Faithful representations of algebras and semigroups

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Plan of the talk

1. Certain classical results
2. Some consequences of representability
3. Certain sufficient conditions
4. Example 1 - a class of quadratic algebras
5. Example 2 - representability of bands and their algebras

Classical results

Theorem (A.I. Malcev, 1943)

Assume that R is an algebra over a field K such that every finitely generated subalgebra B of S admits a faithful representation ϕ_B of degree n over a field extension L_B of K. Then R has a faithful representation of degree n over a field L.

Sketch: every element of the algebra is represented as an $n \times n$ matrix with entries that are independent variables. Then a mixed system of equalities and negations of equalities (involving entries of these matrices) is associated to this situation. A key technical lemma shows that if every finite subsystem of such a system has a solution over a field extension of K then the entire system has a solution over some field L.

The following observation on the size of the involved fields is clear.

Theorem (A.I. Malcev)

If R is finitely generated algebra over a field K and it has a faithful representation in $M_n(L)$ for a field L then it has a faithful representation over a field that is a finitely generated and purely transcendental extension of K.

Theorem (A.I. Malcev)

If R is a commutative finitely generated algebra over a field K then R has a faithful representation over a field L.

Sketch: $R = K[x_1, \ldots, x_n]/M$ for an ideal M. Considering the primary decomposition of M it is enough to deal with the case M is a primary ideal. If $x_1, \ldots, x_r$ is a maximal algebraically independent subset of the set $\{x_1, \ldots, x_n\}$ then let M' be the ideal of $R' = K(x_1, \ldots, x_r)[x_{r+1}, \ldots, x_n]$ generated by the same finite set of polynomials as the ideal M of $K[x_1, \ldots, x_n]$. One shows that $M' \cap K[x_1, \ldots, x_n] = M$, so it is enough to show that R' is representable. But every element of R' is algebraic over $K(x_1, \ldots, x_r)$, so R' has finite dimension over this field, whence it embeds into $M_t(K(x_1, \ldots, x_r))$ for some t.

Some necessary conditions

If R embeds into matrices $M_n(L)$ over a field L then:

- there is a bound on lengths of all chains of annihilator (one-sided) ideals in R, because this property holds in $M_n(L)$ and this property is inherited by subrings)

- R satisfies a polynomial identity
(the standard identity $s_{2n} = \sum_{\sigma \in S_{2n}} \mathrm{sgn}(\sigma) x_{1,\sigma(1)} \cdots x_{2n,\sigma(2n)}$ is satisfied in $M_n(A)$ for every commutative algebra A. Notice: there exist PI algebras that do not satisfy any standard identity)

- every nil subring of R is nilpotent

- the Jacobson radical of R is nilpotent if R is a finitely generated algebra
(more generally: by a theorem of A.Braun (1982) the radical of a finitely generated PI-algebra is nilpotent)

Some further positive results

An improvement of Posner's theorem using central polynomials, discovered by Formanek (1972) and Razmyslov (1973) :

Theorem

Every prime PI-algebra has a ring of classical quotients which is of the form $R(Z(R) \setminus \{0\})^{-1} \cong M_n(D)$ *for a division algebra that has finite dimension over its center* $Z(R)$*. Hence, it embeds into* $M_{nr}(Z(D))$ *for some* $r \geq 1$*.*

Another extreme case (nilpotent versus prime):

Theorem (G.M.Bergman and S.M.Vovsi, 1983)

If R *is nilpotent, then there exists an embedding* $\phi : R \longrightarrow UT_n(A)$ *(into strictly upper triangular matrices) for some* $n \geq 1$ *and a commutative algebra* A*.*

However, in the general case the radical can create problems.

Theorem (A.Z.Anan'in, 1989)

Every finitely generated PI-algebra R that is right noetherian is representable.

This class of algebras is considered as "the correct" generalization of finitely generated commutative algebras.

Special case: R is a finitely generated right module over a finitely generated commutative algebra A.

Warning: there exist finite rings that are not representable in matrices over a commutative ring. For example, (G.Bergman): $R = Hom_{\mathbf{Z}}(G, G)$, for the group $G = \mathbf{Z}_p \times \mathbf{Z}_{p^2}$, for a prime number p, is a ring of cardinality p^5 which is not embeddable in matrices of any size over any commutative ring.

