p})$ $\mathcal{E}^+(\mathfrak{q},\mathfrak{r})\mathcal{E}^+(\mathfrak{p},\mathfrak{q}))$, and consider the relation

$$\mathcal{Q}_3(\mathcal{G}_+[\mathfrak{p},\mathfrak{q},\mathfrak{r}]) = \mathcal{R}_3(X(\mathcal{G}_+[\mathfrak{p},\mathfrak{q},\mathfrak{r}])) \cap (O_3^-(\mathcal{G}[\mathfrak{p},\mathfrak{q},\mathfrak{r}]) \times O_3^+(\mathcal{G}_+[\mathfrak{p},\mathfrak{q},\mathfrak{r}])).$$

It is

$$\mu(\mathcal{R}_3(X(\mathcal{G}_+[\mathfrak{p},\mathfrak{q},\mathfrak{r}]))\ominus\mathcal{Q}_3(\mathcal{G}_+[\mathfrak{p},\mathfrak{q},\mathfrak{r}])) =$$
$$\tfrac{1}{3}\mu((\mathcal{R}_1(X(\mathcal{G}[\mathfrak{p},\mathfrak{q},\mathfrak{r}]))^{\langle 3\rangle})+$$
$$\mathrm{card}(\mathcal{E}^-(\mathfrak{p},\mathfrak{q}))\mathrm{card}(\mathcal{E}^+(\mathfrak{p},\mathfrak{q}))\mu(\mathcal{R}_1(X(\mathcal{G}_+[\mathfrak{p},\mathfrak{q},\mathfrak{r}]))+$$
$$\mu(\mathcal{Q}_2(\mathcal{G}_+[\mathfrak{p},\mathfrak{q},\mathfrak{r}])\otimes\mathcal{R}_1(X(\mathcal{G}_+[\mathfrak{p},\mathfrak{q},\mathfrak{r}]))),$$

which implies that the isomorphism class of the relation $\mathcal{Q}_3(\mathcal{G}_+[\mathfrak{p},\mathfrak{q},\mathfrak{r}])$ is an invariant of the subshifts $X(\mathcal{G}[\mathfrak{p},\mathfrak{q},\mathfrak{r}])$. By (4.9) and (4.11)

$$\mathcal{Q}_3(\mathcal{G}_+[\mathfrak{p},\mathfrak{q},\mathfrak{r}]))\in\rho^\triangle(\mathcal{E}^-(\mathcal{G}[\mathfrak{p},\mathfrak{q}]),\mathcal{E}^+(\mathcal{G}_+[\mathfrak{p},\mathfrak{q},\mathfrak{r}])).$$

and it follows therefore that

$$\mathrm{card}(\mathcal{E}^-(\mathfrak{p},\mathfrak{q}))\mathrm{card}(\mathcal{E}^-(\mathfrak{q},\mathfrak{r})) = D^-(\mathcal{Q}_3(\mathcal{G}_+[\mathfrak{p},\mathfrak{q},\mathfrak{r}])),$$
$$\mathrm{card}(\mathcal{E}^+(\mathfrak{p},\mathfrak{q}))\mathrm{card}(\mathcal{E}^+(\mathfrak{q},\mathfrak{r})) = D^+(\mathcal{Q}_3(\mathcal{G}_+[\mathfrak{p},\mathfrak{q},\mathfrak{r}])),$$

and

$$\mu(\mathcal{R}(\mathfrak{q},\mathfrak{p})) = \frac{\mu(\mathcal{Q}_3(\mathcal{G}_+[\mathfrak{p},\mathfrak{q},\mathfrak{r}])}{D^-(\mathcal{Q}_3(\mathcal{G}_+[\mathfrak{p},\mathfrak{q},\mathfrak{r}]))D^+(\mathcal{Q}_3(\mathcal{G}_+[\mathfrak{p},\mathfrak{q},\mathfrak{r}]))},$$

and it is seen that $\mathrm{card}(\mathcal{E}^-(\mathfrak{q},\mathfrak{r}))$ and $\mathrm{card}(\mathcal{E}^-(\mathfrak{q},\mathfrak{r}))$ as well as the isomorphism class of $\mathcal{R}(\mathfrak{q},\mathfrak{p})$ are invariants of the subshifts $X(G_+[\mathfrak{p},\mathfrak{q},\mathfrak{r}])$.

We note that

$$(4.12) \qquad I_2^-(X(\mathcal{G}_+[\mathfrak{p},\mathfrak{q},\mathfrak{r}])) > I_1^-(X(\mathcal{G}_+[\mathfrak{p},\mathfrak{q},\mathfrak{r}]))(I_1^-(X(\mathcal{G}_+[\mathfrak{p},\mathfrak{q},\mathfrak{r}])) - 1),$$

$$(4.13) \quad I_2^0(X(\mathcal{G}_+[\mathfrak{p},\mathfrak{q},\mathfrak{r}])) > \mathrm{card}(\mathcal{R}_1(X(\mathcal{G}_+[\mathfrak{p},\mathfrak{q},\mathfrak{r}])))+$$
$$\frac{\mathrm{card}(\mathcal{Q}_2(X(\mathcal{G}[\mathfrak{p},\mathfrak{q},\mathfrak{r}])))}{D^-(\mathcal{Q}_2(X(\mathcal{G}_+[\mathfrak{p},\mathfrak{q},\mathfrak{r}])))D^+(\mathcal{Q}_2(X(\mathcal{G}_+[\mathfrak{p},\mathfrak{q},\mathfrak{r}])))}+$$
$$D^-(\mathcal{Q}_2(X(\mathcal{G}_+[\mathfrak{p},\mathfrak{q},\mathfrak{r}])))D^+(\mathcal{Q}_2(X(\mathcal{G}_+[\mathfrak{p},\mathfrak{q},\mathfrak{r}]))),$$

$$(4.14) \qquad \mathcal{Q}_2(X(\mathcal{G}_+[\mathfrak{p},\mathfrak{q},\mathfrak{r}]))\in\rho^\triangle(\mathcal{O}_2^-(\mathcal{G}_+[\mathfrak{p},\mathfrak{q},\mathfrak{r}])),\mathcal{O}_2^+(\mathcal{G}_+[\mathfrak{p},\mathfrak{q},\mathfrak{r}])).$$

The semigroup that is associated to the subshift $X(\mathcal{G}_+[\mathfrak{p},\mathfrak{q},\mathfrak{r}])$ is isomorphic to the $\mathcal{R}$-graph semigroup of a one-vertex $\mathcal{R}$-graph with a relation that is isomorphic to $\widehat{\mathcal{R}}(\mathfrak{p},\mathfrak{p})\oplus\widehat{\mathcal{R}}(\mathfrak{q},\mathfrak{p})\oplus\widehat{\mathcal{R}}(\mathfrak{r},\mathfrak{p})$.

By $\mathcal{G}_0[\mathfrak{p},\mathfrak{q},\mathfrak{r}]$ we denote an $\mathcal{R}$-graph $\mathcal{G}_\mathcal{R}(\{\mathfrak{p},\mathfrak{q},\mathfrak{r}\},\mathcal{E}^-,\mathcal{E}^+)$ such that

$$\mathcal{E}^-(\mathfrak{q},\mathfrak{q}) = \emptyset, \quad \mathcal{E}^-(\mathfrak{r},\mathfrak{r}) = \emptyset, \quad \mathcal{E}^-(\mathfrak{p},\mathfrak{r}) = \emptyset, \quad \mathcal{E}^-(\mathfrak{r},\mathfrak{q}) = \emptyset, \quad \mathcal{E}^-(\mathfrak{q},\mathfrak{p}) = \emptyset,$$

and

$$(4.15) \qquad \mathcal{E}^-(\mathfrak{p},\mathfrak{q}) \neq \emptyset, \quad \mathcal{R}(\mathfrak{p},\mathfrak{q}) = \mathcal{E}^-(\mathfrak{p},\mathfrak{q})\times\mathcal{E}^+(\mathfrak{p},\mathfrak{q}),$$

$$(4.16) \qquad \mathcal{E}^-(\mathfrak{q},\mathfrak{r}) \neq \emptyset, \quad \mathcal{R}(\mathfrak{q},\mathfrak{r}) = \mathcal{E}^-(\mathfrak{q},\mathfrak{r})\times\mathcal{E}^+(\mathfrak{q},\mathfrak{r}),$$

and

$$\mathcal{E}^-(\mathfrak{r},\mathfrak{p}) \neq \emptyset,$$

and such that, in case that $\mathcal{E}^-(\mathfrak{p},\mathfrak{p}) = \emptyset$,

$$(4.17.\mathrm{a}) \qquad \mathcal{R}(\mathfrak{r},\mathfrak{p})\in\rho^{\bigcirc\triangledown}(\mathcal{E}^-(\mathfrak{r},\mathfrak{p}),\mathcal{E}^+(\mathfrak{r},\mathfrak{p}),$$

and in case that $\mathcal{E}^-(\mathfrak{p},\mathfrak{p}) \neq \emptyset$,

$$(4.17.\mathrm{b}) \qquad \mathcal{R}(\mathfrak{r},\mathfrak{p})\in\rho^\triangledown(\mathcal{E}^-(\mathfrak{r},\mathfrak{p}),\mathcal{E}^+(\mathfrak{q},\mathfrak{p})),$$

setting

$$X(\mathcal{G}_0[\mathfrak{p},\mathfrak{q},\mathfrak{r}]) = X(\mathcal{E}^-\cup\mathcal{E}^+,\{\mathfrak{p},\mathfrak{q},\mathfrak{r}\},\mathrm{id}_{\mathcal{E}^-\cup\mathcal{E}^+}).$$

Proposition 4.1. *Let there be given $\mathcal{R}$-graphs*

$$\mathcal{G}_0[\mathfrak{p}, \mathfrak{q}, \mathfrak{r}] = \mathcal{G}_\mathcal{R}(\{\mathfrak{p}, \mathfrak{q}, \mathfrak{r}\}, \mathcal{E}^-, \mathcal{E}^+)$$

and

$$\widetilde{\mathcal{G}}_0[\widetilde{\mathfrak{p}}, \widetilde{\mathfrak{q}}, \widetilde{\mathfrak{r}}])) = \mathcal{G}_{\widetilde{\mathcal{R}}}(\{\widetilde{\mathfrak{p}}, \widetilde{\mathfrak{q}}, \widetilde{\mathfrak{r}}\}, \widetilde{\mathcal{E}}^-, \widetilde{\mathcal{E}}^+).$$

For the subshifts $X(\mathcal{G}_0[\mathfrak{p}, \mathfrak{q}, \mathfrak{r}])))$ and $X(\widetilde{\mathcal{G}}_0[\widetilde{\mathfrak{p}}, \widetilde{\mathfrak{q}}, \widetilde{\mathfrak{r}}])))$ to be topologically conjugate, it is necessary and sufficient that the relations $\mathcal{R}(\mathfrak{p}, \mathfrak{p})$ and $\widetilde{\mathcal{R}}(\widetilde{\mathfrak{p}}, \widetilde{\mathfrak{p}})$ are isomorphic, that the relations $\mathcal{R}(\mathfrak{r}, \mathfrak{p})$ and $\widetilde{\mathcal{R}}(\widetilde{\mathfrak{r}}, \widetilde{\mathfrak{p}})$ are isomorphic, and that

$$(4.18) \qquad \mathrm{card}(\mathcal{E}^-(\mathfrak{p}, \mathfrak{q}))\mathrm{card}(\mathcal{E}^-(\mathfrak{q}, \mathfrak{r})) = \mathrm{card}(\widetilde{\mathcal{E}}^-(\widetilde{\mathfrak{p}}, \widetilde{\mathfrak{q}}))\mathrm{card}(\widetilde{\mathcal{E}}^-(\widetilde{\mathfrak{q}}, \widetilde{\mathfrak{r}})),$$

$$(4.19) \qquad \mathrm{card}(\mathcal{E}^+(\mathfrak{q}, \mathfrak{r}))\mathrm{card}(\mathcal{E}^+(\mathfrak{p}, \mathfrak{q})) = \mathrm{card}(\widetilde{\mathcal{E}}^+(\widetilde{\mathfrak{q}}, \widetilde{\mathfrak{r}}))\mathrm{card}(\widetilde{\mathcal{E}}^+(\widetilde{\mathfrak{p}}, \widetilde{\mathfrak{q}})),$$

$$(4.20) \qquad \mathrm{card}(\mathcal{E}^-(\mathfrak{p}, \mathfrak{q}))\mathrm{card}(\mathcal{E}^+(\mathfrak{p}, \mathfrak{q})) = \mathrm{card}(\widetilde{\mathcal{E}}^-(\widetilde{\mathfrak{p}}, \widetilde{\mathfrak{q}}))\mathrm{card}(\widetilde{\mathcal{E}}^+(\widetilde{\mathfrak{p}}, \widetilde{\mathfrak{q}})).$$

Proof. Note that in the statement of the proposition (4.19) can be replaced by

$$(4.21) \qquad \mathrm{card}(\mathcal{E}^-(\mathfrak{q}, \mathfrak{r}))\mathrm{card}(\mathcal{E}^+(\mathfrak{q}, \mathfrak{r})) = \mathrm{card}(\widetilde{\mathcal{E}}^-(\widetilde{\mathfrak{q}}, \widetilde{\mathfrak{r}}))\mathrm{card}(\widetilde{\mathcal{E}}^+(\widetilde{\mathfrak{q}}, \widetilde{\mathfrak{r}})).$$

Set

$$X = X(\mathcal{G}_0[\mathfrak{p}, \mathfrak{q}, \mathfrak{r}]).$$

The relation $\mathcal{R}(\mathfrak{p}, \mathfrak{p})$ is isomorphic to the relation $\mathcal{R}_1(X)$ and therefore its isomorphism class is an invariant of the subshift X.

