Linear representations of semigroups from 2-categories

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2-categories: definition

Definition. A *2-category* is a category enriched over the monoidal category **Cat** of small categories (in the latter the monoidal structure is induced by the cartesian product).

This means that a 2-category $\mathscr{C}$ is given by the following data:

- objects of $\mathscr{C}$;
- small categories $\mathscr{C}(\mathtt{i},\mathtt{j})$ of morphisms;
- functorial composition $\mathscr{C}(\mathtt{j},\mathtt{k}) \times \mathscr{C}(\mathtt{i},\mathtt{j}) \to \mathscr{C}(\mathtt{i},\mathtt{k})$;
- identity objects $\mathbb{1}_{\mathtt{j}}$;

which are subject to the obvious set of (strict) axioms.

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2-categories: terminology and the first example

Terminology.

- An object in $\mathscr{C}(\mathtt{i},\mathtt{j})$ is called a *1-morphism* of $\mathscr{C}$.
- A morphism in $\mathscr{C}(\mathtt{i},\mathtt{j})$ is called a *2-morphism* of $\mathscr{C}$.
- Composition in $\mathscr{C}(\mathtt{i},\mathtt{j})$ is called *vertical* and denoted $\circ_1$.
- Composition in $\mathscr{C}$ is called *horizontal* and denoted $\circ_0$.

Principal example. The category **Cat** is a 2-category.

- Objects of **Cat** are small categories.
- 1-morphisms in **Cat** are functors.
- 2-morphisms in **Cat** are natural transformations.
- Composition is the usual composition.
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2-categories: over monoids, part 1

A monoid is the same thing as a category with one object.

Indeed: If $\mathcal{C}$ is a category with one object $\clubsuit$, then $\mathcal{C}(\clubsuit, \clubsuit)$ is a monoid under composition.

If $(S, \circ, e)$ is a monoid, we can form a category $\mathcal{C} = \mathcal{C}_{(S,\circ,e)}$ as follows:

- The only object of $\mathcal{C}$ is $\clubsuit$.
- $\mathcal{C}(\clubsuit, \clubsuit) := S$.
- Composition in $\mathcal{C}$ is given by multiplication in S.
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Can we extend $\mathcal{C}$ to a 2-category?

Naive approach to try: Let $X \subset S$ be some submonoid.

For $s, t \in S$ set $\mathrm{Hom}_{\mathcal{C}(\clubsuit,\clubsuit)}(s, t) := \{x \in X : xs = t\}$.

Note! S is just a monoid, not a group, so $\mathrm{Hom}_{\mathcal{C}(\clubsuit,\clubsuit)}(s, t)$ may be empty or it may contain many elements.

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2-categories: over monoids, part 3

Vertical: $xr = s$ and $ys = t$ implies $yxr = t$ OK

Horizontal: $xs = t$ and $x's' = t'$ implies $xsx's' = tt'$

Need: $xx'ss' = tt'$ OK if $X \subset Z(S)$

From now on: X is a submonoid in the center $Z(S)$ of S

All compositions are well-defined!!!

Is this a 2**-category?**

To check: Functoriality of composition.

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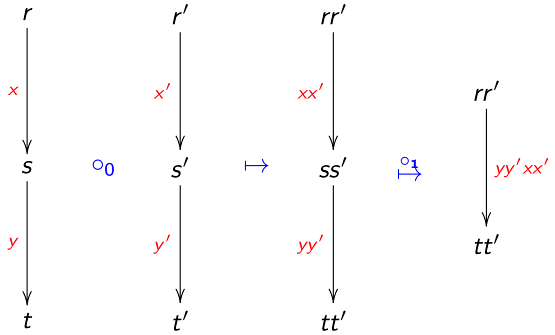
To check: Functoriality of composition.

One way:

Conclusion 1: $(y \circ_0 y') \circ_1 (x \circ_0 x') = yy'xx'$.

