Representation theory of finite semigroups and combinatorial applications

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Introduction

Definition (Representation)

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Definition (Module)

A representation gives a linear action of M on the vector space V by $mv = f(m)v$ for $m \in M, v \in V$. We say that V is an $M-$module.

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- $1v = v$ for all $v \in V$
- $m(nv) = (mn)v$ for all $m, n \in M, v \in V$
- $m(v + w) = mv + mw$, for all $m \in M, v, w \in V$
- $m(cv) = c(mv)$, for all $m \in V, c \in K, v \in V$

then the assignment of $m \in M$ to the function $v \mapsto mv$ is a morphism $f : M \to End(V)$.

The Monoid Algebra

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As algebras, $KM \approx K_0M \times K$.

Definition

(Module Morphism) Let M be a monoid, K a field and V and W M-modules. An M-module morphism is a linear transformation $f : V \to W$ such that $f(mv) = mf(v)$ for all $m \in M$, $v \in V$.

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Every M-module morphism linearly extends to a KM-module morphism and every KM-module morphism restricts to a M-module morphism. This leads to an equivalence of the category ${}_{M}Mod$ of M-modules and ${}_{KM}Mod$ of KM-modules.

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The "modern" definition of the Representation Theory of a monoid M is to "describe" the category ${}_{M}Mod$

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$$\begin{pmatrix} S_1 & 0 & \ldots & 0 \\ 0 & S_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & S_n \end{pmatrix}$$

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2. *KM is isomorphic to a direct product $M_{n_1}(K) \times ... \times M_{n_r}(K)$ of matrix algebras over K.*
3. *The number r is equal to the number of distinct simple M modules and also to the number of conjugacy classes of M and n_i is the dimension of S_i.*

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The second step is via Character Theory, where the character $\chi_f : M \to K$ of a representation $f : M \to M_n(K)$ is $\chi_f(m) =$Trace$(f(m))$. The simple characters form an orthonormal basis for an inner product associated to characters.

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since if M is a finite group, then up to category equivalence, $\mathrm{Obj}({}_M\mathrm{Mod}) = \mathbf{N}^r$ and $\mathrm{Hom}((m_1, \ldots m_r), (n_1, \ldots n_r))$ is the space of $(X_1, \ldots X_r)$, where X_i is an $n_i \times m_i$ matrix over K.

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By choosing a basis for V according to a composition series (the Jordan-Holder Theorem holds for M-modules), a matrix representation is block triangular:

$$\begin{pmatrix} S_1 & T_{1,2} & \dots & T_{1,n} \\ 0 & S_2 & \dots & T_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & S_n \end{pmatrix}$$

where the S_i are simple and $T_{i,j}$ gives information of how to glue S_j and S_i together.

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The simple modules are determined by the Munn-Ponizovsky Theorem, which we recall here. The second part is encoded by the "quiver", which is a combinatorial/homological object associated to KM. (There is another approach via the Krull-Schmidt Theorem that classifies indecomposable modules and minimal morphisms between them).

Munn-Ponizovky Theorem

Theorem (Munn-Ponizovky 1956)

Let M *be a finite monoid. There is a 1-1 correspondence between simple* M*-modules and pairs* (J, V) *where* J *is a regular* $\mathcal{J}$*-class of* M *and* V *is a simple module of a (fixed) maximal subgroup* G *of* J*.*

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Theorem

With the notation above, eS is a simple G-module. (Apex(S), eS) is the Munn-Ponizovsky pair associated to S.

Conversely, let J be a regular $\mathcal{J}$-class of M and let G be a maximal subgroup of J with identity e. Then a simple G-module V induces a simple M-module by induction (or co-induction) via the following steps. This proof scheme summarizes that of O. Ganyushkin, V. Mazorchuk and B. Steinberg, based on a Lemma of Green.

1. Let L be the $\mathcal{L}$-class of e. L acts by partial functions on the left of L (left Schutzenberger representation) and G acts on the right of L by permutations. Thus KL is an $M - G$-bimodule.

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2. Then Ind$(KL) = KL \bigotimes_{KG} V$, the M-module induced by V has a unique maximal submodule (its Radical). The quotient by the Radical of Ind(KL) is the unique simple M module S with Apex(S)=J and $eS = V$.

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3. Dually, if R is the $\mathcal{R}$-class of e, then KR is a $G-M$ bimodule and the M-module Coind(V) = Hom$_{KG}(KR, V)$ has a unique minimal submodule S((its Socle) which is the unique simple M module S with Apex(S)=J and $eS = V$.

Computing the Simple Modules

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2. It follows that KL (KR) is a free right (left) KG module. $KL \approx KG^r$ $(KR \approx KG^l)$, where r is the number of $\mathcal{R}$-classes ($\mathcal{L}$-classes) in L (R).

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1. The matrix $C \bigotimes \rho$ (substitute $\rho(g)$ wherever g appears in C and an $n \times n$ 0-matrix wherever 0 appears in C) defines a linear transformation $f_V = C \bigotimes \rho : V^r \to V^l$. By the previous identifications it is also an M-module morphism $f_V : Ind(V) \to Coind(V)$.

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2. The simple M-module corresponding to V in the Munn-Ponizovsky correspondence is isomorphic to both $Ind(V)/Ker(f_V)$ (Lallement-Petrich) and to $Im(f_V)$ (Rhodes-Zalcstein).

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Theorem

Let M be a finite monoid and K a field that doesn't divide the order of any subgroup of M. Then KM is semisimple if and only if M is regular and every structure matrix C is invertible over the algebra KG, where G is the maximal subgroup of the J-class of C.

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Corollary

Let M be a finite inverse monoid and K a field that doesn't divide the order of any maximal subgroup. Then KM is semisimple.

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The following is due to Okninski-Putcha, as well as Kovacs in the case of the full matrix monoid.

Theorem

Let F be a finite field. Then the full matrix monoid $M_n(F)$ has a semisimple algebra over a field whose characteristic doesn't divide the order of any maximal subgroup. More generally, the same is true for any finite monoid of Lie type.

Basic Algebras

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Theorem (Algebras are basic up to Morita Equivalence)

Let A be a finite dimensional algebra over $\mathbb{K}$. Then there is a unique finite dimensional basic algebra B such that ${}_A$-Mod is equivalent to ${}_B$-Mod.

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1. A *is a finite dimensional basic algebra over* $\mathbb{K}$.
2. $A/\operatorname{rad}(A) \cong \mathbb{K}^n$, *where* $n = dim(A)$.
3. *Every simple module of* A *is 1-dimensional.*
4. A *has a faithful representation by triangular matrices over* $\mathbb{K}$.