A disappointing result:

Example (R.S.Irving, L.W.Small, 1987)

Let $R = K\langle x, y\rangle/(x^2, yxy)$. Then for $V = K[t] \oplus K[t]$ let $f, g \in \mathrm{End}_{K[t]}(V)$ be defined by $f(v_2) = g(v_2) = 0, f(v_1) = v_2, g(v_1) = tv_1$, where v_1, v_2 form a basis of V. Then R embeds into $M_2(K[t])$ via the regular representation. However, R has many homomorphic images that are not embeddable into matrices over a field but satisfy all of the listed above necessary conditions. They are of the form R/I where $I = (xy^ix : i \in J)$, where J are subsets of $\mathbb{N}$ of certain types.

Some nice consequences of representability

Examples of nice (nontrivial!) consequences of representability:

1. For algebras

Theorem (V.T.Markov, 1990)

If R is a representable finitely generated algebra over a field K then the Gelfand-Kirillov dimension of R is an integer ($=$ transcendence degree of certain commutative subalgebra of the field L, where R embeds into $M_n(L)$).

This generalizes the result for prime PI algebras:
$\mathrm{GKdim(R)} = \mathrm{tr\,deg_K(F)}$, where F is the field of fractions of the domain $Z(R)$.

2. For semigroups

Representability of semigroups: certain "pathological" phenomena are not allowed in matrices also for semigroups.

Theorem (Tits alternative, 1972)

Let G be a finitely generated subgroup of the group $GL_n(K)$ over a field K. Then either G is solvable-by-finite or G has a free nonabelian subgroup.

Theorem (J.O., A.Salwa, 1995)

Let S be a a subsemigroup of the multiplicative monoid $M_n(K)$ of matrices over a field K. If K is finitely generated then S has no free non-commutative subsemigroups if and only if S satisfies a semigroup identity.

Semigroup algebras - certain sufficient conditions

A subsemigroup T of a completely 0-simple semigroup M is called uniform if it intersects every nonzero $\mathcal{H}$-class of M.

In many cases (for example, if T has no free noncommutative subsemigroups) T has a completely 0-simple semigroup of quotients (in the sense of Fountain and Gould) that embeds in M.

Theorem (J.O.)

Assume that S is a finitely generated monoid with an ideal chain $S_1 \subseteq S_2 \subseteq \cdots \subseteq S_{n-1} \subseteq S_n = S$ such that S_1 and every Rees factor S_{i+1}/S_i is either nilpotent or a uniform subsemigroup of a Brandt semigroup. If S satisfies the ascending chan condition on right ideals and $\mathrm{GKdim}(K[S])$ is finite then $K[S]$ is right noetherian.

Note: such algebras need not be PI. For example: $K[G]$ for G nilpotent-by-finite.

In the opposite direction we have:

Theorem (J.O.)

If S is a monoid such that $K[S]$ is right noetherian and PI then S is finitely generated. In paricular, by the theorem of Anan'in $K[S]$ is representable.

Therefore from the structure theory of arbitrary semigroups of matrices it follows that S has a chain of the above type with factors nilpotent or uniform subsemigroups of completely 0-simple semigroups (but not necessarily Brandt semigroups).

In some special cases of interest noetherian semigroup algebras must satisfy a polynomial identity:

Theorem (Gateva-Ivanova, Jespers, J.O., 2003)

Assume that the monoid algebra $K[S]$ is right noetherian and $\mathrm{GKdim}(K[S]) < \infty$. Then S is finitely generated and if S has a homogeneous presentation then $K[S]$ satisfies a polynomial identity.

S is defined by a homogeneous presentation if it of the form

$$\langle x_1, \ldots, x_n \mid u_\alpha = w_\alpha, \alpha \in \mathcal{A} \rangle$$

where $u_\alpha = w_\alpha$ are words of the same length for every $\alpha \in \mathcal{A}$. (More generally: relations $w = 0$ are also allowed, but then we deal with the contracted semigroup algebra $K_0[S]$.)

Example 1 - a class of quadratic algebras

$$A = K\langle x_1, \ldots, x_n \mid R\rangle$$

where R is a set of $\binom{n}{2}$ square-free relations of the form

$$x_i x_j = x_{\sigma_i(j)} x_{\gamma_j(i)} \quad \text{with} \quad \sigma_i, \gamma_j \in \mathrm{Sym}_n .$$

Then $A = K[S]$ for a monoid S (called *a nondegenerate monoid of skew type*).