Also set

$$A_{\mathfrak{p},\mathfrak{q}}^- = \mathrm{card}(\mathcal{E}^-(\mathfrak{p}, \mathfrak{q})), \quad A_{\mathfrak{q},\mathfrak{r}}^- = \mathrm{card}(\mathcal{E}^-(\mathfrak{q}, \mathfrak{r})),$$
$$A_{\mathfrak{q},\mathfrak{r}}^+ = \mathrm{card}(\mathcal{E}^+(\mathfrak{q}, \mathfrak{r})), \quad A_{\mathfrak{p},\mathfrak{q}}^+ = \mathrm{card}(\mathcal{E}^+(\mathfrak{p}, \mathfrak{q})),$$

We divide the proof of necessity into two cases. We consider first the case that X has fixed points.

The set $\mathcal{O}_{3,1}^-(X)$ contains precisely the orbits in $O \in \mathcal{O}_3^-(X)$ whose points carry a bi-infinite concatenation of a word in $\mathcal{E}^-(\mathfrak{p}, \mathfrak{p})\mathcal{E}^-(\mathfrak{p}, \mathfrak{q})\mathcal{E}^+(\mathfrak{p}, \mathfrak{q})$. It is

$$\mathrm{card}(\mathcal{O}_{3,1}^-(X)) = \mathrm{card}(\mathcal{E}^-(\mathfrak{p}, \mathfrak{p}))A_{\mathfrak{p},\mathfrak{q}}^- A_{\mathfrak{p},\mathfrak{q}}^+,$$

and it follows that $A_{\mathfrak{p},\mathfrak{q}}^- A_{\mathfrak{p},\mathfrak{q}}^+$ is an invariant of the subshift X.

For the $\mathcal{R}$-graph $\mathcal{G}_0[\mathfrak{p}, \mathfrak{q}, \mathfrak{r}]$ denote by $\mathcal{O}_3^-(\mathcal{G}_0[\mathfrak{p}, \mathfrak{q}, \mathfrak{r}])(\mathcal{O}_3^+(\mathcal{G}_0[\mathfrak{p}, \mathfrak{q}, \mathfrak{r}]))$ the set of $O^- \in \mathcal{O}_3^-(X)$ $(O^+ \in \mathcal{O}_3^+(X))$ that contain the points that carry a bi-infinite concatenation of a word in $\mathcal{E}^-(\mathfrak{p}, \mathfrak{q})\mathcal{E}^-(\mathfrak{q}, \mathfrak{r})\mathcal{E}^-(\mathfrak{r}, \mathfrak{p})$ $(\mathcal{E}^+(\mathfrak{r}, \mathfrak{p})\mathcal{E}^+(\mathfrak{q}, \mathfrak{r})\mathcal{E}^+(\mathfrak{p}, \mathfrak{q}))$, and consider the relation

$$\mathcal{Q}_3(\mathcal{G}_0[\mathfrak{p}, \mathfrak{q}, \mathfrak{r}]) = \mathcal{R}_3(X) \cap (\mathcal{O}_3^-(\mathcal{G}_0[\mathfrak{p}, \mathfrak{q}, \mathfrak{r}]) \times \mathcal{O}_3^+(\mathcal{G}_0[\mathfrak{p}, \mathfrak{q}, \mathfrak{r}])).$$

It is

$$\mu(\mathcal{R}_3(X) \ominus \mathcal{Q}_3(\mathcal{G}_+0[\mathfrak{p}, \mathfrak{q}, \mathfrak{r}])) = \tfrac{1}{3}\mu(\mathcal{R}_1(X)^{\langle 3 \rangle}) + A_{\mathfrak{p},\mathfrak{q}}^- A_{\mathfrak{p},\mathfrak{q}}^+ \mu(\mathcal{R}_1(X)),$$

and it follows that the isomorphism class of the relation $\mathcal{Q}_3(\mathcal{G}_0[\mathfrak{p}, \mathfrak{q}, \mathfrak{r}]))$ is an invariant of the subshift X. By (4.15 - 16) and (17.a)

$$\mathcal{Q}_3(\mathcal{G}_0[\mathfrak{p}, \mathfrak{q}, \mathfrak{r}])) \in \rho^\triangle(\mathcal{O}_3^-(\mathcal{G}_0[\mathfrak{p}, \mathfrak{q}, \mathfrak{r}]), \mathcal{O}_3^+(\mathcal{G}_0[\mathfrak{p}, \mathfrak{q}, \mathfrak{r}])).$$

and it follows therefore that

$$A_{\mathfrak{p},\mathfrak{q}}^- A_{\mathfrak{q},\mathfrak{r}}^- = D^-(\mathcal{Q}_3(\mathcal{G}_0[\mathfrak{p}, \mathfrak{q}, \mathfrak{r}])), \quad A_{\mathfrak{q},\mathfrak{r}}^+ A_{\mathfrak{p},\mathfrak{q}}^+ = D^+(\mathcal{Q}_3(\mathcal{G}_0[\mathfrak{p}, \mathfrak{q}, \mathfrak{r}])),$$

and it is seen that $A_{\mathfrak{p},\mathfrak{q}}^- A_{\mathfrak{q},\mathfrak{r}}^-$ and $A_{\mathfrak{q},\mathfrak{r}}^+ A_{\mathfrak{p},\mathfrak{q}}^+$ are invariants of the subshift X. Also

$$\mu(\mathcal{R}(\mathfrak{r}, \mathfrak{p})) = \frac{\mu(\mathcal{Q}_3(\mathcal{G}_0[\mathfrak{p}, \mathfrak{q}, \mathfrak{r}]))}{A_{\mathfrak{p},\mathfrak{q}}^- A_{\mathfrak{q},\mathfrak{r}}^- A_{\mathfrak{q},\mathfrak{r}}^+ A_{\mathfrak{p},\mathfrak{q}}^+},$$

and it is seen that the isomorphism class of the relation $\mathcal{R}(\mathfrak{r}, \mathfrak{p})$ is an invariant of X.

Consider the case that X has no fixed points. In this case by (4.16) and (4.17.b)

$$\mathcal{R}_3(X) \in \rho^{\triangle}(\mathcal{O}_3^-(X), \mathcal{O}_3^+(X)),$$

and

$$A_{\mathfrak{p},\mathfrak{q}}^- A_{\mathfrak{q},\mathfrak{r}}^- = D^-(\mathcal{R}_3(X)), \quad A_{\mathfrak{q},\mathfrak{r}}^+ A_{\mathfrak{p},\mathfrak{q}}^+ = D^+(\mathcal{R}_3(X)),$$

and

$$\mu(\mathcal{R}(\mathfrak{r},\mathfrak{p})) = \frac{\mu(\mathcal{R}_3(X))}{A_{\mathfrak{p},\mathfrak{q}}^- A_{\mathfrak{q},\mathfrak{r}}^- A_{\mathfrak{q},\mathfrak{r}}^+ A_{\mathfrak{p},\mathfrak{q}}^+},$$

and it is seen that $A_{\mathfrak{p},\mathfrak{q}}^- A_{\mathfrak{q},\mathfrak{r}}^-$ and $A_{\mathfrak{q},\mathfrak{r}}^+ A_{\mathfrak{p},\mathfrak{q}}^+$ as well as the isomorphism class of the relation $\mathcal{R}(\mathfrak{r},\mathfrak{p}))$ are invariants of the subshift X, It is

$$A_{\mathfrak{p},\mathfrak{q}}^- A_{\mathfrak{p},\mathfrak{q}}^+ + A_{\mathfrak{q},\mathfrak{r}}^+ A_{\mathfrak{q},\mathfrak{r}}^- = I_2^0(X) - \mathrm{card}(\mathcal{R}(\mathfrak{r},\mathfrak{p})),$$

which implies that $A_{\mathfrak{p},\mathfrak{q}}^- A_{\mathfrak{p},\mathfrak{q}}^+ + A_{\mathfrak{q},\mathfrak{r}}^+ A_{\mathfrak{q},\mathfrak{r}}^-$, is an invariant of X. Also

$$\begin{aligned}
I_6^0(X) =& \tfrac{1}{3}\{A_{\mathfrak{p},\mathfrak{q}}^- A_{\mathfrak{p},\mathfrak{q}}^+ (A_{\mathfrak{p},\mathfrak{q}}^- A_{\mathfrak{p},\mathfrak{q}}^+ - 1)(A_{\mathfrak{p},\mathfrak{q}}^- A_{\mathfrak{p},\mathfrak{q}}^+ - 2)\} + A_{\mathfrak{p},\mathfrak{q}}^- A_{\mathfrak{p},\mathfrak{q}}^+ (A_{\mathfrak{p},\mathfrak{q}}^- A_{\mathfrak{p},\mathfrak{q}}^+ - 1) + \\
& \tfrac{1}{3}\{A_{\mathfrak{q},\mathfrak{r}}^- A_{\mathfrak{q},\mathfrak{r}}^+ (A_{\mathfrak{q},\mathfrak{r}}^- A_{\mathfrak{q},\mathfrak{r}}^+ - 1)(A_{\mathfrak{q},\mathfrak{r}}^- A_{\mathfrak{q},\mathfrak{r}}^+ - 2)\} + A_{\mathfrak{q},\mathfrak{r}}^- A_{\mathfrak{q},\mathfrak{r}}^+ (A_{\mathfrak{q},\mathfrak{r}}^- A_{\mathfrak{q},\mathfrak{r}}^+ - 1) + \\
& A_{\mathfrak{p},\mathfrak{q}}^- A_{\mathfrak{q},\mathfrak{r}}^- A_{\mathfrak{q},\mathfrak{r}}^+ A_{\mathfrak{p},\mathfrak{q}}^+ (A_{\mathfrak{p},\mathfrak{q}}^- A_{\mathfrak{p},\mathfrak{q}}^+ + A_{\mathfrak{q},\mathfrak{r}}^+ A_{\mathfrak{q},\mathfrak{r}}^-) + \\
& \tfrac{1}{3}\mathrm{card}(\mathcal{R}(\mathfrak{r},\mathfrak{p}))(\mathrm{card}(\mathcal{R}(\mathfrak{r},\mathfrak{p})) - 1)(\mathrm{card}(\mathcal{R}(\mathfrak{r},\mathfrak{p})) - 2) + \\
& \mathrm{card}(\mathcal{R}(\mathfrak{r},\mathfrak{p}))^2 (A_{\mathfrak{p},\mathfrak{q}}^- A_{\mathfrak{p},\mathfrak{q}}^+ + A_{\mathfrak{q},\mathfrak{r}}^+ A_{\mathfrak{q},\mathfrak{r}}^-) + 2\,\mathrm{card}(\mathcal{R}(\mathfrak{r},\mathfrak{p})) A_{\mathfrak{p},\mathfrak{q}}^- A_{\mathfrak{q},\mathfrak{r}}^- A_{\mathfrak{q},\mathfrak{r}}^+ A_{\mathfrak{p},\mathfrak{q}}^+ + \\
& \mathrm{card}(\mathcal{R}(\mathfrak{r},\mathfrak{p})) A_{\mathfrak{p},\mathfrak{q}}^- A_{\mathfrak{p},\mathfrak{q}}^+,
\end{aligned}$$

and it follows from the invariance of $A_{\mathfrak{p},\mathfrak{q}}^- A_{\mathfrak{q},\mathfrak{r}}^-, A_{\mathfrak{q},\mathfrak{r}}^+ A_{\mathfrak{p},\mathfrak{q}}^+$ and of $A_{\mathfrak{p},\mathfrak{q}}^- A_{\mathfrak{p},\mathfrak{q}}^+ + A_{\mathfrak{q},\mathfrak{r}}^+ A_{\mathfrak{q},\mathfrak{r}}^-$ that also $A_{\mathfrak{p},\mathfrak{q}}^- A_{\mathfrak{p},\mathfrak{q}}^+$ is an invariant of the subshift X. This proves necessity.