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Definition

A completely simple semigroup is rectangular if its idempotents form (a necessarily rectangular) band.

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Thus, a completely simple semigroup S is rectangular if and only if it is isomorphic to a Rees matrix semigroup over a group and with structure matrix, the matrix of all 1's. Equivalently, S is isomorphic to the direct product of a group and a rectangular band.

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Definition
A finite monoid is a rectangular monoid if all of its regular $\mathcal{D}$-classes are rectangular completely simple semigroups.

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1. It is well known from group theory that a finite group G has all simple modules 1-dimensional if and only if G is Abelian.
2. One sees without difficult that the structure matrix of a regular $\mathcal{J}$-class has rank 1 if and only if the corresponding principal factor is rectangular.

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Thus all the simple modules with Apex the identity $\mathcal{J}$-class are minimal elements in this partial order and all the simple modules with Apex the minimal ideal of M are maximal elements in this poset. Simple modules with the same Apex are not comparable.

Theorem (Nico 1975, Putcha 1990)

If M is a finite regular monoid, then KM is a quasihereditary algebra with respect to this partial order. An arbitrary finite monoid is a stratified algebra with respect to this partial order.

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Remark

It follows in particular that if M is a finite regular monoid, then KM has finite global dimension.

Coxeter Groups

Definition (Coxeter Group)

A *Coxeter Group* W is given by a set S of generators and relations of the form $s^2 = 1$ for all $s \in S$ and $(st)^{m_{s,t}} = 1$, where $s \neq t \in S$ and $m_{s,t} = m_{t,s} > 1$.

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Example

The symmetric group on n letters, $\mathcal{S}_n$ is a Coxeter group with Coxeter presentation $S = \{s_1, \ldots, s_{n-1}\}$ and relations $s_i^2 = 1, (s_i s_j)^2 = 1, |i-j| > 1, (s_i s_{i+1})^3 = 1$.

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1. The Coxeter Complex and its Left Regular Band
2. The Bruhat Order and its $\mathcal{J}$-trivial monoid.

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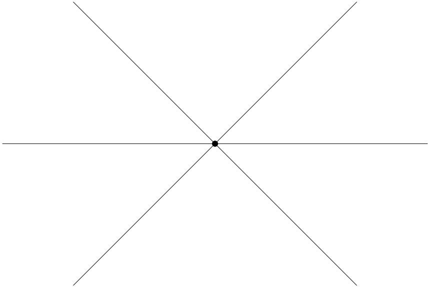
This is called the Coxeter Complex. This complex and all (central) hyperplane arrangements have the structure of a monoid that is a left regular band.

Here is the arrangement associated to $\mathcal{S}_3$.

The faces of the Coxeter Complex of $\mathcal{S}_3$



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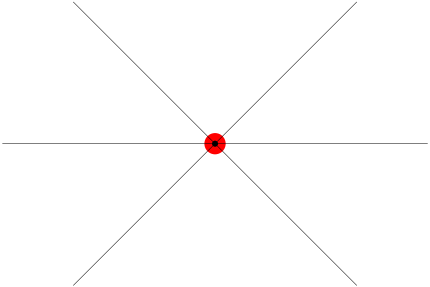

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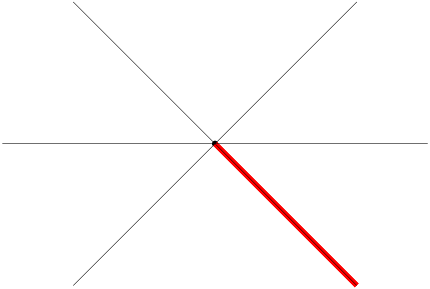

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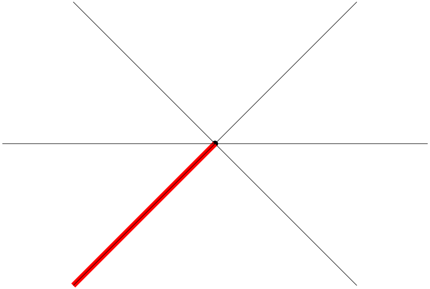

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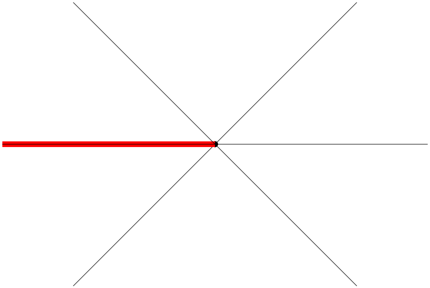

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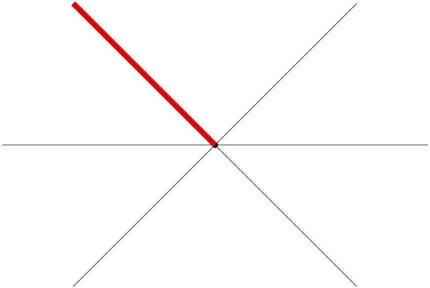

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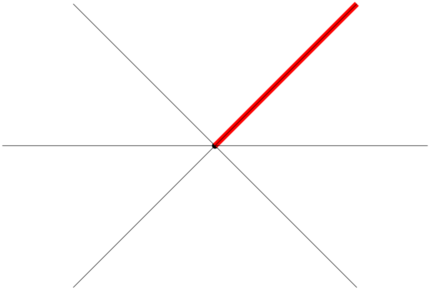

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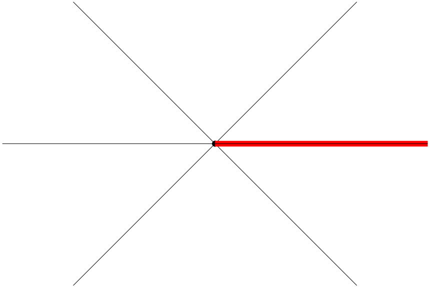

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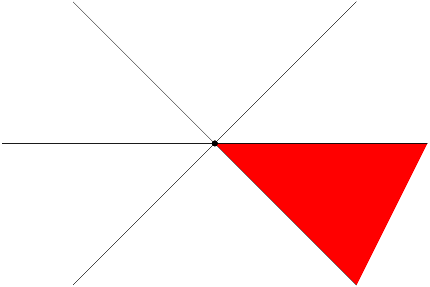