Motivation:
- looking for new constructions of well behaved (noetherian) algebras
- looking for a general framework of set theoretic solutions of the quantum Yang-Baxter equation
- testing difficult open problems concerning finitely presented algebras (on the radical, prime spectrum, ...)

A motivation - the quantum Yang-Baxter equation

Definition

Let Z be a nonempty set and $r : Z^2 \longrightarrow Z^2$ a bijective map. One says that r is a set theoretic solution of the quantum Yang-Baxter equation if it satisfies on Z^3 the relation

$$r_{12}r_{13}r_{23} = r_{23}r_{13}r_{12} \tag{1}$$

where r_{ij} stands for the mapping $Z^3 \longrightarrow Z^3$ obtained by application of r in the components i, j.
The solution is often denoted by (Z, r).

Solving (1) is equivalent to solving the braid equation:
$r_{12}r_{23}r_{12} = r_{23}r_{12}r_{23}$ and we deal with the latter equation only.
The Yang-Baxter equation appeared in statistical mechanics and it turned out to be one of the basic equations in mathematical physics. It lies at the foundations of the theory of quantum groups.

An important problem is to discover all all set theoretic solutions (Drinfeld, 1992).
For example, $r(x_i, x_j) = (x_{\sigma(j)}, x_{\sigma^{-1}(i)})$ is a solution, for every fixed $\sigma \in \mathrm{Sym}_n$, called a permutation solution.

We assume $Z = \{x_1, \ldots, x_n\}$ is a finite set. For a solution (Z, r), let

$$\sigma_i, \gamma_i \colon \{1, \ldots, n\} \longrightarrow \{1, \ldots, n\}$$

be the maps defined by

$$r(x_i, x_k) = (x_{\sigma_i(k)}, x_{\gamma_k(i)}).$$

Definition

Let $Z = \{x_1, x_2, \ldots, x_n\}$, with $n > 1$. A set theoretic solution (Z, r) is called:

(1) involutive if $r^2 = id_{Z \times Z}$,

(2) non-degenerate if all σ_i and all γ_i are bijections,

(3) square free if $r(x_i, x_i) = (x_i, x_i)$ for every i.

Related monoids, groups and algebras

(Z, r) will stand for an involutive non-degenerate square free solution.

Algebraic tools used in this context:

Definition

- Let $S(Z, r)$ (and $G(Z, r)$, respectively) be the monoid and the group defined by the presentation

$$\langle x_1, \ldots, x_n \mid x_i x_j = x_k x_l \text{ if } r(x_i, x_j) = (x_k, x_l) \rangle.$$

  It is called the structure monoid, (resp. the structure group), of the solution (Z, r).
- the group $\langle \sigma_i : i = 1, \ldots, n \rangle \subseteq \mathrm{Sym}_n$, called the involutive Yang-Baxter group of (Z, r).
- $K[S(Z, r)]$ is the K-algebra defined by the above presentation.

Several nice properties of the algebra, similar to these of the polynomial algebra, were proved (using also homological methods developed by Tate and Van den Bergh).

Theorem (T.Gateva-Ivanova, M.Van den Bergh, 2006)

Assume $S(Z,r)$ is the structure monoid of a solution of the Yang-Baxter equation. Then

- *the global dimension of $K[S(Z,r)]$ is n,*
- *$K[S(Z,r)]$ is a noetherian PI-domain,*
- *$S(Z,r)$ has a group of quotients G which is a central localization of $S(Z,r)$.*

A consequence: $S(Z,r)$ has an abelian-by-finite group of quotients $G \cong G(Z,r)$ which is torsion free. One shows also that G is solvable.

In the more general context of algebras of quadratic monoids of skew type:

Theorem (Jespers, JO, 2007)

Let S be a quadratic monoid of skew type. Then the algebra $K[S]$ is noetherian and satisfies a polynomial identity.

The proof is based on several technical observations of combinatorial nature and on the sufficient conditions for noetherianity expressed in terms of a special ideal chain in S with properties of a theorem stated before.
Then one shows that $\mathrm{GKdim}(K[S]) < \infty$ and since defining relations are homogeneous it follows also that $K[S]$ is a PI algebra.
As a consequence (use Anan'in's theorem):

Corollary

$K[S]$ is representable.