To prove sufficiency it is enough to consider the case that

$$\mathfrak{p} = \widetilde{\mathfrak{p}}, \quad \mathfrak{q} = \widetilde{\mathfrak{q}}, \quad \mathfrak{r} = \widetilde{\mathfrak{r}},$$

and that

$$\mathcal{E}^-(\mathfrak{p},\mathfrak{p}) = \widetilde{\mathcal{E}}^-(\mathfrak{p},\mathfrak{p}), \quad \mathcal{E}^+(\mathfrak{p},\mathfrak{p}) = \widetilde{\mathcal{E}}^+(\mathfrak{p},\mathfrak{p}), \quad \mathcal{R}(\mathfrak{p},\mathfrak{p}) = \widetilde{\mathcal{R}}(\mathfrak{p},\mathfrak{p}).$$

$$\mathcal{E}^-(\mathfrak{q},\mathfrak{r}) = \widetilde{\mathcal{E}}^-(\mathfrak{q},\mathfrak{r}), \quad \mathcal{E}^+(\mathfrak{q},\mathfrak{r}) = \widetilde{\mathcal{E}}^+(\mathfrak{q},\mathfrak{r}), \quad \mathcal{R}(\mathfrak{q},\mathfrak{r}) = \widetilde{\mathcal{R}}(\mathfrak{q},\mathfrak{r}).$$

By (4.18 - 21) we can choose bijections

$$\eta_{\mathfrak{r},\mathfrak{q},\mathfrak{r}} : \mathcal{E}^+(\mathfrak{q},\mathfrak{r})\mathcal{E}^-(\mathfrak{q},\mathfrak{r}) \to \widetilde{\mathcal{E}}^+(\mathfrak{q},\mathfrak{r})\widetilde{\mathcal{E}}^-(\mathfrak{q},\mathfrak{r}),$$

$$\eta_{\mathfrak{r},\mathfrak{q},\mathfrak{p}} : \mathcal{E}^+(\mathfrak{q},\mathfrak{r})\mathcal{E}^+(\mathfrak{p},\mathfrak{q}) \to \widetilde{\mathcal{E}}^+(\mathfrak{q},\mathfrak{r})\widetilde{\mathcal{E}}^+(\mathfrak{p},\mathfrak{q}),$$

$$\eta_{\mathfrak{p},\mathfrak{q},\mathfrak{p}} : \mathcal{E}^-(\mathfrak{p},\mathfrak{q})\mathcal{E}^+(\mathfrak{p},\mathfrak{q}) \to \widetilde{\mathcal{E}}^-(\mathfrak{p},\mathfrak{q})\widetilde{\mathcal{E}}^+(\mathfrak{p},\mathfrak{q}),$$

$$\eta_{\mathfrak{p},\mathfrak{q},\mathfrak{r}} : \mathcal{E}^-(\mathfrak{p},\mathfrak{q})\mathcal{E}^-(\mathfrak{q},\mathfrak{r}) \to \widetilde{\mathcal{E}}^-(\mathfrak{p},\mathfrak{q})\widetilde{\mathcal{E}}^-(\mathfrak{q},\mathfrak{r}).$$

A topological conjugacy of $X(\mathcal{G}_0[\mathfrak{p},\mathfrak{q},\mathfrak{r}]])$ onto $X(\widetilde{\mathcal{G}}_0[\widetilde{\mathfrak{p}},\widetilde{\mathfrak{q}},\widetilde{\mathfrak{r}}])$ is given by the mapping that replaces in the points of $X(\mathcal{G}_0[\mathfrak{p},\mathfrak{q}])$ a word in $\mathcal{E}^+(\mathfrak{q},\mathfrak{r})\mathcal{E}^-(\mathfrak{q},\mathfrak{r})$ by its image under $\eta_{\mathfrak{r},\mathfrak{q},\mathfrak{r}}$, a word in $\mathcal{E}^+(\mathfrak{q},\mathfrak{r})\mathcal{E}^+(\mathfrak{p},\mathfrak{q})$ by its image under $\eta_{\mathfrak{r},\mathfrak{q},\mathfrak{p}}$, a word in $\mathcal{E}^-(\mathfrak{p},\mathfrak{q})\mathcal{E}^+(\mathfrak{p},\mathfrak{q})$ by its image under $\eta_{\mathfrak{p},\mathfrak{q},\mathfrak{p}}$, and a word in $\mathcal{E}^-(\mathfrak{p},\mathfrak{q})\mathcal{E}^-(\mathfrak{q},\mathfrak{r})$ by its image under $\eta_{\mathfrak{p},\mathfrak{q},\mathfrak{r}}$. $\qquad\square$

We note that

$$(4.22) \qquad I_2^-(X(\mathcal{G}_0[\mathfrak{p},\mathfrak{q},\mathfrak{r}])) = I_1^-(X(\mathcal{G}_0[\mathfrak{p},\mathfrak{q},\mathfrak{r}]))(I_1^-(X(\mathcal{G}_0[\mathfrak{p},\mathfrak{q},\mathfrak{r}])) - 1),$$

$$(4.23) \qquad I_2^0(X(\mathcal{G}_0[\mathfrak{p},\mathfrak{q},\mathfrak{r}])) > \mathrm{card}(\mathcal{R}_1(X(\mathcal{G}_0[\mathfrak{p},\mathfrak{q},\mathfrak{r}]))).$$

The semigroup that is associated to the subshift $X(\mathcal{G}_0[\mathfrak{p},\mathfrak{q},\mathfrak{r}])$ is isomorphic to the $\mathcal{R}$-graph semigroup of a one-vertex $\mathcal{R}$-graph with a relation that is isomorphic to $\widehat{\mathcal{R}}(\mathfrak{p},\mathfrak{p}) \oplus \widehat{\mathcal{R}}(\mathfrak{r},\mathfrak{p})$.

By $\mathcal{G}_+[\mathfrak{p}, \mathfrak{q}_0, \mathfrak{q}_1]$ we denote an $\mathcal{R}$-graph $\mathcal{G}_\mathcal{R}(\{\mathfrak{p}, \mathfrak{q}_0, \mathfrak{q}_1\}, \mathcal{E}^-, \mathcal{E}^+)$ such that
$$\mathcal{E}^-(\mathfrak{q}_0, \mathfrak{q}_1) = \emptyset, \quad \mathcal{E}^-(\mathfrak{q}_1, \mathfrak{q}_0) = \emptyset$$
and
$$(4.24) \qquad \mathcal{R}(\mathfrak{p}, \mathfrak{q}_0) = \mathcal{E}^-(\mathfrak{p}, \mathfrak{q}_0) \times \mathcal{E}^+(\mathfrak{p}, \mathfrak{q}_0), \quad \mathcal{R}(\mathfrak{p}, \mathfrak{q}_1) = \mathcal{E}^-(\mathfrak{p}, \mathfrak{q}_1) \times \mathcal{E}^+(\mathfrak{p}, \mathfrak{q}_1),$$

$$(4.25) \ \ \mathcal{R}(\mathfrak{q}_0, \mathfrak{p}) \in \rho^\triangledown(\mathcal{E}^-(\mathfrak{q}_0, \mathfrak{p}), \mathcal{E}^+(\mathfrak{q}_0, \mathfrak{p})), \quad \mathcal{R}(\mathfrak{q}_1, \mathfrak{p}) \in \rho^\triangledown(\mathcal{E}^-(\mathfrak{q}_1, \mathfrak{p}), \mathcal{E}^+(\mathfrak{q}_1, \mathfrak{p})),$$
such that
$$\mathcal{E}^-(\mathfrak{p}, \mathfrak{q}_0) \neq \emptyset, \quad \mathcal{E}^-(\mathfrak{p}, \mathfrak{q}_1)) \neq \emptyset,$$
and such that the pair
$$(\mathrm{card}(\mathcal{E}^-(\mathfrak{p}, \mathfrak{q}_0)), \mathrm{card}(\mathcal{E}^-(\mathfrak{p}, \mathfrak{q}_1)))$$
is lexicographically larger than the pair
$$(\mathrm{card}(\mathcal{E}^+(\mathfrak{p}, \mathfrak{q}_0)), \mathrm{card}(\mathcal{E}^+(\mathfrak{p}, \mathfrak{q}_1))),$$
setting
$$X(\mathcal{G}_+[\mathfrak{p}, \mathfrak{q}_0, \mathfrak{q}_1]) = X(\mathcal{E}^- \cup \mathcal{E}^+, \{\mathfrak{p}, \mathfrak{q}_0, \mathfrak{q}_1\}, \mathrm{id}_{\mathcal{E}^- \cup \mathcal{E}^+}).$$

The relation $\mathcal{R}_1(X(\mathcal{G}_+[\mathfrak{p}, \mathfrak{q}_0, \mathfrak{q}_1]))$ is isomorphic to the relation $\mathcal{R}(\mathfrak{p}, \mathfrak{p})$, and therefore its isomorphism class is an invariant of the subshifts $X(\mathcal{G}_+[\mathfrak{p}, \mathfrak{q}_0, \mathfrak{q}_1])$. For an $\mathcal{R}$-graph $\mathcal{G}_+[\mathfrak{p}, \mathfrak{q}_0, \mathfrak{q}_1]$ denote by $\mathcal{O}_{(0)}^-(\mathcal{G}_+[\mathfrak{p}, \mathfrak{q}_0, \mathfrak{q}_1])$ $(\mathcal{O}_{(1)}^-(\mathcal{G}_+[\mathfrak{p}, \mathfrak{q}_0, \mathfrak{q}_1]))$ the set of $O^- \in \mathcal{O}_2^-(X(\mathcal{G}_+[\mathfrak{p}, \mathfrak{q}_0, \mathfrak{q}_1]))$ that contain the points that carry a bi-infinite concatenation of a word in $\mathcal{E}^-(\mathfrak{p}, \mathfrak{q}_0) \ \mathcal{E}^-(\mathfrak{q}_0, \mathfrak{p}) \ (\mathcal{E}^-(\mathfrak{p}, \mathfrak{q}_1)\mathcal{E}^-(\mathfrak{q}_1, \mathfrak{p}))$, and set

$$\mathcal{Q}_2^{(0)}(\mathcal{G}_+[\mathfrak{p}, \mathfrak{q}_0, \mathfrak{q}_1])) =$$
$$\mathcal{R}_2(X(\mathcal{G}_+[\mathfrak{p}, \mathfrak{q}_0, \mathfrak{q}_1])) \cap (\mathcal{O}_{(0)}^-(\mathcal{G}_+[\mathfrak{p}, \mathfrak{q}_0, \mathfrak{q}_1])) \times \mathcal{O}_{(0)}^+(\mathcal{G}_+[\mathfrak{p}, \mathfrak{q}_0, \mathfrak{q}_1]))),$$