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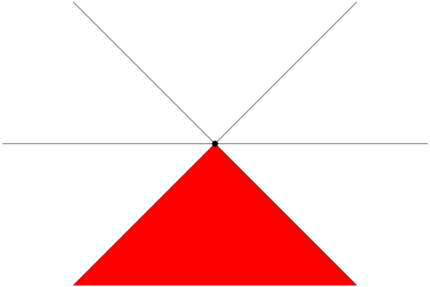

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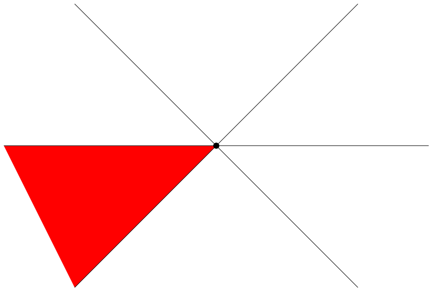

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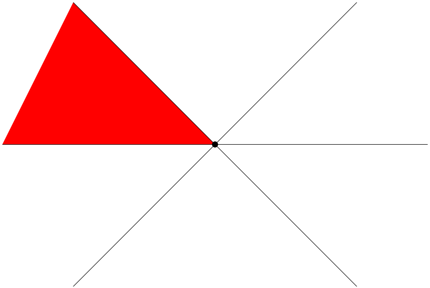

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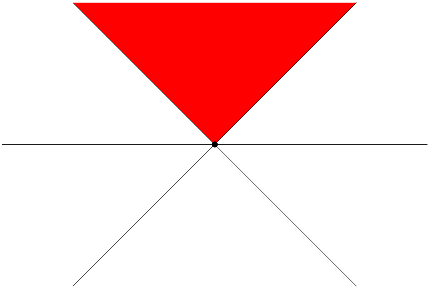

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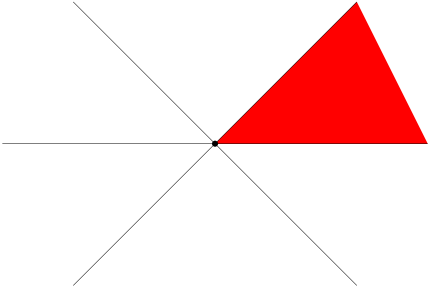

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The Cayley graph of $\mathcal{S}_3$ relative to Coxeter generators

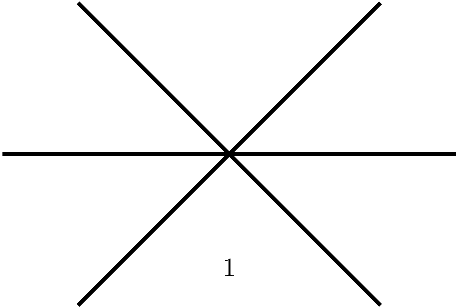

Figure: The Cayley graph of $\mathcal{S}_3$ is the dual graph of the chambers relative to the reflections defining the group.

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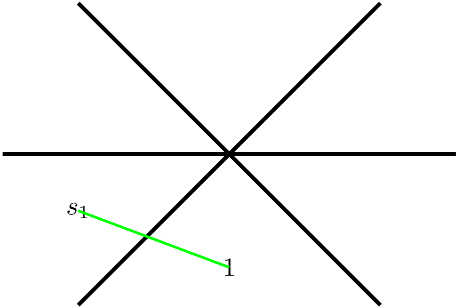

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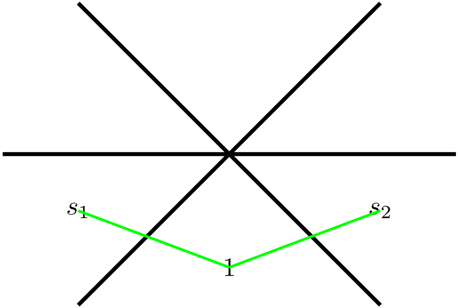

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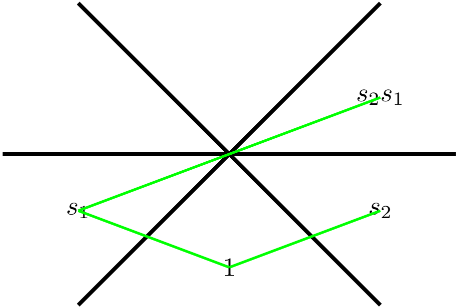

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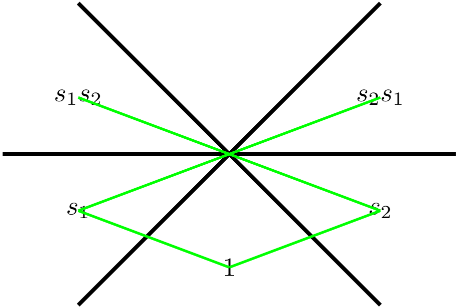

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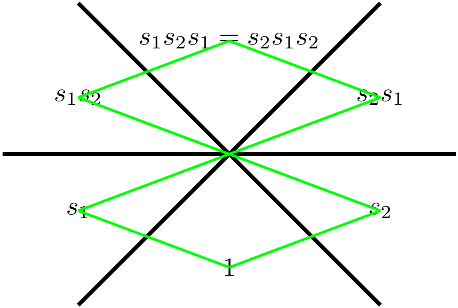

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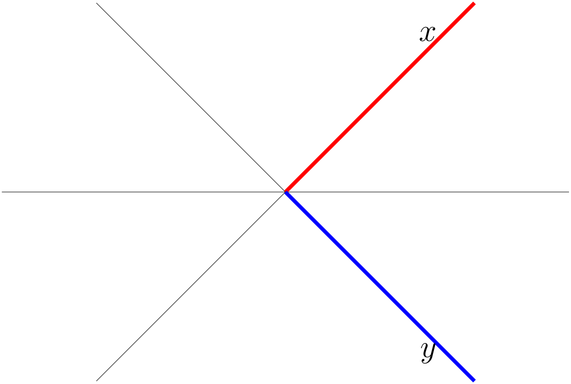

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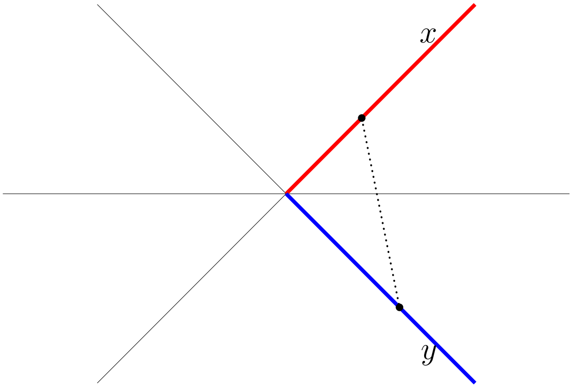