One way:



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2-categories: over monoids, part 4: functoriality, part 1

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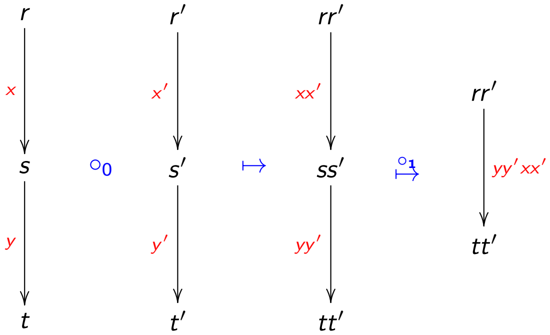

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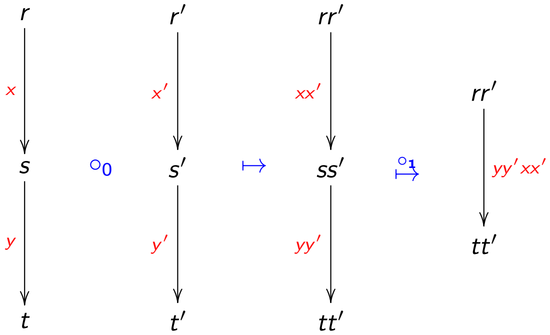

Conclusion 1: $(y \circ_0 y') \circ_1 (x \circ_0 x') = yy'xx'$.

Another way:

Conclusion 2: $(y \circ_1 x) \circ_0 (y' \circ_1 x') = yxy'x'$.

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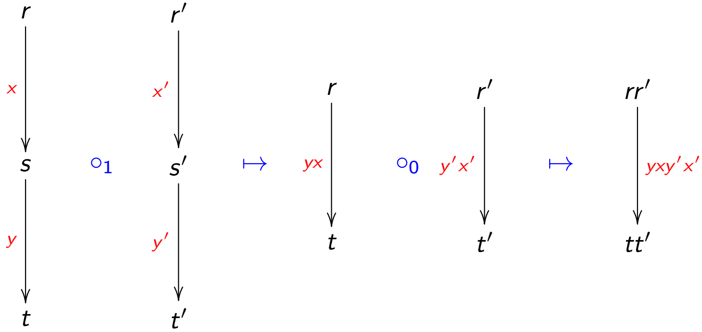

Conclusion 2: $(y \circ_1 x) \circ_0 (y' \circ_1 x') = yxy'x'$.

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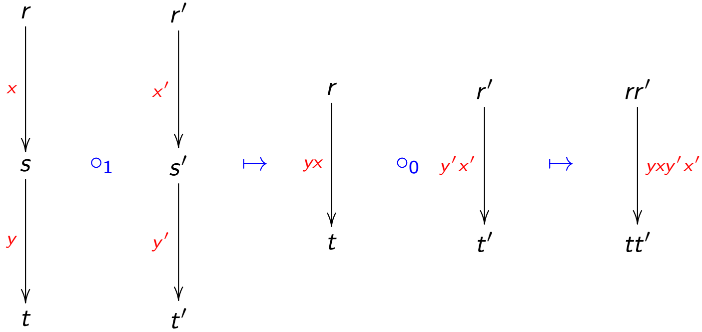

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2-categories: over monoids, part 5: functoriality, part 2

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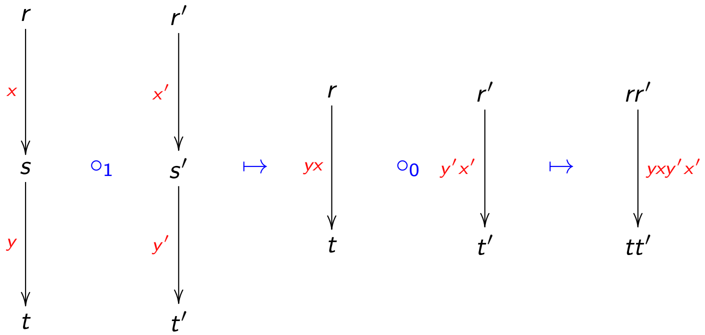

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Need: the *interchange law* $(y \circ_1 x) \circ_0 (y' \circ_1 x') = (y \circ_0 y') \circ_1 (x \circ_0 x')$.

In our case: $yxy'x' = yy'xx' \ \forall \ x, y, x', y' \in X$ OK since $X \subset Z(S)$.

Claim. The above defines on $\mathcal{C}$ the structure of a 2-category if and only if $X \subset Z(S)$.