$$\mathcal{Q}_2^{(1)}(\mathcal{G}_+[\mathfrak{p}, \mathfrak{q}_0, \mathfrak{q}_1])) =$$
$$\mathcal{R}_2(X(\mathcal{G}_+[\mathfrak{p}, \mathfrak{q}_0, \mathfrak{q}_1])) \cap (\mathcal{O}_{(1)}^-(\mathcal{G}_+[\mathfrak{p}, \mathfrak{q}_0, \mathfrak{q}_1])) \times \mathcal{O}_{(1)}^+(\mathcal{G}_+[\mathfrak{p}, \mathfrak{q}_0, \mathfrak{q}_1]))),$$
and
$$\mathcal{Q}_2(\mathcal{G}_+[\mathfrak{p}, \mathfrak{q}_0, \mathfrak{q}_1])) = \mathcal{Q}_2^{(0)}(\mathcal{G}_+[\mathfrak{p}, \mathfrak{q}_0, \mathfrak{q}_1])) \cup \mathcal{Q}_2^{(1)}(\mathcal{G}_+[\mathfrak{p}, \mathfrak{q}_0, \mathfrak{q}_1])).$$
It is
$$\mu(\mathcal{R}_2(X(\mathcal{G}_+[\mathfrak{p}, \mathfrak{q}_0, \mathfrak{q}_1]) \ominus \mathcal{Q}_2(\mathcal{G}_+[\mathfrak{p}, \mathfrak{q}_0, \mathfrak{q}_1]))) = \tfrac{1}{2}\mu(\mathcal{R}_1(X(\mathcal{G}_+[\mathfrak{p}, \mathfrak{q}_0, \mathfrak{q}_1]))^{\langle 2 \rangle}),$$
which implies that the isomorphism class of $\mathcal{Q}_2(\mathcal{G}_+[\mathfrak{p}, \mathfrak{q}_0, \mathfrak{q}_1])$ is an invariant of $X(\mathcal{G}_+[\mathfrak{p}, \mathfrak{q}_0, \mathfrak{q}_1])$. Also observe that by (4.25) the sets
$$\mathcal{O}_{(0)}^-(\mathcal{G}_+[\mathfrak{p}, \mathfrak{q}_0, \mathfrak{q}_1]), \mathcal{O}_{(0)}^+(\mathcal{G}_+[\mathfrak{p}, \mathfrak{q}_0, \mathfrak{q}_1]),$$
and
$$\mathcal{O}_{(1)}^-(\mathcal{G}_+[\mathfrak{p}, \mathfrak{q}_0, \mathfrak{q}_1]), \mathcal{O}_{(1)}^+(\mathcal{G}_+[\mathfrak{p}, \mathfrak{q}_0, \mathfrak{q}_1]),$$
are uniquely determined as the sets $\mathcal{O}_{(0)}^-, \mathcal{O}_{(0)}^+$, and $\mathcal{O}_{(1)}^-, \mathcal{O}_{(1)}^+$, such that
$$\mathcal{O}_{(0)}^- \cup \mathcal{O}_{(1)}^- = \mathcal{O}_{(0)}^-(\mathcal{G}_+[\mathfrak{p}, \mathfrak{q}_0, \mathfrak{q}_1]) \cup \mathcal{O}_{(1)}^-(\mathcal{G}_+[\mathfrak{p}, \mathfrak{q}_0, \mathfrak{q}_1]),$$
$$\mathcal{O}_{(0)}^+ \cup \mathcal{O}_{(1)}^+ = \mathcal{O}_{(0)}^+(\mathcal{G}_+[\mathfrak{p}, \mathfrak{q}_0, \mathfrak{q}_1]) \cup \mathcal{O}_{(1)}^+(\mathcal{G}_+[\mathfrak{p}, \mathfrak{q}_0, \mathfrak{q}_1]),$$
and
$$(\mathcal{O}_{(0)}^- \times \mathcal{O}_{(0)}^+) \cap \mathcal{Q}(\mathcal{G}_+[\mathfrak{p}, \mathfrak{q}_0, \mathfrak{q}_1]) \in \rho^\triangle(\mathcal{O}_{(0)}^-, O_{(0)}^+),$$
$$(\mathcal{O}_{(1)}^- \times \mathcal{O}_{(1)}^+) \cap \mathcal{Q}(\mathcal{G}_+[\mathfrak{p}, \mathfrak{q}_0, \mathfrak{q}_1]) \in \rho^\triangle(\mathcal{O}_{(1)}^-, \mathcal{O}_{(1)}^+),$$
and it follows that also the isomorphism classes of the relation $\mathcal{Q}_2^{(0)}(\mathcal{G}_+[\mathfrak{p}, \mathfrak{q}_0, \mathfrak{q}_1])$ and of the relation $\mathcal{Q}^{(1)}(\mathcal{G}_+[\mathfrak{p}, \mathfrak{q}_0, \mathfrak{q}_1])$ are invariants of $X(\mathcal{G}_+[\mathfrak{p}, \mathfrak{q}_0, \mathfrak{q}_1])$. By (4.24)
$$\mathrm{card}(\mathcal{E}^-(\mathfrak{p}, \mathfrak{q}_0)) = D^-(\mathcal{Q}_2^{(0)}(\mathcal{G}_+[\mathfrak{p}, \mathfrak{q}_0, \mathfrak{q}_1]))),$$

$$\mathrm{card}(\mathcal{E}^+(\mathfrak{p}, \mathfrak{q}_0)) = D^+(\mathcal{Q}_2^{(0)}(\mathcal{G}_+[\mathfrak{p}, \mathfrak{q}_0, \mathfrak{q}_1]))),$$

$$\mathrm{card}(\mathcal{E}^-(\mathfrak{p}, \mathfrak{q}_1)) = D^-(\mathcal{Q}_2^{(1)}(\mathcal{G}_+[\mathfrak{p}, \mathfrak{q}_0, \mathfrak{q}_1]))),$$

$$\mathrm{card}(\mathcal{E}^+(\mathfrak{p}, \mathfrak{q}_1)) = D^+(\mathcal{Q}_2^{(1)}(\mathcal{G}_+[\mathfrak{p}, \mathfrak{q}_0, \mathfrak{q}_1]))),$$

and

$$\mu(\mathcal{R}(\mathfrak{q}_0, \mathfrak{p})) = \frac{\mu(\mathcal{Q}_2^{(0)}(\mathcal{G}_+[\mathfrak{p}, \mathfrak{q}_0, \mathfrak{q}_1])))}{D^-(\mathcal{Q}_2^{(0)}(\mathcal{G}_+[\mathfrak{p}, \mathfrak{q}_0, \mathfrak{q}_1])))D^+(\mathcal{Q}_2^{(0)}(\mathcal{G}_+[\mathfrak{p}, \mathfrak{q}_0, \mathfrak{q}_1])))},$$

$$\mu(\mathcal{R}(\mathfrak{q}_1, \mathfrak{p})) = \frac{\mu(\mathcal{Q}_2^{(1)}(\mathcal{G}_+[\mathfrak{p}, \mathfrak{q}_0, \mathfrak{q}_1])))}{D^-(\mathcal{Q}_2^{(1)}(\mathcal{G}_+[\mathfrak{p}, \mathfrak{q}_0, \mathfrak{q}_1])))D^+(\mathcal{Q}_2^{(1)}(\mathcal{G}_+[\mathfrak{p}, \mathfrak{q}_0, \mathfrak{q}_1])))},$$

and therefore also the isomorphism classes of the relations $\mathcal{R}(\mathfrak{q}_0, \mathfrak{p})$ and $\mathcal{R}(\mathfrak{q}_1, \mathfrak{p})$ are invariants of $X(\mathcal{G}_+[\mathfrak{p}, \mathfrak{q}_0, \mathfrak{q}_1])$.

We note that

$$(4.27) \qquad I_2^-(X(\mathcal{G}_+[\mathfrak{p}, \mathfrak{q}_0, \mathfrak{q}_1])) > I_1^-(X(\mathcal{G}_+[\mathfrak{p}, \mathfrak{q}_0, \mathfrak{q}_1]))(I_1^-(X(\mathcal{G}_+[\mathfrak{p}, \mathfrak{q}_0, \mathfrak{q}_1]) - 1),$$

$$(4.28) \qquad \mathcal{Q}_2(X(\mathcal{G}_+[\mathfrak{p}, \mathfrak{q}_0, \mathfrak{q}_1]))) \notin \rho^{\triangle}(\mathcal{O}_2^-(\mathcal{G}_+[\mathfrak{p}, \mathfrak{q}_0, \mathfrak{q}_1]), \mathcal{O}_2^+(\mathcal{G}_+[\mathfrak{p}, \mathfrak{q}_0, \mathfrak{q}_1])).$$

The semigroup that is associated to the subshift $X(\mathcal{G}_+[\mathfrak{p}, \mathfrak{q}_0, \mathfrak{q}_1])$ is isomorphic to the $\mathcal{R}$-graph semigroup of a one-vertex $\mathcal{R}$-graph with a relation that is isomorphic to $\widehat{\mathcal{R}}(\mathfrak{p}, \mathfrak{p}) \oplus \widehat{\mathcal{R}}(\mathfrak{q}_0, \mathfrak{p}) \oplus \widehat{\mathcal{R}}(\mathfrak{q}_1, \mathfrak{p})$.

By $\mathcal{G}_0[\mathfrak{p}, \mathfrak{q}_0, \mathfrak{q}_1]$ we denote an $\mathcal{R}$-graph $\mathcal{G}_{\mathcal{R}}(\{\mathfrak{p}, \mathfrak{q}_0, \mathfrak{q}_1\}, \mathcal{E}^-, \mathcal{E}^+)$ such that

$$\mathcal{E}^-(\mathfrak{q}_0, \mathfrak{q}_1) = \emptyset, \quad \mathcal{E}^-(\mathfrak{q}_1, \mathfrak{q}_0) = \emptyset,$$

and

$$\mathcal{E}^-(\mathfrak{p}, \mathfrak{q}_0) \neq \emptyset, \quad \mathcal{E}^-(\mathfrak{p}, \mathfrak{q}_1) \neq \emptyset,$$

$$(4.29) \qquad \mathcal{R}(\mathfrak{p}, \mathfrak{q}_0) = \mathcal{E}^-(\mathfrak{p}, \mathfrak{q}_0) \times \mathcal{E}^+(\mathfrak{p}, \mathfrak{q}_0), \quad \mathcal{R}(\mathfrak{p}, \mathfrak{q}_1) = \mathcal{E}^-(\mathfrak{p}, \mathfrak{q}_1) \times \mathcal{E}^+(\mathfrak{p}, \mathfrak{q}_1),$$

$$(4.30) \quad \mathcal{R}(\mathfrak{q}_0, \mathfrak{p}) \in \rho^{\triangledown}(\mathcal{E}^-(\mathfrak{q}_0, \mathfrak{p}), \mathcal{E}^+(\mathfrak{q}_0, \mathfrak{p})), \quad \mathcal{R}(\mathfrak{q}_1, \mathfrak{p}) \in \rho^{\triangledown}(\mathcal{E}^-(\mathfrak{q}_1, \mathfrak{p}), \mathcal{E}^+(\mathfrak{q}_1, \mathfrak{p})),$$

and such that

$$\mathrm{card}(\mathcal{E}^-(\mathfrak{p}, \mathfrak{q}_0)) = \mathrm{card}(\mathcal{E}^-(\mathfrak{p}, \mathfrak{q}_1)), \qquad \mathrm{card}(\mathcal{E}^+(\mathfrak{p}, \mathfrak{q}_0)) = \mathrm{card}(\mathcal{E}^+(\mathfrak{p}, \mathfrak{q}_1)),$$

setting

$$X(\mathcal{G}_0(\{\mathfrak{p}, \mathfrak{q}_0, \mathfrak{q}_1\}) = X(\mathcal{E}^- \cup \mathcal{E}^+, \{\mathfrak{p}, \mathfrak{q}_0, \mathfrak{q}_1\}, \mathrm{id}_{\mathcal{E}^- \cup \mathcal{E}^+}).$$

Proposition 4.2. *Let there be given $\mathcal{R}$-graphs*

$$\mathcal{G}_0[\mathfrak{p}, \mathfrak{q}_0, \mathfrak{q}_1] = \mathcal{G}_{\mathcal{R}}(\{\mathfrak{p}, \mathfrak{q}_0, \mathfrak{q}_1\}, \mathcal{E}^-, \mathcal{E}^+)$$

and

$$\widetilde{\mathcal{G}}_0[\widetilde{\mathfrak{p}}, \widetilde{\mathfrak{q}}_0, \widetilde{\mathfrak{q}}_1] = \mathcal{G}_{\widetilde{\mathcal{R}}}(\{\widetilde{\mathfrak{p}}, \widetilde{\mathfrak{q}}_0, \widetilde{\mathfrak{q}}_1\}, \widetilde{\mathcal{E}}^-, \widetilde{\mathcal{E}}^+).$$