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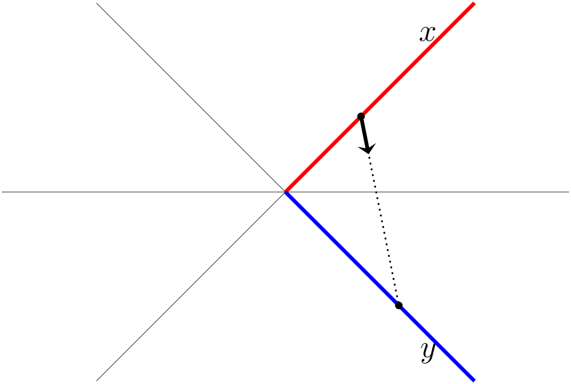

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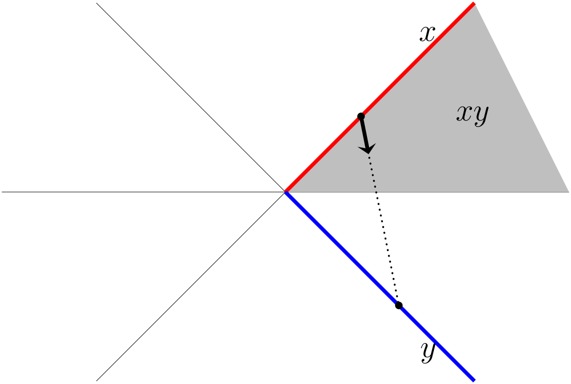


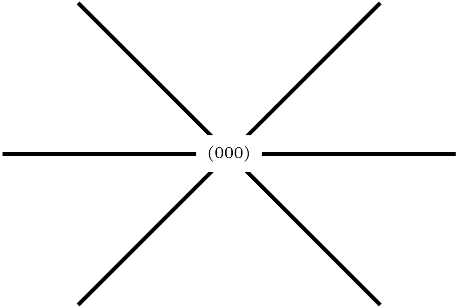
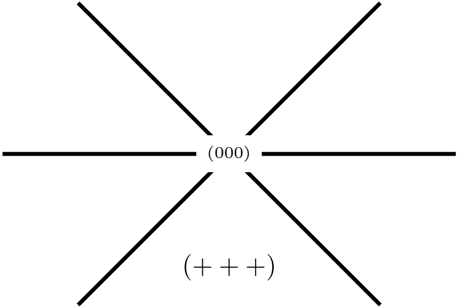


Figure: The sign sequences of the faces of the hyperplane arrangement in $\mathbb{R}^2$ consisting of three distinct lines. The geometric product is just multiplication in $\{0, +, -\}^3$.



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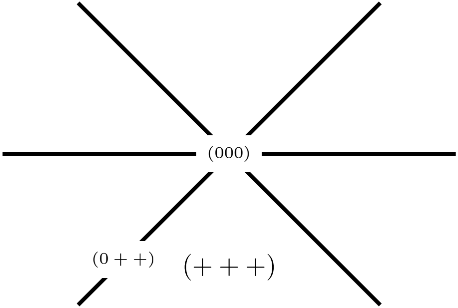

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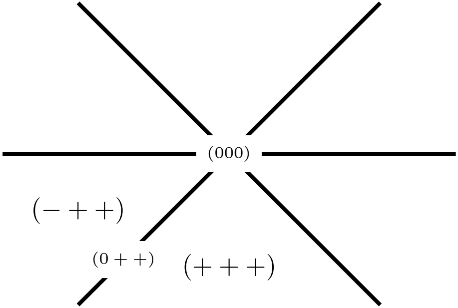

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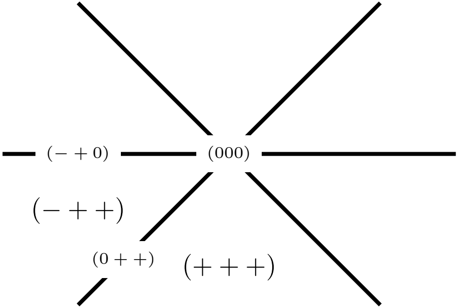

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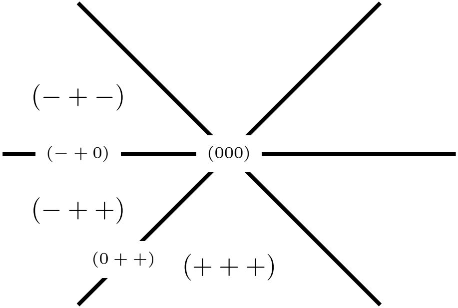

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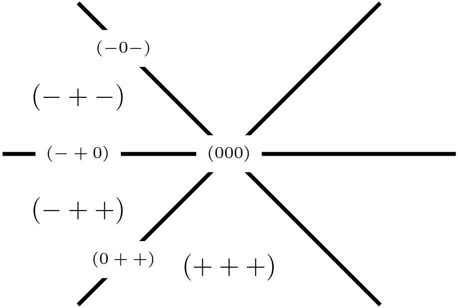

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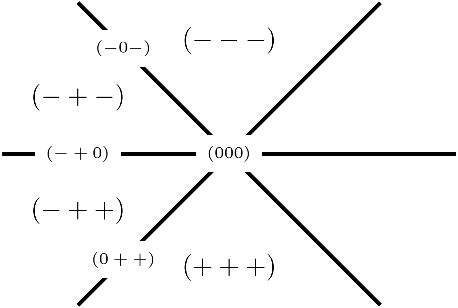

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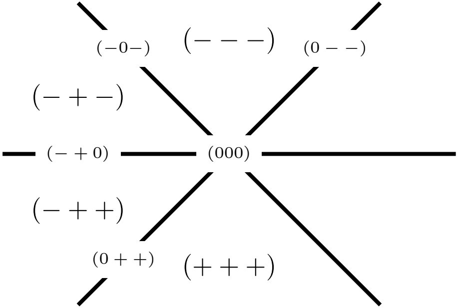

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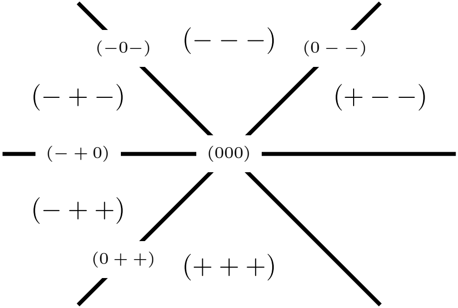