2-categories: over monoids, part 6: the interchange law

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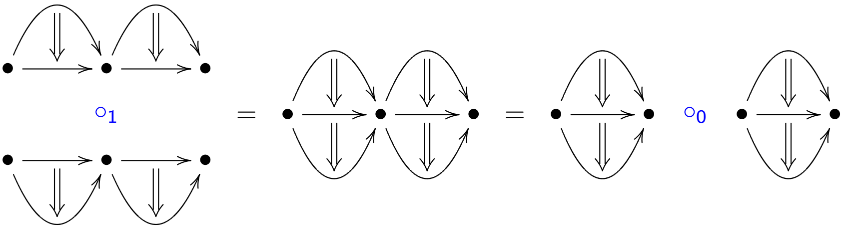

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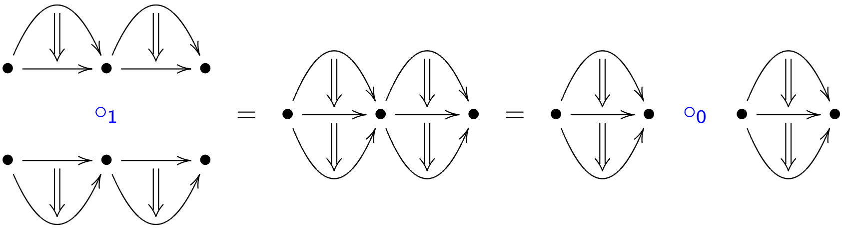

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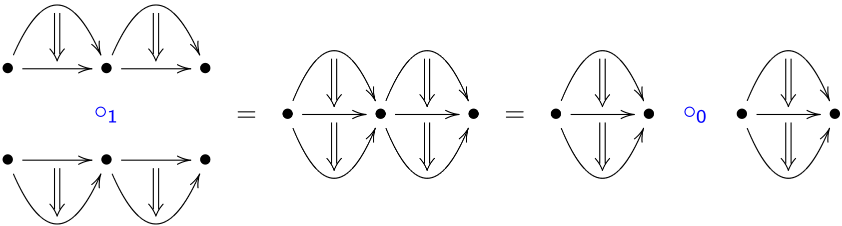

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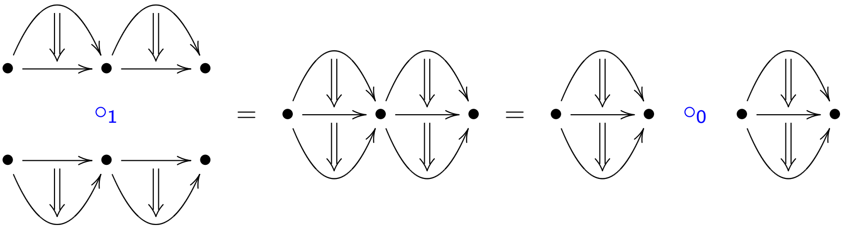

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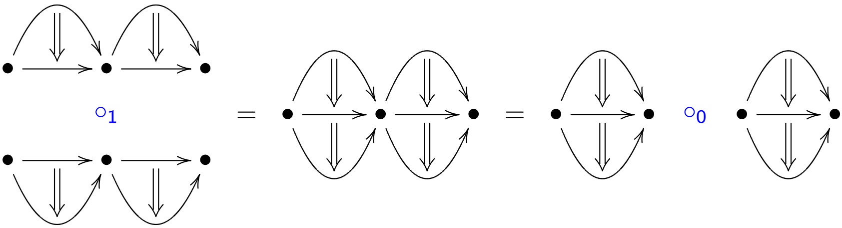

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S — monoid

$\leq$ — compatible order on S (i.e. $a \leq b$ implies $as \leq bs$ and $sa \leq sb$)

Define $\mathcal{C}_{(S,\leq)}$ — 2-category via

- $\mathcal{C}_{(S,\leq)}$ has one object $\clubsuit$
- 1-morphisms: $\mathcal{C}_{(S,\leq)}(\clubsuit, \clubsuit) = S$
- 2-morphisms: for $s, t \in S$ set $\mathrm{Hom}(s,t) = \begin{cases} (s,t), & s \leq t; \\ \varnothing, & \text{else.} \end{cases}$
- horizontal composition is given by multiplication in S;
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2-categories: over monoids, part 7: ordered monoids

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2-representations: 2-functors, part 1

$\mathscr{A}$ and $\mathscr{C}$ — two 2-categories

Definition. A *2-functor* $\mathrm{F} : \mathscr{A} \to \mathscr{C}$ is a functor which sends 1-morphisms to 1-morphisms and 2-morphisms to 2-morphisms in a way that is coordinated with all the categorical structures (domains, codomains, identities and compositions).