For the subshifts $X(\mathcal{G}_0[\widetilde{\mathfrak{p}}, \widetilde{\mathfrak{q}}_0, \widetilde{\mathfrak{q}}_1])$ and $X(\widetilde{\mathcal{G}}_0[\widetilde{\mathfrak{p}}, \widetilde{\mathfrak{q}}_0, \widetilde{\mathfrak{q}}_1])$ to be topologically conjugate it is necessary and sufficient that the relation $\mathcal{R}(\mathfrak{p}, \mathfrak{p})$ is isomorphic to the relation $\widetilde{\mathcal{R}}(\widetilde{\mathfrak{p}}, \widetilde{\mathfrak{p}})$ and that

$$\mu(\mathcal{R}(\mathfrak{q}_0, \mathfrak{p})) + \mu(\mathcal{R}(\mathfrak{q}_1, \mathfrak{p})) = \mu(\mathcal{R}(\widetilde{\mathfrak{q}}_0, \widetilde{\mathfrak{p}})) + \mu(\mathcal{R}(\widetilde{\mathfrak{q}}_1, \widetilde{\mathfrak{p}})).$$

Proof. We prove necessity. We denote the common value of $\mathrm{card}(\mathcal{E}^-(\mathfrak{p}, \mathfrak{q}_0))$ and $\mathrm{card}(\mathcal{E}^-(\mathfrak{p}, \mathfrak{q}_1))$ $(\mathrm{card}(\mathcal{E}^+(\mathfrak{p}, \mathfrak{q}_0))$ and $\mathrm{card}(\mathcal{E}^+(\mathfrak{p}, \mathfrak{q}_1)))$ by $A^-(A^+)$. The relation $\mathcal{R}_1(X(\mathcal{G}_0[\mathfrak{p}, \mathfrak{q}_0, \mathfrak{q}_1]))$ is isomorphic to the relation $\mathcal{R}(\mathfrak{p}, \mathfrak{p})$, and therefore its isomorphism class is an invariant of the subshifts $X(\mathcal{G}_+[\mathfrak{p}, \mathfrak{q}_0, \mathfrak{q}_1])$. For an $\mathcal{R}$-graph

$\mathcal{G}_0[\mathfrak{p}, \mathfrak{q}_0, \mathfrak{q}_1]$ denote by $\mathcal{O}_{(0)}^-$ ($\mathcal{O}_{(1)}^-$) the set of $O^- \in \mathcal{O}_2^-(X(\mathcal{G}_+[\mathfrak{p}, \mathfrak{q}_0, \mathfrak{q}_1]))$ that contain the points that carry a bi-infinite concatenation of a word in $\mathcal{E}^-(\mathfrak{p}, \mathfrak{q}_0)\mathcal{E}^-(\mathfrak{q}_0, \mathfrak{p})$ ($\mathcal{E}^-(\mathfrak{p}, \mathfrak{q}_1)\mathcal{E}^-(\mathfrak{q}_1, \mathfrak{p})$), and set

$$\mathcal{Q}_2(\mathcal{G}_0[\mathfrak{p}, \mathfrak{q}_0, \mathfrak{q}_1])) =$$
$$(\mathcal{R}_2(X(\mathcal{G}_0[\mathfrak{p}, \mathfrak{q}_0, \mathfrak{q}_1])) \cap (\mathcal{O}_{(0)}^- \times \mathcal{O}_{(0)}^+) \cup (\mathcal{R}_2(X(\mathcal{G}_0[\mathfrak{p}, \mathfrak{q}_0, \mathfrak{q}_1])) \cap (\mathcal{O}_{(1)}^- \times \mathcal{O}_{(1)}^+).$$

It is

$$\mu(\mathcal{R}_2(X(\mathcal{G}_0[\mathfrak{p}, \mathfrak{q}_0, \mathfrak{q}_1]) \ominus \mathcal{Q}_2(\mathcal{G}_0[\mathfrak{p}, \mathfrak{q}_0, \mathfrak{q}_1]))) = \tfrac{1}{2}\mu(\mathcal{R}_1(X(\mathcal{G}_0[\mathfrak{p}, \mathfrak{q}_0, \mathfrak{q}_1]))^{\langle 2 \rangle}),$$

which implies that the isomorphism class of $\mathcal{Q}_2(\mathcal{G}_0[\mathfrak{p}, \mathfrak{q}_0, \mathfrak{q}_1])$ is an invariant of the subshifts $X(\mathcal{G}_+[\mathfrak{p}, \mathfrak{q}_0, \mathfrak{q}_1])$. By (4.29 - 30)

$$\mathcal{Q}_2(\mathcal{G}_0[\mathfrak{p}, \mathfrak{q}_0, \mathfrak{q}_1])) \in \rho^\triangle(\mathcal{O}_{(0)}^- \cup \mathcal{O}_{(1)}^-, \mathcal{O}_{(0)}^+ \cup \mathcal{O}_{(1)}^+),$$

and therefore

$$A^- = D^-(\mathcal{Q}_2(\mathcal{G}_0[\mathfrak{p}, \mathfrak{q}_0, \mathfrak{q}_1])), \quad A^+ = D^+(\mathcal{Q}_2(\mathcal{G}_0[\mathfrak{p}, \mathfrak{q}_0, \mathfrak{q}_1])),$$

and

$$\mu(\mathcal{R}(\mathfrak{q}_0, \mathfrak{p}) \cup \mathcal{R}(\mathfrak{q}_1, \mathfrak{p})) = \frac{\mu(\mathcal{Q}_2(\mathcal{G}_0[\mathfrak{p}, \mathfrak{q}_0, \mathfrak{q}_1])}{A^- A^+},$$

and it is seen that the isomorphism class of the relation $\mathcal{R}(\mathfrak{q}_0, \mathfrak{p}) \oplus \mathcal{R}(\mathfrak{q}_0, \mathfrak{p})$ is an invariant of the subshifts $X(\mathcal{G}_+[\mathfrak{p}, \mathfrak{q}_0, \mathfrak{q}_1])$.

For the proof of sufficiency choose any $\mathcal{R}_\circ^\circ \in \Xi$ such that

$$\mu_{\mathcal{R}_\circ^\circ}(\mathcal{R}(\mathfrak{q}_0, \mathfrak{p})) + \mu_{\mathcal{R}_\circ^\circ}(\mathcal{R}(\mathfrak{q}_1, \mathfrak{p})) > 0.$$

Without loss of generality one can assume that $\mu_{\mathcal{R}_\circ^\circ}(\mathcal{R}(\mathfrak{q}_0, \mathfrak{p})) > 0$. Let $\mathcal{E}_\circ^- \subset \mathcal{E}^-(\mathfrak{q}_0, \mathfrak{p})), \mathcal{E}_\circ^+ \subset \mathcal{E}^+(\mathfrak{q}_0, \mathfrak{p}))$ be such that $\mathcal{R}(\mathfrak{q}_0, \mathfrak{p}) \cap (\mathcal{E}_\circ^- \times \mathcal{E}_\circ^+)$ is isomorphic to $\mathcal{R}_\circ^\circ$.

We prove that the subshift $X(\mathcal{G}_0[\mathfrak{p}, \mathfrak{q}_0, \mathfrak{q}_1])$ is topologically conjugate to a subshift $X(\widehat{\mathcal{G}}_0[\mathfrak{p}, \mathfrak{q}_0, \mathfrak{q}_1])$, where

$$\widehat{\mathcal{G}}_0[\mathfrak{p}, \mathfrak{q}_0, \mathfrak{q}_1] = \mathcal{G}_{\widehat{\mathcal{R}}}(\{\mathfrak{p}, \mathfrak{q}_0, \mathfrak{q}_1\}, \widehat{\mathcal{E}}^-, \widehat{\mathcal{E}}^+)$$

is an $\mathcal{R}$-graph such that $\widehat{\mathcal{R}}(\mathfrak{q}_0, \mathfrak{p})$ is isomorphic to $\mathcal{R}_\circ^\circ$, and such that

$$\mu_{\mathcal{R}_\circ^\circ}(\widehat{\mathcal{R}}(\mathfrak{q}_1, \mathfrak{p})) = \mu_{\mathcal{R}_\circ^\circ}(\mathcal{R}(\mathfrak{q}_0, \mathfrak{p})) + \mu_{\mathcal{R}_\circ^\circ}(\mathcal{R}(\mathfrak{q}_1, \mathfrak{p})) - 1,$$

and

$$\mu_{\mathcal{R}_\circ}(\widehat{\mathcal{R}}(\mathfrak{q}_1, \mathfrak{p})) = \mu_{\mathcal{R}_\circ}(\mathcal{R}(\mathfrak{q}_0, \mathfrak{p})) + \mu_{\mathcal{R}_\circ}(\mathcal{R}(\mathfrak{q}_1, \mathfrak{p})), \quad \mathcal{R}_\circ \neq \mathcal{R}_\circ^\circ.$$

It is possible to set

$$\widehat{\mathcal{E}}^-(\mathfrak{q}_0, \mathfrak{p}) = \mathcal{E}_\circ^-, \quad \widehat{\mathcal{E}}^-(\mathfrak{q}_1, \mathfrak{p}) = \mathcal{E}^-(\mathfrak{q}_1, \mathfrak{p}) \cup (\mathcal{E}^-(\mathfrak{q}_0, \mathfrak{p}) \setminus \mathcal{E}_\circ^-),$$

$$\widehat{\mathcal{E}}^+(\mathfrak{q}_0, \mathfrak{p}) = \mathcal{E}_\circ^+, \quad \widehat{\mathcal{E}}^+(\mathfrak{q}_1, \mathfrak{p}) = \mathcal{E}^+(\mathfrak{q}_1, \mathfrak{p}) \cup (\mathcal{E}^+(\mathfrak{q}_0, \mathfrak{p}) \setminus \mathcal{E}_\circ^+),$$

and

$$\widehat{\mathcal{R}}(\mathfrak{q}_1, \mathfrak{p}) = \mathcal{R}(\mathfrak{q}_0, \mathfrak{p}) \cap ((\mathcal{E}^-(\mathfrak{q}_0, \mathfrak{p}) \setminus \mathcal{E}_\circ^-) \times (\mathcal{E}^+(\mathfrak{q}_0, \mathfrak{p}) \setminus \mathcal{E}_\circ^+)) \oplus \mathcal{R}(\mathfrak{q}_1, \mathfrak{p}).$$

We choose bijections

$$\psi^- : \mathcal{E}^-(\mathfrak{p}, \mathfrak{q}_0) \to \mathcal{E}^-(\mathfrak{p}, \mathfrak{q}_1), \quad \psi^+ : \mathcal{E}^+(\mathfrak{p}, \mathfrak{q}_0) \to \mathcal{E}^+(\mathfrak{p}, \mathfrak{q}_1).$$

A topological conjugacy of $X(\mathcal{G}_0[\mathfrak{p}, \mathfrak{q}_0, \mathfrak{q}_1])$ onto $X(\widehat{\mathcal{G}}_0[\mathfrak{p}, \mathfrak{q}_0, \mathfrak{q}_1])$ is given by the mapping that replaces in the points of $X(\mathcal{G}_0[\mathfrak{p}, \mathfrak{q}_0, \mathfrak{q}_1])$ every symbol $e^- \in \mathcal{E}^-(\mathfrak{p}, \mathfrak{q}_0)$ that is followed by a symbol in $\mathcal{E}^-(\mathfrak{q}_0, \mathfrak{p})$ by $\psi^-(e^-)$, and every symbol $e^+ \in \mathcal{E}^+(\mathfrak{p}, \mathfrak{q}_0)$ that is preceded by a symbol in $\mathcal{E}^+(\mathfrak{q}_0, \mathfrak{p})$ by $\psi^+(e^+)$. $\qquad\square$