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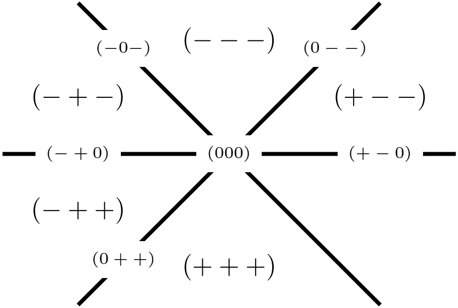

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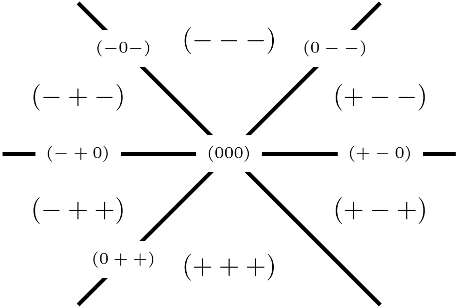

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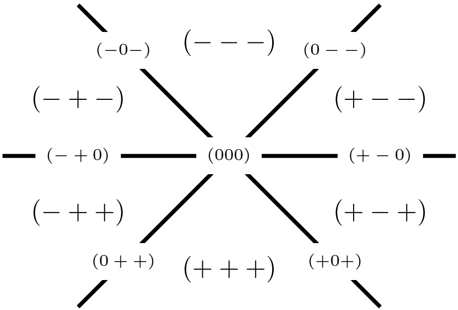

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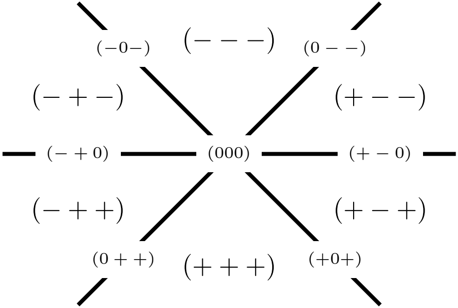

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All hyperplane arrangement LRBs are **submonoids** of $\{0, +, -\}^n$, where $n =$ the number of hyperplanes.

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Remarks

- *Informally: identities say ignore "repetitions".*
- *We consider only finite monoids here.*

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Let B be a band. The following are equivalent:

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7. *If $f : B \to \Lambda(B)$ is the map to the maximal semilattice image, then $f^{-1}(l)$ is left zero for all $l \in \Lambda(B)$.*
8. *B divides $\{0, +, -\}^n$, for some n, where $\{0, +, -\}$ is the monoid with identity 0 and left zero ideal $\{+, -\}$.*

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8. *B divides $\{0, +, -\}^n$, for some n, where $\{0, +, -\}$ is the monoid with identity 0 and left zero ideal $\{+, -\}$. That is, LRB is the variety of monoids generated by the monoid $\{0, +, -\}$.*

Representation Theory of LRBs

- Simple $\mathbb{K}B$-modules and its Jacobson Radical

Let $\Lambda(B)$ denote the lattice of principal left ideals of B, ordered by inclusion:

$$\Lambda(B) = \{Bb : b \in B\} \qquad Ba \cap Bb = B(ab)$$

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Bruhat Order

Let W be a Coxeter group with generators S.

Definition

A word x over S is reduced if it is a shortest length representative for some $w \in W$.

Theorem

(Tits). Let x, y be reduced representatives for an element $w \in W$. Then there is a series of Braid Moves that change x to y.

Definition

Let $y = s_1 ... s_n$ be a word over S. A subword of y is a word $x = s_{i_1} ... s_{i_k}$ where $1 \leq i_1 \leq ... i_k \leq n$

Subwords and the Definition of Bruhat Order

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Let W be a Coxeter group with generators S and let $u, w \in W$. The following conditions are equivalent.

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Definition

Let $u, w \in W$. Define $u \leq w$ if some reduced word for u is a subword of some reduced word for w.

Fact: $\leq$ is a partial order on W called the Bruhat order, with the identity element as minimal element.

The Bruhat Order of S_3

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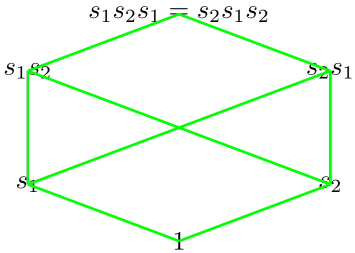

Subword Order, Partially Ordered Monoids and $\mathcal{J}$-Trivial Monoids

In Algebraic Automata Theory, the relationship between subword order of free monoids, partially ordered monoids in which the identity is the minimal element and finite $\mathcal{J}$-trivial monoids is part of the celebrated theorem of Imre Simon which we recall here.

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Definition

A monoid M is $\mathcal{J}$-trivial if distinct elements generate distinct principal two sided ideals. That is, for all $m, n \in M, MmM = MnM$ if and only if $m = n$.

Simon's Theorem

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4. M *is a subdirect product of finite syntactic monoids* $M(L)$, *where* L *is a language over some finite alphabet* S *and* L *is a Boolean combination of languages of the form* $S^* s_1 ... S^* s_n S^*, s_i \in S, i = 1, ..., n.$

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Remark

The name "0-Hecke monoid" comes from the fact that the algebra $\mathbb{K}\mathcal{H}(W)$ over a field $\mathbb{K}$ is the Hecke algebra with parameter $q = 0$.

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3. *$\mathcal{H}(W)$ is an ordered monoid with respect to the Bruhat order on W and thus $\mathcal{H}(W)$ is a $\mathcal{J}$-trivial monoid.*

Other incarnations of the 0-Hecke monoid

The monoid $\mathcal{H}(W)$ has been discovered a number of times in a number of fields. Here are some isomorphic descriptions of this monoid:

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4. Mazorchuck, Steinberg $\mathcal{H}(W)$ is isomorphic to the monoid generated by Tits folds on the Coxeter complex of W.

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If we define a product $*$ on $E(M)$ by:

$$e * f = (ef)^{\omega}$$

where x^{ω} is the unique idempotent in the subsemigroup generated by an element x of a finite semigroup, then we have the following Theorem.

Theorem

Let M be a $\mathcal{J}$-trivial monoid and let $\mathbb{K}$ be a field.

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For each $X \in E(M)$, the corresponding simple module is 1 dimensional and is given by the following action.

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We see then that $\mathbb{K}M$, like the algebra of an LRB is a basic algebra: All of its simple modules are 1 dimensional.
Equivalently, $\mathbb{K}M$ has a faithful representation by triangular matrices.

Basic Algebras

Let $\mathbb{K}$ be an algebraically closed field.

