Inverse semigroups and groupoids via quantales

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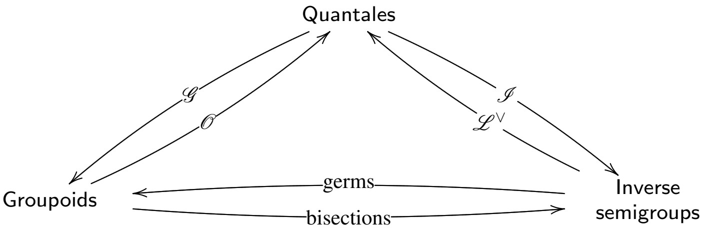
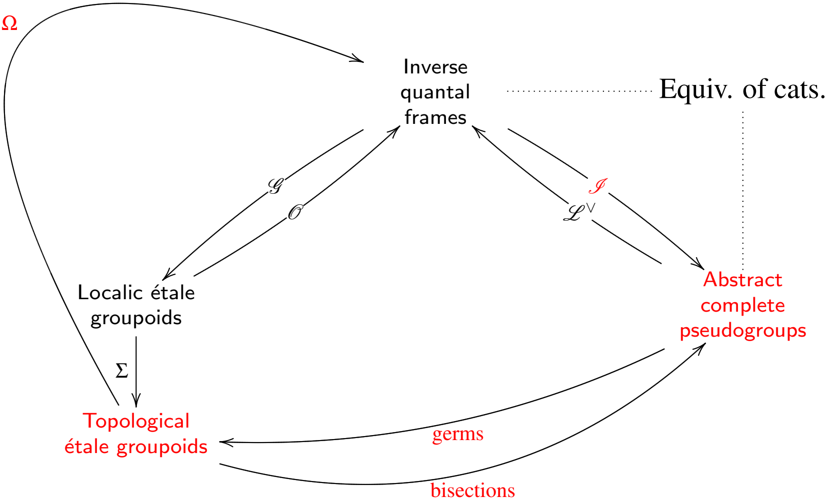
ROADMAP


Quantales
$\mathscr{G}$
$\mathscr{O}$
$\mathscr{I}$
$\mathscr{L}^{\vee}$
Groupoids
germs
bisections
Inverse
semigroups

Roadmap



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Localic étale groupoids
Abstract complete pseudogroups
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Plan

- PART I: Locales
  - Point-free topology as commutative algebra: sup-lattices ("abelian groups"), locales ("commutative rings")
  - Relation to pointset topology
- PART II: Groupoids
  - Groupoids and inverse semigroups as noncommutative algebra: quantales ("noncommutative rings")
  - Relation to topological groupoids

LOCALES

SUP-LATTICES

Sup-lattices (aka complete lattices) are the "abelian groups" of point-free topology.

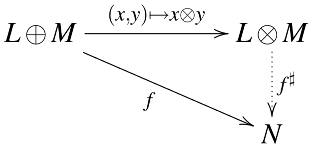
$a+b$ becomes $a \vee b$ and there are sum-operations of unbounded arity:

$$\bigvee_i a_i$$

Homomorphisms: $f\left(\bigvee_i a_i\right) = \bigvee_i f(a_i)$.

Direct sum $L \oplus M$ (= cartesian product) is both a product and a coproduct.

Tensor product = image of universal "bilinear" map:

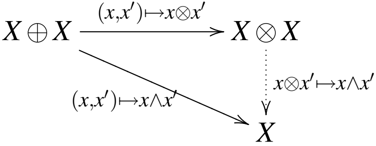


Locales are sup-lattices satisfying the following distributivity property:

$$x \wedge \bigvee_i y_i = \bigvee_i x \wedge y_i$$

Motivating example: the topology $\Omega(S)$ of a space S.

A locale X is a commutative (and idempotent) "ring":



Locales

A **locale homomorphism** $\varphi : X \to Y$ (= "ring homomorphism") is a sup-lattice homomorphism such that:

$$\begin{aligned} \varphi(x_1 \wedge x_2) &= \varphi(x_1) \wedge \varphi(x_2) \\ \varphi(1_X) &= 1_Y \end{aligned}$$

Motivating example: a continuous map of topological spaces $f : S \to T$ yields a homomorphism

$$f^{-1} : \Omega(T) \to \Omega(S) .$$

Definition

Category $\mathbf{Loc}$:

- **Objects:** locales
- **Arrows:** a "continuous map" $f : X \to Y$ is a homomorphism $f^* : Y \to X$.