We note that

$$(4.31) \qquad I_2^- (X(\mathcal{G}_0[\mathfrak{p}, \mathfrak{q}_0, \mathfrak{q}_1])) > I_1^- (X(\mathcal{G}_0[\mathfrak{p}, \mathfrak{q}_0, \mathfrak{q}_1]))(I_1^- (X(\mathcal{G}_0[\mathfrak{p}, \mathfrak{q}_0, \mathfrak{q}_1])) - 1),$$

$$(4.32) \quad I_2^0(X(\mathcal{G}_0[\mathfrak{p}, \mathfrak{q}_0, \mathfrak{q}_1])) = \mathrm{card}(\mathcal{R}_1(X(\mathcal{G}_0[\mathfrak{p}, \mathfrak{q}_0, \mathfrak{q}_1]))) +$$
$$\frac{\mathrm{card}(\mathcal{Q}(X(\mathcal{G}_0[\mathfrak{p}, \mathfrak{q}_0, \mathfrak{q}_1])))}{D^-(\mathcal{Q}_2(X(\mathcal{G}_0[\mathfrak{p}, \mathfrak{q}_0, \mathfrak{q}_1])))D^+(\mathcal{Q}_2(X(\mathcal{G}_0[\mathfrak{p}, \mathfrak{q}_0, \mathfrak{q}_1])))} +$$
$$2D^-(\mathcal{Q}(X(\mathcal{G}_0[\mathfrak{p}, \mathfrak{q}_0, \mathfrak{q}_1])))D^+(\mathcal{Q}_2(X(\mathcal{G}_0[\mathfrak{p}, \mathfrak{q}_0, \mathfrak{q}_1]))),$$

$$(4.33) \qquad \mathcal{Q}_2(\mathcal{G}_0[\mathfrak{p}, \mathfrak{q}_0, \mathfrak{q}_1]) \in \rho^\triangle(\mathcal{O}_2^-(\mathcal{G}_0[\mathfrak{p}, \mathfrak{q}_0, \mathfrak{q}_1]), \mathcal{O}_2^+(\mathcal{G}_0[\mathfrak{p}, \mathfrak{q}_0, \mathfrak{q}_1]).$$

The semigroup that is associated to the subshift $X(\mathcal{G}_0[\mathfrak{p}, \mathfrak{q}_0, \mathfrak{q}_1])$ can be described just like the semigroup that is associated to the subshift $X(\mathcal{G}_+[\mathfrak{p}, \mathfrak{q}_0, \mathfrak{q}_1])$, as the $\mathcal{R}$-graph semigroup of a one-vertex $\mathcal{R}$-graph with a relation that is isomorphic to $\widehat{\mathcal{R}}(\mathfrak{p}, \mathfrak{p}) \oplus \widehat{\mathcal{R}}(\mathfrak{q}_0, \mathfrak{p}) \oplus \widehat{\mathcal{R}}(\mathfrak{q}_1, \mathfrak{p})$.

Denote by $\mathcal{G}_+^{(+)}[\mathfrak{p}, \mathfrak{q}, \mathfrak{r}](\mathcal{G}_+^{(0)}[\mathfrak{p}, \mathfrak{q}, \mathfrak{r}])$ an $\mathcal{R}$-graph of type $\mathcal{G}_+[\mathfrak{p}, \mathfrak{q}, \mathfrak{r}])$ such that $\mathcal{E}^-(\mathfrak{p}, \mathfrak{p}) \neq \emptyset(\mathcal{E}^-(\mathfrak{p}, \mathfrak{p}) = \emptyset)$, and denote by $\mathcal{G}_0^{(+)}[\mathfrak{p}, \mathfrak{q}_0, \mathfrak{q}_1]$ $(\mathcal{G}_0^0[\mathfrak{p}, \mathfrak{q}_0, \mathfrak{q}_1])$ an $\mathcal{R}$-graph of type $\mathcal{G}_0[\mathfrak{p}, \mathfrak{q}_0, \mathfrak{q}_1]$ such that $\mathcal{E}^-(\mathfrak{p}, \mathfrak{p}) \neq \emptyset$ $(\mathcal{E}^-(\mathfrak{p}, \mathfrak{p}) = \emptyset)$. We note that

$$(4.34) \qquad \mathrm{card}(\mathcal{O}_{3,1}(X(\mathcal{G}_+^{(+)}[\mathfrak{p}, \mathfrak{q}, \mathfrak{r}])) = \mathrm{card}(\mathcal{E}^-(\mathfrak{p}, \mathfrak{q}))\mathrm{card}(\mathcal{E}^+(\mathfrak{p}, \mathfrak{q})),$$

$$(4.35) \qquad \mathrm{card}(\mathcal{O}_{3,1}(X(\mathcal{G}_0^{(+)}[\mathfrak{p}, \mathfrak{q}_0, \mathfrak{q}_1])) = 2\,\mathrm{card}(\mathcal{E}^-(\mathfrak{p}, \mathfrak{q}_0))\mathrm{card}(\mathcal{E}^+(\mathfrak{p}, \mathfrak{q}_0)),$$

$$(4.36) \qquad\qquad I_3^- (X(\mathcal{G}_+^0[\mathfrak{p}, \mathfrak{q}, \mathfrak{r}])) > 0,$$

$$(4.37) \qquad\qquad I_3^- (X(\mathcal{G}_0^0[\mathfrak{p}, \mathfrak{q}_0, \mathfrak{q}_1])) = 0,$$

$$(4.38) \quad I_3^- (X(\mathcal{G}_+^{(+)}[\mathfrak{p}, \mathfrak{q}, \mathfrak{r}])) >$$
$$(I_2^- (X(\mathcal{G}_+^{(+)}[\mathfrak{p}, \mathfrak{q}, \mathfrak{r}])) - (I_1^- (X(\mathcal{G}_+^{(+)}[\mathfrak{p}, \mathfrak{q}, \mathfrak{r}])) - 1)^2)I_1^- (X(\mathcal{G}_+^{(+)}[\mathfrak{p}, \mathfrak{q}, \mathfrak{r}])) +$$
$$\mathrm{card}(\mathcal{E}^-(\mathfrak{p}, \mathfrak{q}))\mathrm{card}(\mathcal{E}^+(\mathfrak{p}, \mathfrak{q}))I_1^- (X(\mathcal{G}_+^{(+)}[\mathfrak{p}, \mathfrak{q}, \mathfrak{r}])) +$$
$$\tfrac{1}{3}I_1^- (X(\mathcal{G}_+^{(+)}[\mathfrak{p}, \mathfrak{q}, \mathfrak{r}]))(I_1^- (X(\mathcal{G}_+^{(+)}[\mathfrak{p}, \mathfrak{q}, \mathfrak{r}])) - 1)(I_1^- (X(\mathcal{G}_+^{(+)}[\mathfrak{p}, \mathfrak{q}, \mathfrak{r}])) - 2),$$

$$(4.39) \quad I_3^- (X(\mathcal{G}_0^+[\mathfrak{p}, \mathfrak{q}_0, \mathfrak{q}_1])) >$$
$$(I_2^- (X(\mathcal{G}_0^+[\mathfrak{p}, \mathfrak{q}_0, \mathfrak{q}_1])) - (I_1^- (X(\mathcal{G}_0^+[\mathfrak{p}, \mathfrak{q}_0, \mathfrak{q}_1])) - 1)^2)I_1^- (X(\mathcal{G}_0^+[\mathfrak{p}, \mathfrak{q}_0, \mathfrak{q}_1])) +$$
$$2\,\mathrm{card}(\mathcal{E}^-(\mathfrak{p}, \mathfrak{q}))\mathrm{card}(\mathcal{E}^+(\mathfrak{p}, \mathfrak{q}))I_1^- (X(\mathcal{G}_0^+[\mathfrak{p}, \mathfrak{q}_0, \mathfrak{q}_1])) +$$
$$\tfrac{1}{3}I_1^- (X(\mathcal{G}_0^+[\mathfrak{p}, \mathfrak{q}_0, \mathfrak{q}_1]))(I_1^- (X(\mathcal{G}_0^+[\mathfrak{p}, \mathfrak{q}_0, \mathfrak{q}_1])) - 1)(I_1^- (X(\mathcal{G}_0^+[\mathfrak{p}, \mathfrak{q}_0, \mathfrak{q}_1])) - 2).$$

Taking into account, that the isomorphism class of the relation $\mathcal{Q}_2$ is determined by the isomorphism class of the relations $\mathcal{R}_1$ and $\mathcal{R}_2$ by

$$\mu(\mathcal{Q}_2) = \mu(\mathcal{R}_2) - \tfrac{1}{2}\mu(\mathcal{R}_1^{\langle 2 \rangle}),$$

it is seen from (4.1 - 2), (4.5 - 7), (4.12 - 14), (4.22 - 23), (4.27 - 28), and $(4.31 - 39)$ that the invariants $I_1^-, I_2^-, I_3^-, I_2^0$, and $\mathrm{card}(\mathcal{O}_{3,1})$ distinguish between the subshifts of the various types of $\mathcal{R}$-graphs $\mathcal{G}[\mathfrak{p}]$, $\mathcal{G}[\mathfrak{p}, \mathfrak{q}]$ and $\mathcal{G}_+[\mathfrak{p}, \mathfrak{q}, \mathfrak{r}], \mathcal{G}_0[\mathfrak{p}, \mathfrak{q}, \mathfrak{r}]$ and also $\mathcal{G}_+[\mathfrak{p}, \mathfrak{q}_0, \mathfrak{q}_1]$, $\mathcal{G}_0[\mathfrak{p}, \mathfrak{q}_0, \mathfrak{q}_1]$. Also observe, that the family of $\mathcal{R}$-graphs with up to three vertices, that we have considered, appears naturally as a sub-family of a family $\mathcal{R}$-graphs, that are unions of $\mathcal{R}$ graphs that have a ladder structure with gaps, and that have a common base point.

5. Examples II

We describe now the strongly connected directed graphs with three vertices, with one of the vertices having a single incoming edge, that have Markov-Dyck shifts with an associated semigroup, that is the inverse semigroup of a strongly connected directed graph with two vertices, the vertex set of this two-vertex graph to be denoted by $\{\alpha, \beta\}$, its edge set by $\mathcal{F}$ and its adjacency matrix by $T = \begin{pmatrix} T_{\alpha\alpha} & T_{\alpha\beta} \\ T_{\beta\alpha} & T_{\beta\beta} \end{pmatrix}$, such that

$$T_{\alpha\alpha} + T_{\beta\alpha} > 1, \quad T_{\alpha\beta} + T_{\beta\beta} > 1,$$

where we can assume that T is normalized such that $T_{\alpha\alpha} > T_{\beta\beta}$ or $T_{\alpha\alpha} = T_{\beta\beta}, T_{\alpha\beta} \geq T_{\beta\alpha}$. We will specify the directed graphs by their adjacency matrices A, denoting the graph by $\mathcal{G}(A)$, and its Markov-Dyck shift by $MD(\mathcal{G}(A))$. The graphs come in two sets. A first set contains the graphs with vertices $\mathfrak{p}_{\alpha(0)}, \mathfrak{p}_{\alpha(1)}$, and $\mathfrak{p}_\beta$, and with an adjacency matrix $A^{(\alpha)}(T, \Delta^{(\alpha)}, \Delta_\alpha)$ that is given by

$$(5.1) \quad \begin{pmatrix} A^{(\alpha)}(\mathfrak{p}_{\alpha(0)}, \mathfrak{p}_{\alpha(0)}) & A^{(\alpha)}(\mathfrak{p}_{\alpha(0)}, \mathfrak{p}_{\alpha(1)}) & A^{(\alpha)}(\mathfrak{p}_{\alpha(0)}, \mathfrak{p}_\beta) \\ A^{(\alpha)}(\mathfrak{p}_{\alpha(1)}, \mathfrak{p}_{\alpha(0)}) & A^{\alpha)}(\mathfrak{p}_{\alpha(1)}, \mathfrak{p}_{\alpha(1)}) & A^{(\alpha)}(\mathfrak{p}_{\alpha(1)}, \mathfrak{p}_\beta) \\ A^{(\alpha)}(\mathfrak{p}_\beta, \mathfrak{p}_{\alpha(0)}) & A^{(\alpha)}(\mathfrak{p}_\beta, \mathfrak{p}_{\alpha(1)}) & A^{(\alpha)}(\mathfrak{p}_\beta, \mathfrak{p}_\beta) \end{pmatrix} =$$

$$\begin{pmatrix} T_{\alpha\alpha} - \Delta^{(\alpha)} & 1 & \Delta_\alpha \\ \Delta^{(\alpha)} & 0 & T_{\alpha\beta} - \Delta_\alpha \\ T_{\beta\alpha} & 0 & T_{\beta\beta} \end{pmatrix},$$

$$0 \leq \Delta^{(\alpha)} \leq T_{\alpha\alpha}, \ 0 \leq \Delta_\alpha \leq T_{\alpha\beta}, \ \Delta^{(\alpha)} + T_{\alpha\beta} - \Delta_\alpha > 0.$$

The second set contains the graphs with vertices $\mathfrak{p}_{\beta(0)}, \mathfrak{p}_{\beta(1)}$, and $\mathfrak{p}_\alpha$, and with an adjacency matrix

$$A^{(\beta)}(T, \Delta^{(\beta)}, \Delta_\beta)), \quad 0 \leq \Delta^{(\beta)} \leq T_{\beta\beta}, 0 \leq \Delta_\beta \leq T_{\alpha\beta}, \Delta^{(\beta)} + T_{\beta\alpha} - \Delta_\beta > 0,$$

that is obtained from (5.1) by interchanging alpha with beta. By Theorem (2.3) $MD(\mathcal{G}(A^{(\alpha)}(T, \Delta^{(\alpha)}, \Delta_\alpha)))$ and $MD(\mathcal{G}(A^{(\beta)}(T, \Delta^{(\beta)}, \Delta_\beta)))$ have Property (A) and their associated semigroup is by Theorem (3.3) the inverse semigroup of the directed graph with adjacency matrix T.