Theorem

The following conditions are equivalent.

1. A *is a finite dimensional basic algebra over* $\mathbb{K}$*.*
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Theorem

Every finite dimensional algebra over $\mathbb{K}$ *is Morita equivalent to a unique basic algebra.*

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Theorem

Let M be a finite monoid and $\mathbb{K}$ an algebraically closed field of characteristic 0. The following conditions are equivalent.

1. $\mathbb{K}M$ *is basic.*
2. M *is a rectangular monoid and every subgroup of* M *is Abelian.*

Examples of Finite Monoids with Basic Algebras

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- This has been done by: Bidigare, Hanlon and Rockmore; Diaconis and Brown; Brown; Björner; Diaconis and Athanasiadis; and Chung and Graham.

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- long-term behavior: favorite books move to the front

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Others:

Björner, Athanasiadis-Diaconis, Chung-Graham, . . .

Free Partially-Commutative LRB

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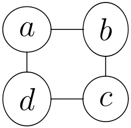
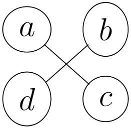
- LRB-version of the Cartier-Foata *free partially-commutative monoid* (aka *trace monoids*).

Acyclic orientations

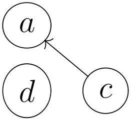
Elements of $F(G)$ correspond to acyclic orientations of induced subgraphs of the complement $\overline{G}$.

Example

$G =$ 

$\overline{G} =$ 

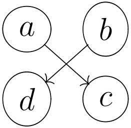
Acyclic orientation on induced subgraph on vertices $\{a, d, c\}$:



In $F(G)$: $cad = cda = dca$ (c comes before a since $c \to a$)

Random walk on $F(G)$

States: acyclic orientations of the complement $\overline{G}$



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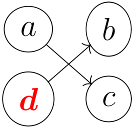

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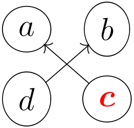

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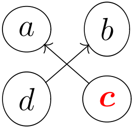

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Athanasiadis-Diaconis (2010): studied this chain using a different LRB (graphical arrangement of G)

The (Karnofsky)-Rhodes Expansion of a Semilattice

If Λ is a semilattice let $\Delta(\Lambda) = \{x_1 > x_2 \ldots > x_k | x_i \in \Lambda\}$ be the set of chains in Λ.

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- It is an LRB whose $\mathcal{R}$ order has Hasse diagram a tree and $\mathcal{L}$ order is the Hasse diagram of Λ.

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$$31245 \leftrightarrow \emptyset < \{3\} < \{3,1\} < \{3,1,2\} < \{3,1,2,4\} < \{3,1,2,4,5\}$$

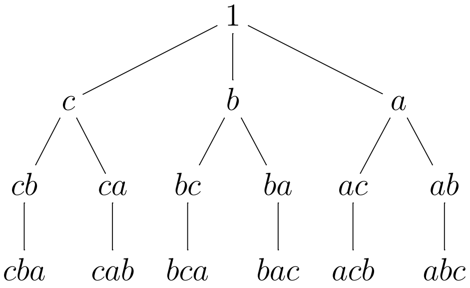
- Many of the LRBs on combinatorial structures are submonoids of (Karnofsky)-Rhodes expansions of semilattices.

Poset of a LRB

B is a partially-ordered set via its $\mathcal{R}$-order:

$$a \leq b \quad \Leftrightarrow \quad ba = a$$

Example: $F(\{a, b, c\})$



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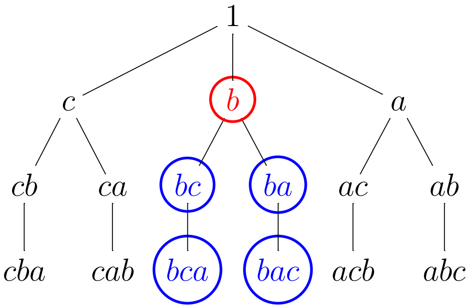
$$B_{[Ba,Bb)} = \left\{ x \in B : x < b \text{ and } Ba \leq Bx \right\}$$

Example: $B(abc) \subseteq Bb$



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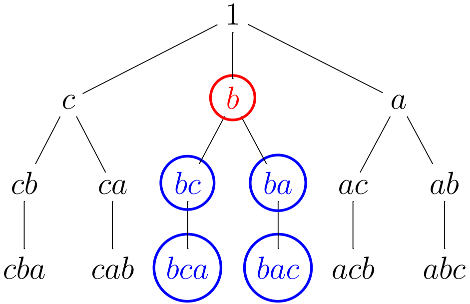

$$B_{[Babc,Bb)} = \{bc, ba, bca, bac\}$$

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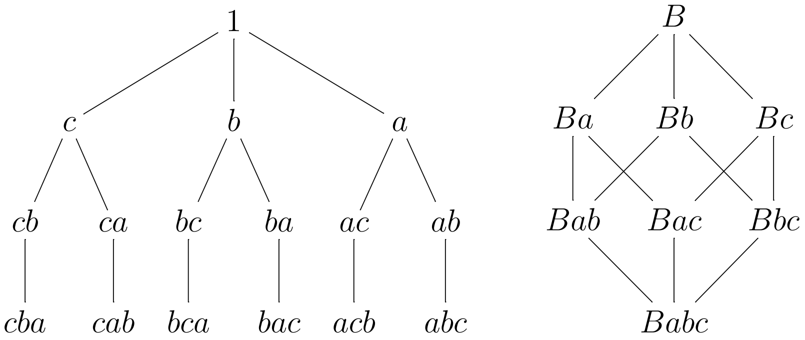
Poset and $\Lambda(B)$ for $B = F(\{a, b, c\})$



Quiver of $\mathbb{K}B$

The *(Ext)-quiver* of an algebra A is the digraph Q_A with:

- vertex set the simple A-modules S_X
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Theorem

Let A be a basic finite dimensional algebra. Then A is a quotient of the path algebra $P = \mathbb{K}Q_A$ of its quiver Q_A by an ideal I such that $(P^+)^n \subseteq I \subseteq (P^+)^2$, for some $n \geq 2$, where (P^+) is the ideal of positive length paths. Conversely, every such algebra is a finite dimensional basic algebra.

Corollary. Let B be a finite LRB. The quiver of $\mathbb{K}B$ has vertex set $\Lambda(B)$. The number of arrows $X \to Y$ is 0 if $X \nless Y$; otherwise, it is one less than the number of connected components of $\Delta B_{[X,Y)}$.