Theorem (Cedo, JO, 2012)

Let S be a monoid of skew type. Then $S \subseteq M_m(L)$ for a field extension L of K and some $m \geq 1$ and:

- (i) *S intersects finitely many $\mathcal{H}$-classes of the multiplicative monoid $M_m(L)$.*
- (ii) *If H is a maximal subgroup of $M_m(L)$ such that $S \cap H \neq \emptyset$ then $S \cap H$ generates a finitely generated abelian-by-finite subgroup of H which is the group of quotients of $S \cap H$.*
- (iii) *If e, f are idempotents such that $e \in H_1, f \in H_2$, for some maximal subgroups H_1, H_2 of $M_m(L)$ intersecting S, then $ef = fe$.*
- (iv) *There exists a positive integer N such that $a^N b^N = b^N a^N$ for every $a, b \in S$.*

Linearity of S was crucial for proving statement (iv) of the above theorem, which in turn is the key part of the proof of the following corollary.

Corollary

There exists $N \geq 1$ such that the monoid $C = \langle s^N \mid s \in S \rangle$ is commutative and finitely generated. Moreover,

$$S = \bigcup_{f \in F} fC = \bigcup_{f \in F} Cf$$

for a finite set $F \subseteq S$.
Moreover, C is a disjoint union of cancellative semigroups.

Compare with the special case of the structure monoids of solutions of Y-B equation (they have abelian-by-finite groups of quotients).

An application - automaton algebras

Let $A = K\langle x_1, \ldots, x_n\rangle / J$ be a finitely generated algebra and $\pi : K\langle x_1, \ldots, x_n\rangle \longrightarrow A$ the natural homomorphism.
Let $<$ be a well order on the free monoid $X = \langle x_1, \ldots, x_n\rangle$ compatible with the product; for example a deg-lex order. Let $I = \{w \in X \mid w \text{ is a leading term in some } f \in K\langle X\rangle, \pi(f) = 0\}$.
Define $N(A) = X \setminus I$, the set of *normal words*. Then $N(A)$ represents a basis of A.
Definition (Ufnarovskii) A is an automaton algebra if $N(A)$ is a regular language, that is, $N(A)$ is the set of elements of X recognized by a finite automaton.
If I is a finitely generated ideal of X (equivalently, A has a finite Gröbner basis) then this is the case. Note: if J is a finitely generated ideal (so A is finitely presented), then I is not finitely generated, in general.
Warning: this notion depends on the chosen presentation of the algebra; it also depends on the chosen ordering of the monoid X.

It is believed that in the class of automaton algebras (they are expected to have better combinatorial properties) certain nice structural results are also possible. In particular, some of the general open problems on finitely generated algebras might have a positive solution. Some results confirming this expectation are known.

Problem Let A be a quadratic algebra of skew type. Is A an automaton algebra?

A reason for stating this problem:

Theorem (W.Rump, 2005)

Let $S = S(X, r)$ be the structure monoid of a (square-free, nonndegenerate, involutive) solution of the Yang-Baxter equation. Then, after renumbering the generators $S = \{x_1^{i_1} \cdots x_n^{i_n} \mid i_j \geq 0\}$, and every element of S is uniquely presented in the above form (normal form of elements of S).

So $A = K[S]$ is an automaton algebra.

Theorem (F.Cedo, J.O., 2012)

If $A = K[S]$ is an algebra of skew type then the set $N(S)$ of normal forms of elements of S (with respect to any deg-lex order) is a regular language. So A is an automaton algebra.

The proof is of a combinatorial nature but it strongly relies on condition (iv) proved above.

Example 2 - representability of bands

S - a semigroup with operation written multiplicatively.
S is a band if $a^2 = a$ for every $a \in S$.

Motivating problem (L. Livshits, G. Macdonald, B. Mathes and H. Radjavi, 1998):
Find conditions on a band S in order that S embeds into the multiplicative semigroup $M_n(L)$ of $n \times n$ matrices over a field L for some $n \geq 1$.

Note: here we deal with semigroups that are not finitely generated. Namely:

1. if S is finitely generated, then S is finite, whence it is representable,

2. if $K[S]$ is right noetherian then S is finite.

A commutative band is called a semilattice.

Theorem

Let S be a band. Then $S = \bigcup_{\alpha \in \Gamma} S_\alpha$, a disjoint union, where Γ is a semilattice, each S_α is a rectangular band and $S_\alpha S_\beta \subseteq S_{\alpha\beta}$.