One has

$$(\alpha 1) \qquad I_1^-(MD(\mathcal{G}(A^{(\alpha)}(T, \Delta^{(\alpha)}, \Delta_\alpha)))) = T_{\alpha\alpha} + T_{\beta\beta} - \Delta^{(\alpha)},$$

$$(\beta 1) \qquad I_1^-(MD(\mathcal{G}(A^{(\beta)}(T, \Delta^{(\beta)}, \Delta_\beta)))) = T_{\beta\beta} + T_{\alpha\alpha} - \Delta^{(\beta)},$$

$$(\alpha 2) \quad I_2^-(MD(\mathcal{G}(A^{(\alpha)}(T, \Delta^{(\alpha)}, \Delta_\alpha)))) =$$

$$T_{\alpha\alpha}(T_{\alpha\alpha} - 1) + T_{\beta\beta}(T_{\beta\beta} - 1) + \Delta^{(\alpha)} + \Delta_\alpha T_{\beta\alpha},$$

$$(\beta 2) \quad I_2^-(MD(\mathcal{G}(A^{(\beta)}(T, \Delta^{(\beta)}, \Delta_\beta)))) =$$

$$T_{\beta\beta}(T_{\beta\beta} - 1) + T_{\alpha\alpha}(T_{\alpha\alpha} - 1) + \Delta^{(\beta)} + T_{\alpha\beta}\Delta_\beta.$$

It follows from ($\alpha 1$) that for

$$0 \leq \Delta^{(\alpha)}, \widetilde{\Delta}^{(\alpha)} \leq T_{\alpha\alpha}, 0 \leq \Delta_\alpha, \widetilde{\Delta}_\alpha \leq T_{\alpha\beta}, \Delta^{(\alpha)} \neq \widetilde{\Delta}^{(\alpha)},$$

the subshifts $MD(\mathcal{G}(A^{(\alpha)}(T, \Delta^{(\alpha)}, \Delta_\alpha))$ and $MD(\mathcal{G}(A^{(\alpha)}(T, \widetilde{\Delta}^{(\alpha)}, \widetilde{\Delta}_\alpha))$ are not topologically conjugate, and it follows from ($\alpha 2$) that for

$$0 \leq \Delta^{(\alpha)} \leq T_{\alpha\alpha}, 0 \leq \Delta_\alpha, \widetilde{\Delta}_\alpha \leq T_{\alpha\beta}, \Delta_\alpha \neq \widetilde{\Delta}_\alpha,$$

the subshifts $MD(\mathcal{G}(A^{(\alpha)}(T, \Delta^{(\alpha)}, \Delta_\alpha))$ and $MD(\mathcal{G}(A^{(\alpha)}(T, \Delta^{(\alpha)}, \widetilde{\Delta}_\alpha))$ are not topologically conjugate. The symmetric argument that uses $(\beta 1)$ and $(\beta 2)$, gives the same result.

Denote for $k \in \mathbb{N}$ by $I_k^-(\alpha)(I_k^-(\beta))$ the number of orbits of length k with negative multiplier in $\{f \in \mathcal{F} : s(f) = t(f) = \alpha\}(\{f \in \mathcal{F} : s(f) = t(f) = \beta\})$. One has

$$(1\alpha\alpha) \qquad I_1^-(\alpha)(MD(\mathcal{G}(A^{(\alpha)}(T, \Delta^{(\alpha)}, \Delta_\alpha)))) = T_{\alpha\alpha} - \Delta^{(\alpha)},$$

$$(1\alpha\beta) \qquad I_1^-(\alpha)(MD(\mathcal{G}(A^{(\beta)}(T, \Delta^{(\beta)}, \Delta_\beta)))) = T_{\alpha\alpha},$$

$$(1\beta\alpha) \qquad I_1^-(\beta)(MD(\mathcal{G}(A^{(\alpha)}(T, \Delta^{(\alpha)}, \Delta_\alpha)))) = T_{\beta\beta},$$

$$(1\beta\beta) \qquad I_1^-(\beta)(MD(\mathcal{G}(A^{(\beta)}(T, \Delta^{(\beta)}, \Delta_\beta)))) = T_{\beta\beta} - \Delta^{(\beta)},$$

$$(3\alpha\alpha) \qquad I_3^-(\alpha)(MD(\mathcal{G}(A^{(\alpha)}(T, 0, \Delta_\alpha)))) = T_{\alpha\alpha}(T_{\alpha\alpha} + 1 + \Delta_\alpha),$$

$$(3\alpha\beta) \qquad I_3^-(\alpha)(MD(\mathcal{G}(A^{(\beta)}(T, 0, \Delta_\beta)))) = T_{\alpha\alpha}(T_{\alpha\alpha} + T_{\alpha\beta}),$$

$$(3\beta\alpha) \qquad I_3^-(\beta)(MD(\mathcal{G}(A^{(\alpha)}(T, 0, \Delta_\alpha)))) = T_{\beta\beta}(T_{\beta\beta} + T_{\beta\alpha}),$$

$$(3\beta\beta) \qquad I_3^-(\beta)(MD(\mathcal{G}(A^{(\beta)}(T, 0, \Delta_\beta)))) = T_{\beta\beta}(T_{\beta\beta} + 1 + \Delta_\beta),$$

$$(5\alpha\alpha) \quad I_5^-(\alpha)(MD(\mathcal{G}(A^{(\alpha)}(T, 0, \Delta_\alpha)))) =$$
$$T_{\alpha\alpha}\left((T_{\alpha\alpha} + 1 + \Delta_\alpha)^2 + T_{\alpha\alpha}(T_{\alpha\alpha} + 1 + \Delta_\alpha) + T_{\alpha\beta} - \Delta_\alpha + \Delta_\alpha(T_{\beta\beta} + T_{\beta\alpha})\right),$$

$$(5\alpha\beta) \quad I_5^-(\alpha)(MD(\mathcal{G}(A^{(\beta)}(T, 0, \Delta_\beta)))) =$$
$$T_{\alpha\alpha}\left((T_{\alpha\alpha} + T_{\alpha\beta})^2 + T_{\alpha\alpha}(T_{\alpha\alpha} + T_{\alpha\beta}) + T_{\alpha\beta}(T_{\beta\beta} + \Delta_\beta)\right),$$

$$(5\beta\alpha) \quad I_5^-(\beta)(MD(\mathcal{G}(A^{(\alpha)}(T, 0, \Delta_\alpha)))) =$$
$$T_{\beta\beta}\left((T_{\beta\beta} + T_{\beta\alpha})^2 + T_{\beta\beta}(T_{\beta\beta} + T_{\beta\alpha}) + T_{\beta\alpha}(T_{\alpha\alpha} + \Delta_\alpha)\right),$$

$$(5\beta\beta) \quad I_5^-(\beta)(MD(\mathcal{G}(A^{(\beta)}(T, 0, \Delta_\beta)))) =$$
$$T_{\beta\beta}\left((T_{\beta\beta} + 1 + \Delta_\beta)^2 + T_{\beta\beta}(T_{\beta\beta} + 1 + \Delta_\beta) + T_{\beta\alpha} - \Delta_\beta + \Delta_\beta(T_{\alpha\alpha} + T_{\alpha\beta})\right).$$

In the case that $T_{\alpha\alpha} = T_{\beta\beta}, T_{\alpha\beta} = T_{\beta\alpha}$, and $\Delta^{(\alpha)} = \Delta^{(\beta)}, \Delta_\alpha = \Delta_\beta$, the graphs $\mathcal{G}(A^{(\alpha)}(T, \Delta^{(\alpha)}, \Delta_\alpha)))$ and $\mathcal{G}(A^{(\beta)}(T, \Delta^{(\beta)}, \Delta_\beta))$ are isomorphic, and therefore the subshifts $MD(\mathcal{G}(A^{(\alpha)}(T, \Delta^{(\alpha)}, \Delta_\alpha)$ and $MD(\mathcal{G}(A^{(\beta)}(T, \Delta^{(\beta)}, \Delta_\beta)$ are topologically conjugate.

In the case that $T_{\alpha\alpha} > T_{\beta\beta}$ or $T_{\alpha\alpha} = T_{\beta\beta}, T_{\alpha\beta} > T_{\beta\alpha}$, an automorphism of the graph with adjacency matrix T leaves the vertices α and β fixed and therefore permutes the sets $\{f \in \mathcal{F} : s(e) = t(e) = \alpha\}$ and $\{f \in \mathcal{F} : s(e) = t(e) = \beta\}$. It follows then for this case from [Kr3, Corollary 3.2] and [HIK, Proposition 4.2] that $I_k^-(\alpha), I_k^-(\beta), k \in \mathbb{N}$, are invariants. Therefore by $(1\alpha\alpha)$ and $(1\alpha\beta)$ no subshift

$$MD(\mathcal{G}(A^{(\alpha)}(T, \Delta^{(\alpha)}, \Delta_\alpha))), \quad 0 < \Delta^{(\alpha)} \le T_{\alpha\alpha}, 0 \le \Delta_\alpha \le T_{\alpha\beta},$$

is topologically conjugate to any of the subshifts

$$MD(\mathcal{G}(A^{(\beta)}(T, \Delta^{(\beta)}, \Delta_\beta))), \quad 0 \le \Delta^{(\beta)} \le T_{\beta\beta}, \, 0 \le \Delta_\beta \le T_{\beta\alpha}, \Delta^{(\beta)} + T_{\beta\alpha} - \Delta_\beta > 0,$$

and by $(1\beta\beta)$ and $(1\beta\alpha)$ no subshift

$$MD(\mathcal{G}(A^{(\beta)}(T, \Delta^{(\beta)}, \Delta_\beta))), \quad 0 < \Delta^{(\beta)} \le T_{\beta\beta}, 0 \le \Delta_\beta \le T_{\beta\alpha},$$

is topologically conjugate to any of the subshifts

$$MD(\mathcal{G}(A^{(\alpha)}(T, \Delta^{(\alpha)}, \Delta_\alpha))), \quad 0 \le \Delta^{(\alpha)} \le T_{\alpha\alpha}, \, 0 \le \Delta_\alpha \le T_{\alpha\beta}, \Delta^{(\alpha)} + T_{\alpha\beta} - \Delta_\alpha > 0.$$

We prove that under the hypothesis that $T_{\alpha\alpha} > T_{\beta\beta}$ or $T_{\alpha\alpha} = T_{\beta\beta}, T_{\alpha\beta} > T_{\beta\alpha}$, no subshift

$$MD(\mathcal{G}(A^{(\alpha)}(T, 0, \Delta_\alpha)))$$

is topologically conjugate to a subshift

$$MD(\mathcal{G}(A^{(\beta)}(T, 0, \Delta_\beta))).$$

Assume the contrary. Then it follows from $(1\alpha\alpha)$ and $(1\beta\beta)$ (or from $(1\alpha\beta)$ and $(1\beta\alpha)$) and $(\alpha 2)$ and $(\beta 2)$ that

(5.2) $$\Delta_\alpha T_{\beta\alpha} = T_{\alpha\beta}\Delta_\beta.$$

In the the case that

$$T_{\beta\beta} = 0.$$

one has by $(5\beta\alpha)$ and $(5\beta\beta)$ that

$$(T_{\alpha\alpha} - 1)(T_{\beta\alpha} - \Delta_\beta) = 0.$$

$T_{\beta\alpha} = \Delta_\beta$ is impossible, and for the case that $T_{\alpha\alpha} = 1$ one has from $(3\alpha\alpha)$ and $(3\alpha\beta)$ that

$$1 + \Delta_\alpha = T_{\alpha\beta},$$

and then from (15.2 that $T_{\alpha\beta} = \Delta_\alpha$, which is also impossible.