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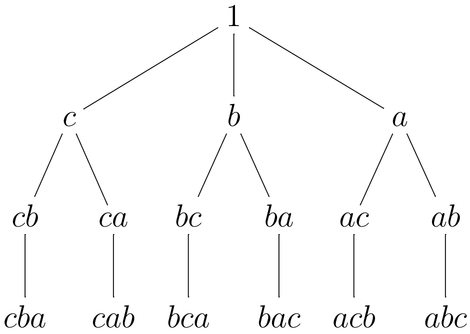
Proof. For $X < Y$:

$$\operatorname{Ext}^1_{\mathbb{K}B}(S_X, S_Y) = \widetilde{H}^0\big(\Delta B_{[X,Y)}, \mathbb{K}\big)$$

Computing the quiver of $B = F(\{a, b, c\})$



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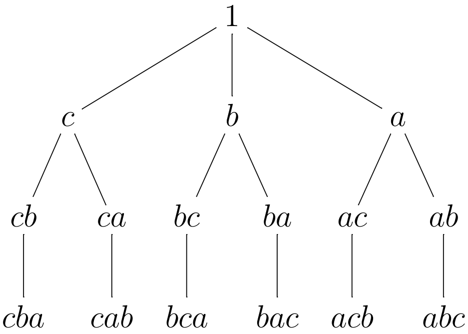

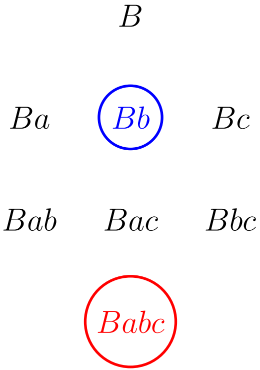

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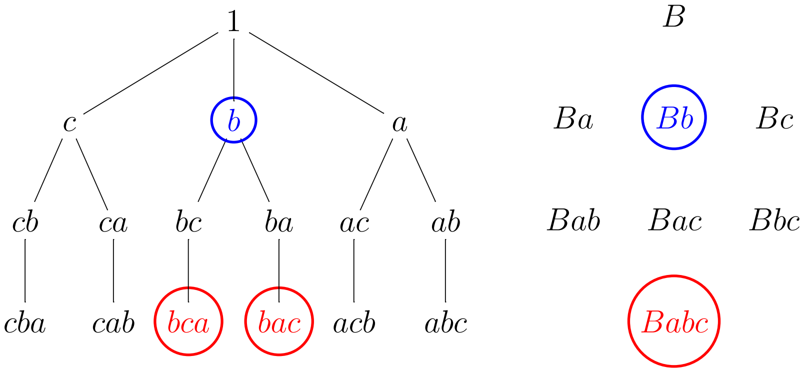

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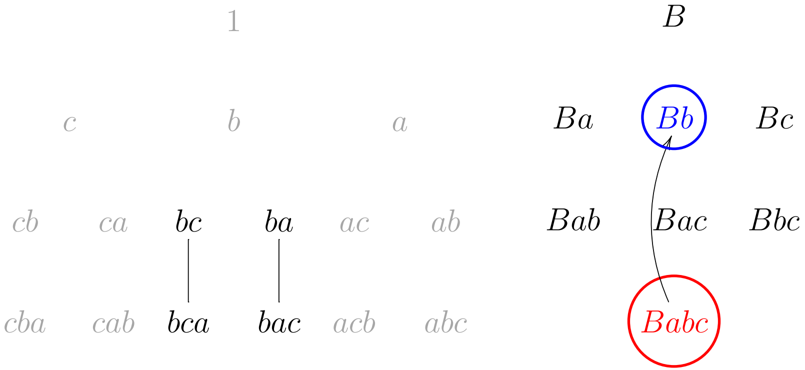

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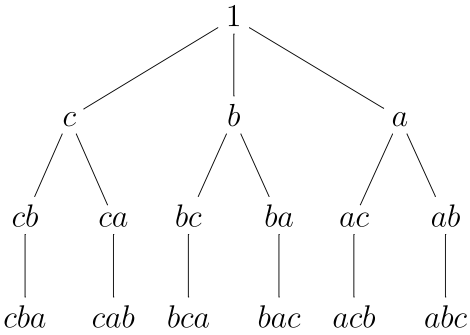

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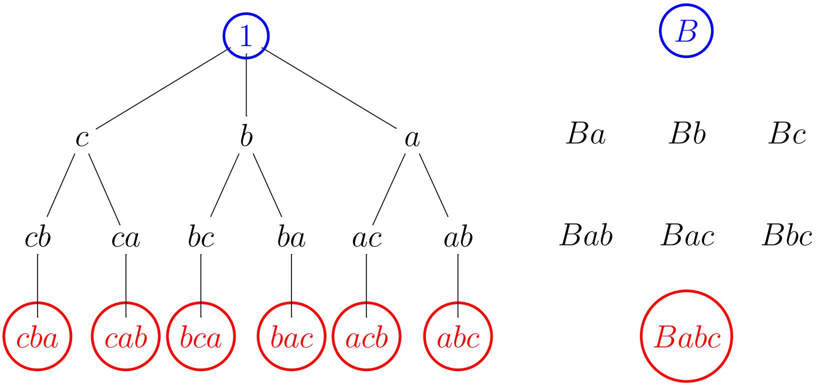

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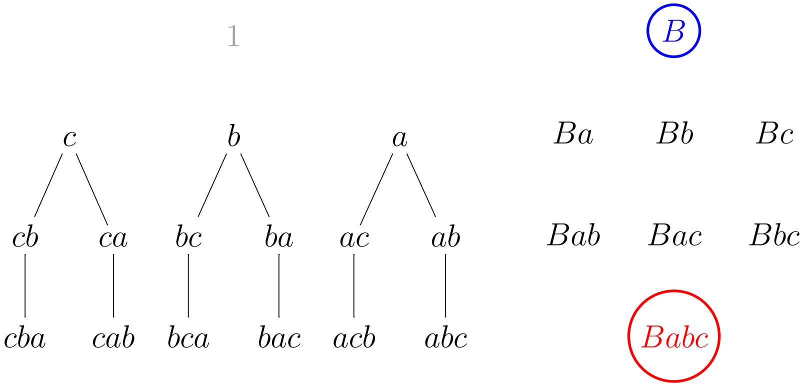