We say that S is a semilattice of rectangular bands (recall: they defined by the identity $xyz = xz$). This yields a Γ-gradation on S. Actually, the relation ρ defined by

$$(a, b) \in \rho \quad \text{if} \quad aba = a \text{ and } bab = b$$

is a congruence on S, the sets S_α are the ρ-classes and $S/\rho = \Gamma$. The subbands S_α are called the components of S.

For a regular semigroup S a general structure theorem for linear semigroups easily leads to the following

Lemma

If $S \subseteq M_n(L)$ is a band then S has at most 2^n components.

Theorem

If $S \subseteq M_n(L)$ is a band then S is triangularizable.

This follows easily from the fact that the semigroup algebra $L[S]$ modulo its (nilpotent) radical is isomorphic to L^r, where r is the number of components of S.

Proposition

Let S be a rectangular band. Then S embeds into $M_3(L)$ for a field L. Actually, for every field K, the semigroup algebra $K[S]$ embeds into the algebra $M_3(L)$ for a field L (depending on K and on S).

A concrete matrix embedding of S is of the form

$$x \mapsto \begin{pmatrix} 0 & \mu(xf) & \mu(xf)\eta(fx) \\ 0 & 1 & \eta(fx) \\ 0 & 0 & 0 \end{pmatrix} \in M_3(L)$$

where $f \in S = SffS$ is a fixed element and $\mu : Sf \longrightarrow L, \eta : fS \longrightarrow L$ are injective maps. It is easy to see that, for any field K, choosing the field L big enough and choosing appropriate maps μ, η, we also get an embedding of algebras $K[S] \hookrightarrow M_3(L)$.

There exists a band with two components $S = E \cup F$ which is not linear. In this example, E a left zero and F a right zero semigroup. Examples with both E and F right zero semigroups are also known.

Example

Let $S = E \cup F$, a disjoint union, where
$E = \{e_i \mid i \geq 1\}, X = \{f_i \mid i \geq 1\}, X' = \{f_i' \mid i \geq 1\}$ and
$F = X \cup X'$. Define the operation:

$ee' = e$ and $ff' = f'$ for $e, e' \in E, f, f' \in F$,

$x'e = x'$ for $x' \in X', e \in E$

$ef = f$ for $e \in E, f \in F$,

$f_j e_i = f_j'$ for $j = 1, \ldots, i-1$

$f_j e_i = f_i'$ for $j \geq i$.

Then S is a band with components E, F and F is an ideal of S.
The right annihilator

$$\mathrm{r}_S(f_j - f_j') = \{x \in S \mid f_j x = f_j' x\}$$

of $f_j - f_j'$ in S contains e_i with $i \geq j$ and does not contain $e_1, \ldots, e_{j-1}$. It follows that there is an infinite descending chain of such annihilators. Therefore S is not a linear semigroup.

Results

1. Triangular embeddings over commutative rings

For a K-algebra A, denote by $T_n(A)$ the algebra of upper triangular matrices $m = (m_{ij})$ over A with $m_{ii} \in K$ for every $i = 1, \ldots, n$.

Theorem (F.Cedo, J.O., 2007)

Let $S = \bigcup_{\gamma \in \Gamma} S_\gamma$ be a band with finitely many components. Then the Jacobson radical $J(K[S])$ of $K[S]$ is nilpotent and $K[S]$ embeds into $T_n(A)$ for a commutative K-algebra A.

The proof relies on the result of Bergman and Vovsi and an induction on the number of components of S.

Corollary

The semigroup algebra $K[S]$ of a band S with finitely many components satisfies a standard polynomial identity.

2. An embedding of a band S with two components into $M_n(A)$

This is based on a commutative algebra A naturally associated to a band S.
For any field K, we construct a commutative algebra A such that $K[S]$ embeds into the ring $T_7(A)$ of upper triangular matrices over $\bar{R}$ with diagonal in K. This A is a semigroup algebra $K[C]$ of a commutative semigroup C.