In the the case that

$$T_{\beta\beta} > 0.$$

It follows from $(3\alpha\alpha)$ and $(3\alpha\beta)$, and from $(3\alpha\alpha)$ and $(3\alpha\beta)$, that

$$1 + \Delta_\alpha = T_{\alpha\beta}, \quad 1 + \Delta_\beta = T_{\beta\alpha}.$$

Fom this one has by (5.2) that

$$\Delta_\alpha = \Delta_\beta, \quad T_{\alpha\beta} = T_{\beta\alpha},$$

and from this one has by $(5\alpha\alpha)$ and $(5\alpha\beta)$ (or by $(5\beta\beta)$ and $(5\beta\alpha)$), that

$$T_{\alpha\alpha} = T_{\beta\beta},$$

which contradicts the hypothesis.

Consider now also the matrix T as the adjacency matrix of a directed graph $\mathcal{G}(T)$ with its Markov-Dyck shift $MD(\mathcal{G}(T))$. The subshift $MD(\mathcal{G}(T))$ has Property (A) and its associated semigroup is the inverse semigroup of $\mathcal{G}(T)$. One has

$$I_2^0(\mathcal{G}(T)) = T_{\alpha\alpha} + T_{\alpha\beta} + T_{\beta\alpha} + T_{\beta\beta},$$

and

$$I_2^0(MD(\mathcal{G}(A^{(\alpha)}(T, \Delta^{(\alpha)}, \Delta_\alpha)) = I_2^0(MD(\mathcal{G}^{(\beta)}(T, \Delta^{(\beta)}, \Delta_\beta)) =$$
$$T_{\alpha\alpha} + T_{\alpha\beta} + T_{\beta\alpha} + T_{\beta\beta} + 1.$$

It follows that the subshift $MD(\mathcal{G}(T))$ is not topologically conjugate to any of the subshifts $MD(\mathcal{G}^{(\alpha)}(T, \Delta^{(\alpha)}, \Delta_\alpha)$, nor to any of the subshifts $MD(\mathcal{G}^{(\beta)}(T, \Delta^{(\beta)}, \Delta_\beta)$.

6. $\mathcal{R}$-GRAPH SEMIGROUPS ASSOCIATED TO SUBSHIFTS WITH PROPERTY (A)

There exist finite strongly connected directed graphs $\mathcal{G}$ whose inverse semigroup $\mathcal{S}_{\mathcal{G}}$ is associated to topologically transitive subshifts with Property (A) that are not topologically conjugate to $\mathcal{S}_{\mathcal{G}}$-presentations [Kr3, Section 8]. However, as is seen from the following theorem, the class of $\mathcal{R}$-graph semigroups that are associated to topologically transitive subshifts with Property (A) coincides with the class of $\mathcal{R}$-graph semigroups that are associated to $\mathcal{S}_{\mathcal{R}}(\mathfrak{P}, \mathcal{E}^-, \mathcal{E}^+)$ presentations with Property (A).

For a semigroup (with zero) $\mathcal{S}$ we set

$$[F] = \{F' \in \mathcal{S} : \Gamma(F') = \Gamma(F)\}.$$

The set $[\mathcal{S}] = \{[F] : F \in \mathcal{S}\}$ with the product given by

$$[G][H] = [GH], \quad G, H \in \mathcal{S},$$

is a semigroup.

Theorem 6.1. *The $\mathcal{R}$-graph semigroup $\mathcal{S}_{\mathcal{R}}(\mathfrak{P}, \mathcal{E}^-, \mathcal{E}^+)$ is associated to a topologically transitive subshift with Property (A) if and only if the $\mathcal{R}$-graph $\mathcal{G}_{\mathcal{R}}(\mathfrak{P}, \mathcal{E}^-, \mathcal{E}^+)$ satisfies conditions $(a), (b), (c)$ and (d).*

Proof. The $\mathcal{R}$-graph $\mathcal{G}_{\mathcal{R}}(\mathfrak{P}, \mathcal{E}^-, \mathcal{E}^+)$ satisfies Condition (a) if the projection of $\mathcal{S}(\mathfrak{P}, \mathcal{E}^-, \mathcal{E}^+)$ onto $[\mathcal{S}(\mathfrak{P}, \mathcal{E}^-, \mathcal{E}^+)]$ is an isomorphism, and this is by Theorem 2.3 of [HK] necessary for the $\mathcal{R}$-graph semigroup $\mathcal{S}(\mathfrak{P}, \mathcal{E}^-, \mathcal{E}^+)$ to be associated to a topologically transitive subshift with Property (A).

To prove necessity of conditions $(b), (c)$ and (d), let $X \subset \Sigma^{\mathbb{Z}}$ be a subshift with property (A), and let $\mathcal{S}(\mathfrak{P}, \mathcal{E}^-, \mathcal{E}^+)$ be an $\mathcal{R}$-graph semigroup that is associated to X.

Let $(e_i)_{0 \leq i \leq I}, I \in \mathbb{N}$, be a cycle in $\mathcal{E}_{\mathcal{R}}^-$, and let $\mathfrak{p} = s(e_0^-)$. Let $p \in P(A(X))$ be a representative of $\mathfrak{p}$, and let $y \in Y_X$ be a representative of $\prod_{0 \leq i \leq I} e_i$ that is left asymptotic and right asymptotic to the orbit of p. The density of Y_X in X [HK, Lemma 2.1] implies that $y \in A(X)$. Therefore $[y]_{\approx} = [p]_{\approx}$, which contradicts $\prod_{0 \leq i \leq I} e_i \neq \mathbf{1}_{\mathfrak{p}}$. This proves necessity of Condition $(b-)$. The proof of the necessity of Condition $(b+)$ is symmetric.

Let $\mathfrak{p} \in \mathfrak{P}^{(1)}$ such that

$$(1) \qquad\qquad\qquad\qquad \eta(\mathfrak{p}) \neq \mathfrak{p},$$

and let $e^- \in \mathcal{E}_{\mathcal{R}}^-(\mathfrak{p})$ and $e^+ \in \mathcal{E}_{\mathcal{R}}^+(\mathfrak{p})$. Let $p \in P(A(X))$ be a representative of $\mathfrak{p}$, and let $q \in P(A(X))$ be a representative of $\eta(\mathfrak{p})$. Also, let $y(-) \in Y_X$ be a representative of e^-, that is left asymptotic to the orbit of q and right asymptotic to the orbit of p and let $y(+) \in Y_X$ be a representative of e^+, that is left asymptotic to the orbit of p and right asymptotic to the orbit of q. The density of Y_X in X implies that $y(-)$ and $y(+)$ are in $A(X)$. Therefore $[p]_{\approx} = [q]_{\approx}$ and this contradicts (1). This proves necessity of Condition (c).

Let $\mathfrak{q}, \mathfrak{r} \in \mathfrak{P}, \mathfrak{q} \neq \mathfrak{r}$, let $(e_i^-)_{0 \leq i \leq I(-)}, I(-) \in \mathbb{N}$, be a path in $\mathcal{E}_{\mathcal{R}}^-$ from $\mathfrak{q}$ to $\mathfrak{r}$ and let $(e_i^+)_{0 \leq i \leq I(+)}, I(+) \in \mathbb{N}$, be a path in $\mathcal{E}_{\mathcal{R}}^+$ from $\mathfrak{q}$ to $\mathfrak{r}$. Let $q \in P(A(X))$ be a representative of $\mathfrak{q}$, and let $r \in P(A(X))$ be a representative of $\mathfrak{r}$. Also let $y(-) \in Y_X$ be a representative of $\prod_{0 \leq i \leq I(-)} e_i^-$, that is left asymptotic to the orbit of q and right asymptotic to the orbit of r and let $y(+) \in Y_X$ be a representative of $\prod_{0 \leq i \leq I(+)} e_i^+$, that is left asymptotic to the orbit of q and right asymptotic to the orbit of r. The density of Y_X in X implies that $y(-)$ and $y(+)$ are in $A(X)$, and this means that the images under the projection of $\mathcal{S}(\mathfrak{P}, \mathcal{E}^-, \mathcal{E}^+)$ onto $[\mathcal{S}(\mathfrak{P}, \mathcal{E}^-, \mathcal{E}^-)]$ of $\prod_{0 \leq i \leq I(-)} e_i^-$ and $\prod_{0 \leq i \leq I(+)} e_i^+$ are the same. This proves necessity of Condition (d).

Sufficiency is assured by Theorem (3.3), from which it follows, that for $\mathcal{R}$-graphs $\mathcal{G}_{\mathcal{R}}(\mathfrak{P}, \mathcal{E}^-, \mathcal{E}^+)$, that satisfy conditions $(a), (b), (c)$ and (d) the subshifts $X(\mathcal{E}^- \cup \mathcal{E}^+, \mathfrak{P}, \mathrm{id}_{\mathcal{E}^- \cup \mathcal{E}^+})$ have Property (A) and have as their associated semigroup the $\mathcal{R}$-graph semigroup $\mathcal{S}_{\mathcal{R}}(\mathfrak{P}, \mathcal{E}^-, \mathcal{E}^+)$. $\qquad\square$

Examples of $\mathcal{R}$-graphs $\mathcal{G}_{\mathcal{R}}(\mathfrak{P}, \mathcal{E}^-, \mathcal{E}^+)$ that satisfy conditions $(a), (b), (c)$ and (d), are

$$\mathfrak{P} = \{\mathfrak{p}\}, \quad \mathcal{E}^- = \{\alpha^-, \beta^-\}, \quad \mathcal{E}^+ = \{\alpha^+, \beta^+\},$$
$$\mathcal{R} = \{(\alpha^-, \alpha^+), (\alpha^-, \beta^+), (\beta^-, \alpha^+)\},$$

and

$$\mathfrak{P} = \{\mathfrak{p}\}, \quad \mathcal{E}^- = \{\alpha^-, \beta^-, \gamma^-\}, \quad \mathcal{E}^+ = \{\alpha^+, \beta^+, \gamma^+\},$$
$$\mathcal{R} = (\mathcal{E}^- \times \mathcal{E}^+) \setminus \{(\alpha^-, \alpha^+), (\beta^-, \beta^+)(\gamma^-, \gamma^+)\}.$$

Conditions $(a), (b), (c)$ and (d) are satisfied in the case of the graph inverse semigroups of finite directed graphs in which every vertex has at least two incoming edges. This is the case that was considered in [HIK]. More generally, these conditions are also satisfied in the case of $\mathcal{R}$-graph semigroups $\mathcal{S}_{\mathcal{R}}(\mathfrak{P}, \mathcal{E}^-, \mathcal{E}^+)$ where the $\mathcal{R}$-graph $\mathcal{G}_{\mathcal{R}}(\mathfrak{P}, \mathcal{E}^-, \mathcal{E}^+)$ is such that

$$\mathcal{E}^-(\mathcal{R}(\eta(\mathfrak{p}), \mathfrak{p})) = \emptyset, \ \mathcal{E}^+(\mathcal{R}(\eta(\mathfrak{p}), \mathfrak{p})) = \emptyset, \quad \mathfrak{p} \in \mathfrak{P}^{(1)}.$$

This is the case that was considered in [Kr4].

For $\mathcal{R}$-graph semigroups $\mathcal{S}_{\mathcal{R}}(\mathfrak{P}, \mathcal{E}^-, \mathcal{E}^+)$ that satisfy conditions $(a), (b), (c)$ and (d) the subshifts $X(\mathcal{E}^- \cup \mathcal{E}^+, \mathfrak{P}, \mathrm{id}_{\mathcal{E}^- \cup \mathcal{E}^+})$ are topologically conjugate if and only if their associated semigroups are isomorphic, if and only if the $\mathcal{R}$-graphs $\mathcal{G}_{\mathcal{R}}(\mathfrak{P}, \mathcal{E}^-, \mathcal{E}^+)$ are isomorphic (see [Kr3, Kr4]). By this remark and by the results of section 4 and 5 we have also shown that the Markov-Dyck shifts that arise from strongly connected directed graphs with up to three vertices are topologically conjugate if and only if the directed graphs are isomorphic.

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Toshihiro Hamachi
Faculty of Mathematics
Kyushu University
744 Motooka, Nishi-ku, Fukuoka 819-0395
Japan
hamachi@math.kyushu-u.ac.jp

Wolfgang Krieger
Institute for Applied Mathematics
University of Heidelberg
Im Neuenheimer Feld 294, 69120 Heidelberg
Germany
krieger@math.uni-heidelberg.de