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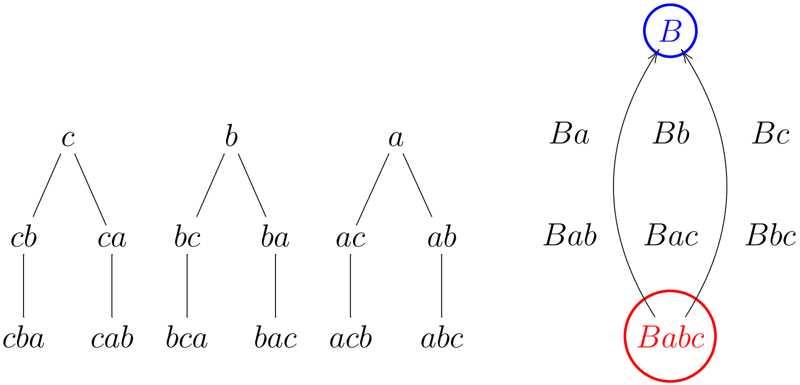

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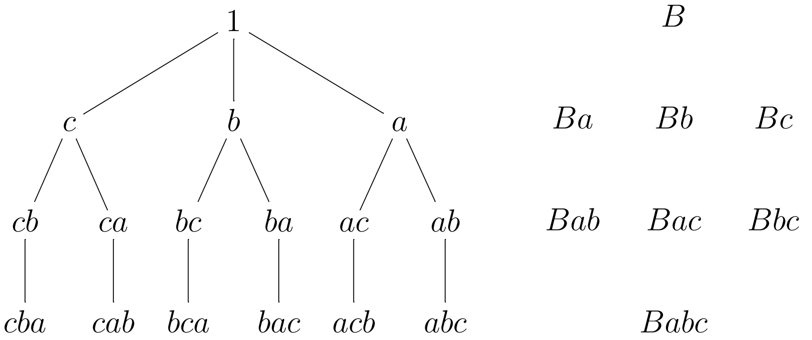

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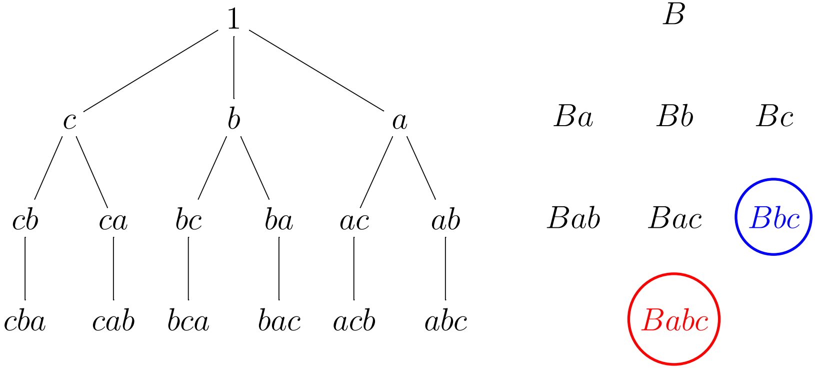

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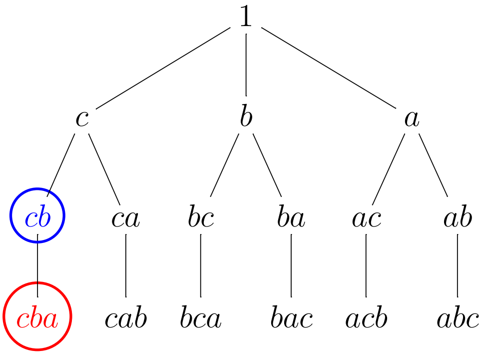

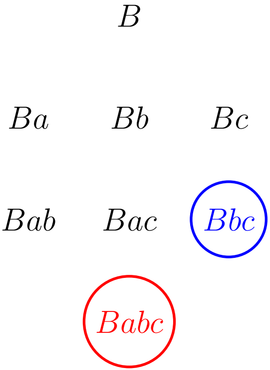

Computing the quiver of $B = F(\{a, b, c\})$

1 $\qquad$ B

$c \quad b \quad a$ $\qquad$ $Ba \quad Bb \quad Bc$

$cb \quad ca \quad bc \quad ba \quad ac \quad ab$ $\qquad$ $Bab \quad Bac \quad Bbc$

$cba \quad cab \quad bca \quad bac \quad acb \quad abc$ $\qquad$ $Babc$

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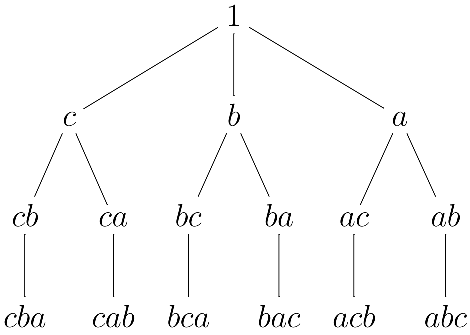

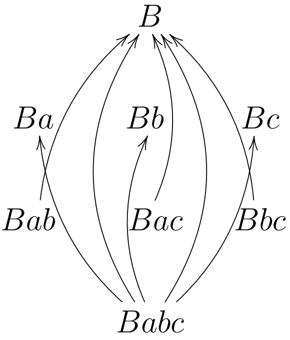

Global dimension

Let A be a finite dimensional algebra.

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- It is known that every finite regular monoid has an algebra of finite global dimension.

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5. $\mathbb{K}F(G)$ is hereditary iff G is chordal, that is, has no induced cycles greater than length 3.

Outline of Proof

An Eckmann-Shapiro–type lemma reduces to the case:

$$\begin{aligned}
& \operatorname{Ext}^n_{\mathbb{K}B}(S_{\widehat{0}}, S_{\widehat{1}}) & \\
= \;& H^n(B, S_{\widehat{1}}) & \text{(monoid cohomology)} \\
= \;& H^{n-1}(B, \mathbb{K}^{B_{[\widehat{0},\widehat{1})}}) & \text{(dimension shift)} \\
= \;& H^{n-1}(B \ltimes B_{[\widehat{0},\widehat{1})}, \mathbb{K}) & \text{(Eckmann-Shapiro)} \\
= \;& H^{n-1}(|B \ltimes B_{[\widehat{0},\widehat{1})}|, \mathbb{K}) & \text{(classifying space)} \\
= \;& H^{n-1}(\Delta(B_{[\widehat{0},\widehat{1})}), \mathbb{K}) & \text{(Quillen's Theorem A)}
\end{aligned}$$

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Theorem

An LRB B embeds into $\{0, +, -\}^n$ for some n iff B is geometric. That is, the quasivariety generated by $\{0, +, -\}$ is the quasivariety of geometric LRBs.

Remark

Mark Sapir proved on the other hand that there are a continuum of quasivarieties of LRBs generated by finite LRBs.