Example. For $\mathtt{i} \in \mathscr{C}$ the functor $\mathscr{C}(\mathtt{i}, \_) : \mathscr{C} \to \mathbf{Cat}$ sends

- an object $\mathtt{j} \in \mathscr{C}$ to the category $\mathscr{C}(\mathtt{i}, \mathtt{j})$,
- a 1-morphism $F \in \mathscr{C}(\mathtt{j}, \mathtt{k})$ to the functor $F \circ \_ : \mathscr{C}(\mathtt{i}, \mathtt{j}) \to \mathscr{C}(\mathtt{i}, \mathtt{k})$,
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$\mathscr{A}$ and $\mathscr{C}$ — two 2-categories

Definition. A *2-functor* $\mathrm{F} : \mathscr{A} \to \mathscr{C}$ is a functor which sends 1-morphisms to 1-morphisms and 2-morphisms to 2-morphisms in a way that is coordinated with all the categorical structures (domains, codomains, identities and compositions).

Example. For $\mathtt{i} \in \mathscr{C}$ the functor $\mathscr{C}(\mathtt{i}, {}_{-}) : \mathscr{C} \to$ **Cat** sends

- an object $\mathtt{j} \in \mathscr{C}$ to the category $\mathscr{C}(\mathtt{i}, \mathtt{j})$,
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2-representations: 2-functors, part 2

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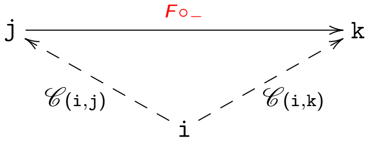

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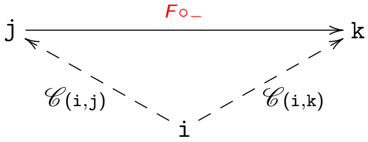

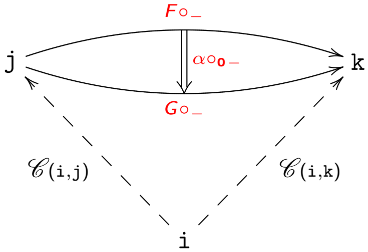

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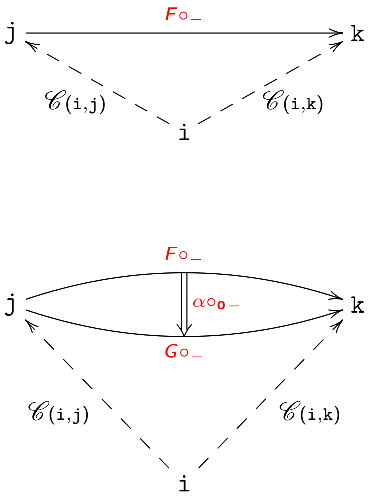

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"Classical" 2-representations:

- in **Cat**;
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Definition. The (split) *Grothendieck group* $[\mathcal{A}]$ of a small additive category $\mathcal{A}$ is the quotient of the free abelian group generated by objects of $\mathcal{A}$ modulo relations $[X] - [Y] - [Z]$ whenever $X \cong Y \oplus Z$ in $\mathcal{A}$.

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Main point: Forget the 2-level.

Note: For $\Bbbk$-linear categories indecomposability is defined on the 2-level (an object in indecomposable iff its endomorphism algebra is local).

Assume: $\mathscr{C}$ — locally finitary; $\mathcal{F}$ — 2-representation of $\mathscr{C}$ s.t.

- object $\mathtt{i} \mapsto$ additive (abelian, triangulated) category $\mathcal{C}_{\mathtt{i}}$
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Then: The category $[\mathscr{C}]$ acts on $[\mathcal{C}]$

In particular: If $\mathscr{C}$ has 1 object $\clubsuit$ then the monoid $[\mathscr{C}](\clubsuit, \clubsuit)$ acts on the abelian group $[\mathcal{C}]$

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Decategorification: advantages

Assume: $\mathscr{C}$ is 2-represented on $\mathcal{C}$

Decategorify: $[\mathscr{C}]$ acts on $[\mathcal{C}]$

Main point: $\mathcal{C}$ has non-trivial structure

Example 1: The group $[\mathcal{C}]$ might have many natural bases (e.g. given by simple, injective, projective or tilting modules).