This leads to the questions:

1. Can representability of S (over a field) can be characterized in terms of the algebra A?

2. Can representability of $K[S]$ be expressed in terms of A?

We illustrate the idea with the case where S has two components. Let S be a band with two components F, E such that F is an ideal of S. We may assume that

$$F = \{f_{i,j} \mid i \in I, j \in J\}, E = \{e_{a,b} \mid a \in A, b \in B\},$$

and

$$f_{i,j}f_{k,l} = f_{i,l} \quad \text{and} \quad e_{a,b}e_{c,d} = e_{a,d},$$

for all $i, k \in I$, $j, l \in J$, $a, c \in A$ and $b, d \in B$. Let $\alpha\colon A \times B \times I \to I$ be the map defined by

$$e_{a,b}f_{i,j} = f_{\alpha(a,b,i),j}.$$

Let $\beta\colon J \times A \times B \to J$ be the map defined by

$$f_{i,j}e_{a,b} = f_{i,\beta(j,a,b)}.$$

Let $X = \{x_i \mid i \in I\}$, $Y = \{y_j \mid j \in J\}$,
$Z = \{z_a \mid a \in A\}$, $Z' = \{z'_a \mid a \in A\}$,
$T = \{t_b \mid b \in B\}$, $T' = \{t'_b \mid b \in B\}$
be disjoint sets of commuting indeterminates over K. Let $R = K[X \cup Y \cup Z \cup Z' \cup T \cup T']$ be the corresponding polynomial ring. Let M be the ideal of R generated by:

(i) $t_b x_i - t_d x_k$, for all $i, k \in I$ and all $b, d \in B$ such that there exists $a \in A$ satisfying $\alpha(a, b, i) = \alpha(a, d, k)$,

(ii) $z_a t_b x_i - z_c t_d x_k$, for all $i, k \in I$, $a, c \in A$ and $b, d \in B$ such that $\alpha(a, b, i) = \alpha(c, d, k)$,

(iii) $y_j z'_a - y_l z'_c$, for all $j, l \in J$ and all $a, c \in A$ such that there exists $b \in B$ satisfying $\beta(j, a, b) = \beta(l, c, b)$,

(iv) $y_j z'_a t'_b - y_l z'_c t'_d$, for all $j, l \in J$, $a, c \in A$ and $b, d \in B$ such that $\beta(j, a, b) = \beta(l, c, d)$.

Let $\bar{R} = R/M$. Then $\rho \colon S \to T_7(\bar{R})$ given below is an embedding:

$$\rho(e_{a,b}) = \begin{pmatrix} 0 & \overline{z_a} & \overline{z_a t_b} & 0 & 0 & 0 & 0 \\ 0 & 1 & \overline{t_b} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \overline{z'_a} & \overline{z'_a t'_b} \\ 0 & 0 & 0 & 0 & 0 & 1 & \overline{t'_b} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

for all $a \in A$ and $b \in B$,

$$\rho(f_{i,j}) = \begin{pmatrix} 0 & 0 & 0 & \overline{z_a t_b x_k} & 0 & \overline{z_a t_b x_k y_l z'_c} & \overline{z_a t_b x_k y_l z'_c t'_d} \\ 0 & 0 & 0 & \overline{t_b x_k} & 0 & \overline{t_b x_k y_l z'_c} & \overline{t_b x_k y_l z'_c t'_d} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & \overline{y_l z'_c} & \overline{y_l z'_c t'_d} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

for all $f_{i,j} \in EFE$ such that $\alpha(a, b, k) = i$ and $\beta(l, c, d) = j$,

$$\rho(f_{i,j}) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \overline{x_i} & 0 & \overline{x_i y_l z'_c} & \overline{x_i y_l z'_c t'_d} \\ 0 & 0 & 0 & 1 & 0 & \overline{y_l z'_c} & \overline{y_l z'_c t'_d} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

for all $f_{i,j} \in FE \setminus EFE$ such that $\beta(l, c, d) = j$,

$$\rho(f_{i,j}) = \begin{pmatrix} 0 & 0 & 0 & \overline{z_a t_b x_k} & \overline{z_a t_b x_k y_j} & 0 & 0 \\ 0 & 0 & 0 & \overline{t_b x_k} & \overline{t_b x_k y_j} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & \overline{y_j} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

for all $f_{i,j} \in EF \setminus EFE$ such that $\alpha(a, b, k) = i$,

$$\rho(f_{i,j}) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \overline{x_i} & \overline{x_i y_j} & 0 & 0 \\ 0 & 0 & 0 & 1 & \overline{y_j} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

for all $f_{i,j} \in F \setminus (EF \cup FE)$.

Conclusion (from the point of view of Malcev's approach): every finitely generated subsemigroup of a band is finite, hence representable; however it seems very difficult to find conditions under which all such representations have a bounded dimension. Though S is embeddable in matrices over a commutative algebra.