Example 2: The category $\mathcal{C}$ could have stratifications, e.g. by Gelfand-Kirillov dimension of objects. This gives rise to filtrations on $[\mathcal{C}]$.

Example 3: The category $\mathcal{C}$ could be graded, which would give a "layered upgrade" of $[\mathcal{C}]$ (e.g. Jones polynomial $\to$ Khovanov homology).

Decategorification: advantages

Assume: $\mathscr{C}$ is 2-represented on $\mathcal{C}$

Decategorify: $[\mathscr{C}]$ acts on $[\mathcal{C}]$

Main point: $\mathcal{C}$ has non-trivial structure

Example 1: The group $[\mathcal{C}]$ might have many natural bases (e.g. given by simple, injective, projective or tilting modules).

Example 2: The category $\mathcal{C}$ could have stratifications, e.g. by Gelfand-Kirillov dimension of objects. This gives rise to filtrations on $[\mathcal{C}]$.

Example 3: The category $\mathcal{C}$ could be graded, which would give a "layered upgrade" of $[\mathcal{C}]$ (e.g. Jones polynomial $\to$ Khovanov homology).

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Hecke-Kiselman semigroups: definition

Assume: Γ — simple digraph (no loops or multiple edges in the same direction)

Definition: The *Hecke-Kiselman* monoid $\mathbf{HK}_\Gamma$ has generators e_i where i is a vertex of Γ and relations

- $e_ie_je_i = e_je_ie_j$ if $i \rightleftarrows j$;
- $e_ie_je_i = e_je_ie_j = e_ie_j$ if $i \longrightarrow j$;
- $e_ie_j = e_je_i$ if $i \qquad j$.

Examples:

- Γ — no edges $\Rightarrow$ $\mathbf{HK}_\Gamma$ is the Boolean of Γ_0;
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Hecke-Kiselman semigroups: Catalan monoid

Catalan monoid: $\mathbf{C}_n$ — order preserving (i.e. $a \leq b \Rightarrow f(a) \leq f(b)$) and order decreasing (i.e. $f(a) \leq a$) transformations of $\{0, 1, \dots, n\}$.

$|\mathbf{C}_n| = \frac{1}{n+1}\binom{2n}{n}$ — the n-th Catalan number

$\Gamma = \Gamma_n := \quad 1 \longrightarrow 2 \longrightarrow \cdots \longrightarrow n$

Theorem (A. Solomon): $\mathbf{HK}_{\Gamma_n} \cong \mathbf{C}_n$

Standard effective representations Φ of $\mathbf{C}_n$:

$\mathbf{v} = (v_1, v_2, \dots, v_n)$ basis of $\Bbbk^n$, action

$$e_i(v_j) = \begin{cases} v_j, & j \neq i; \\ v_{j-1}, & j = i > 1; \\ 0, & j = i = 1. \end{cases}$$

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Path categories

Γ — acyclic quiver (no loops but multiple edges allowed)

$\Bbbk\Gamma$ — path category of Γ

- objects: vertices of Γ
- morphisms: linear combinations of paths in Γ
- composition: concatenation of paths

Representation of $\Bbbk\Gamma$ — functor to $\Bbbk$-vector spaces, i.e.

- objects $\mapsto$ vector space
- paths in $\Gamma \mapsto$ linear map
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$\Bbbk\Gamma\text{-mod}$ — category of locally finite dimensional representations (morphisms= natural transformations of functors)

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Γ — acyclic quiver (no loops but multiple edges allowed)

$\Bbbk\Gamma$ — path category of Γ

- objects: vertices of Γ
- morphisms: linear combinations of paths in Γ
- composition: concatenation of paths

Representation of $\Bbbk\Gamma$ — functor to $\Bbbk$-vector spaces, i.e.

- objects $\mapsto$ vector space
- paths in Γ $\mapsto$ linear map
- concatenation of paths $\mapsto$ composition of linear maps

$\Bbbk\Gamma\text{-}\mathrm{mod}$ — category of locally finite dimensional representations (morphisms= natural transformations of functors)

Projection functors

Γ — acyclic quiver, $i \in \Gamma$

$\mathrm{F}_i : \Bbbk\Gamma\text{-mod} \to \Bbbk\Gamma\text{-mod}$ — *projection* functor
"factor out the maximal possible $\Bbbk\Gamma$-invariant subspace at vertex i"

Theorem (Grensing). Projections functors satisfy:

- $\mathrm{F}_i\mathrm{F}_j \cong \mathrm{F}_j\mathrm{F}_i$ if i and j are not connected in Γ;
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Difficulty. Projections functors are not exact.

Fact. Projections functors send injectives to injectives.

Way out. Let G_i be the unique left exact functor whose action on the additive category of injective modules is isomorphic to that of F_i

Fact: G_i is exact

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Categorification of the Catalan monoid

Γ — acyclic quiver, Θ — underlying simple digraph

Definition: 2-category $\mathscr{C}_{\Theta,\Gamma}$.

- Object: $\clubsuit := \Bbbk\Gamma\text{-mod}$;
- 1-morphisms: Endofunctors on $\Bbbk\Gamma\text{-mod}$ isomorphic to a direct sum of direct summands of compositions of the G_i'th
- 2-morphisms: natural transformations of functors

The 2-category $\mathscr{C}_{\Theta,\Gamma}$ is given by its *defining 2-representation*, that is a functorial action on $\Bbbk\Gamma\text{-mod}$.

Theorem (Grensing-M): $[\mathscr{C}_{\Gamma_n,\Gamma_n}](\clubsuit,\clubsuit) \cong \mathbb{Z}[\mathbf{C}_n]$.

Corollary: In the basis of simple modules, the action of $[\mathscr{C}_{\Gamma_n,\Gamma_n}](\clubsuit,\clubsuit)$ on $[\Bbbk\Gamma\text{-mod}]$ gives Φ.

Consequence: In the basis of projective (injective) modules we get two new (but equivalent) effective linear representations of $\mathbb{C}_n$.

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Other Hecke-Kiselman monoids

Γ, Θ as above

Fact: Mapping e_i to G_i gives a weak functorial action of $\mathbf{HK}_\Theta$ on $\Bbbk\Gamma\text{-mod}$.

Example: From [Kudryavtseva-M] it follows that if Θ is the full graph on $\{1, 2, \dots, n\}$ oriented from smaller to bigger vertices (i.e. $\mathbf{HK}_\Theta$ is the Kiselman semigroup), then there exists Γ such that this action is faithful.

Difficulty: Composition of the G_i's may decompose!

Problem: What are indecomposable 1-morphisms in $\mathscr{C}_{\Theta,\Gamma}$?

Known full answer: For Γ_n any composition of the G_i's is indecomposable.

Known partial answer: For a Dynkin quiver of type A and any orientation, indecomposable 1-morphisms in $\mathscr{C}_{\Theta,\Gamma}$ form a monoid T (under composition) generated by idempotents (each $\to \bullet \to$ contributes with one generator and each $\to \bullet \leftarrow$ and $\leftarrow \bullet \to$ with two generators). There is a presentation for T and a realization of $\mathbf{HK}_\Theta$ inside $\mathbb{Z}[T]$.

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Known full answer: For Γ_n any composition of the G_i's is indecomposable.

Known partial answer: For a Dynkin quiver of type A and any orientation, indecomposable 1-morphisms in $\mathscr{C}_{\Theta,\Gamma}$ form a monoid T (under composition) generated by idempotents (each $\to \bullet \to$ contributes with one generator and each $\to \bullet \leftarrow$ and $\leftarrow \bullet \to$ with two generators). There is a presentation for T and a realization of $\mathbf{HK}_\Theta$ inside $\mathbb{Z}[T]$.

Other Hecke-Kiselman monoids

Γ, Θ as above

Fact: Mapping e_i to G_i gives a weak functorial action of $\mathbf{HK}_\Theta$ on $\Bbbk\Gamma\text{-mod}$.

Example: From [Kudryavtseva-M] it follows that if Θ is the full graph on $\{1, 2, \dots, n\}$ oriented from smaller to bigger vertices (i.e. $\mathbf{HK}_\Theta$ is the Kiselman semigroup), then there exists Γ such that this action is faithful.

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