The assignments $S \mapsto \Omega(S)$ and $f \mapsto f^{-1}$ define a functor $\Omega : \mathbf{Top} \to \mathbf{Loc}$.

Homeomorphism = isomorphism f in $\mathbf{Loc}$ — i.e. f^* has an inverse.

Sublocale $s : S \rightarrowtail X$ = surjective locale homomorphism $s^* : X \twoheadrightarrow S$ (up to an isomorphism).

Restriction of $f : X \to Y$ to a sublocale $s : S \rightarrowtail X$ is

$$S \overset{s}{\rightarrowtail} X \xrightarrow{f} Y, \qquad \text{“}f|_S\text{”} = f \circ s : S \to Y$$

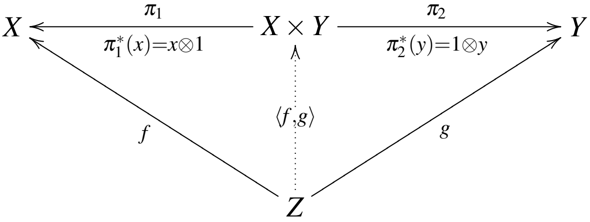
Image of $f : X \to Y$ = sublocale of Y defined by $f^* : Y \twoheadrightarrow f^*(Y)$.

Open sublocales are of the form $s_x = (-) \wedge x : X \twoheadrightarrow \downarrow(x)$ $\sim$ “subspace x with relative topology”.

$f : X \to Y$ is a **local homeomorphism** $\iff$ $\exists$ family (x_i) in X such that $\bigvee_i x_i = 1$ (an “open cover of X”) and each restriction $f \circ s_{x_i}$ defines a homeomorphism of locales whose image is an open sublocale of Y.

Coproduct of locales $X + Y$ = direct sum $X \oplus Y$.

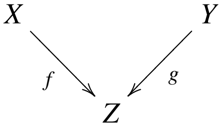
Product of locales $X \times Y$ = tensor product as sup-lattices $X \otimes Y$:



$$\langle f, g\rangle^*(x \otimes y) = f(x) \wedge g(y)$$

LOCALES

Consider the following maps in **Loc**:



For topological spaces the **pullback** (aka **fibered product**) $X \times_Z Y$ would be the subspace of the product space $X \times Y$ consisting of those pairs (x, y) such that $f(x) = g(y)$.

For locales the **pullback** is a sublocale of $X \otimes Y$. Algebraically it is a quotient.

The homomorphisms $f^* : Z \to X$ and $g^* : Z \to Y$ turn both X and Y into Z-**modules**, by change of "base ring":

$$z \cdot x := f^*(z) \wedge x \qquad z \cdot y := g^*(z) \wedge y$$

The pullback coincides with $X \otimes_Z Y$ (= sup-lattice quotient determined by the relations $z \cdot x \otimes y = x \otimes z \cdot y$).

Locales versus topology

The "locale with one point" is

$$\mathbf{2} \;:=\; \begin{array}{c} 1 \\ | \\ 0 \end{array}$$

A **point** of a locale X is a map $p : \mathbf{2} \to X$.

Let $\mathscr{P} = \{\text{points of } X\}$.

For each $x \in X$ define $U_x = \{p \in \mathscr{P} \mid p^*(x) = 1\}$.

The **spectrum** of X is the topological space $\Sigma(X) := (\mathscr{P}, \mathscr{T})$ whose topology is $\mathscr{T} = \{U_x \mid x \in X\}$.

This defines a functor $\Sigma : \mathbf{Loc} \to \mathbf{Top}$, where for each map $f : X \to Y$ the continuous map $\Sigma(f) : \Sigma(X) \to \Sigma(Y)$ is defined by

$$\Sigma(f)(p) = f \circ p \,.$$

Locales versus topology

So we have two functors: $\mathbf{Top} \underset{\Sigma}{\overset{\Omega}{\rightleftarrows}} \mathbf{Loc}$.

The assignment $x \mapsto U_x$ is a surjective homomorphism $X \to \Omega(\Sigma(X))$.

Definition

X is **spatial** if this is an isomorphism.

There are also continuous maps $S \to \Sigma(\Omega(S))$ given by:

$$s \mapsto \tilde{s} \ , \qquad \text{where } \tilde{s}(U) = 1 \text{ iff } s \in U.$$

Definition

S is **sober** if $s \mapsto \tilde{s}$ is a homeomorphism.

Important example: *Hausdorff spaces are sober.*

Σ is right adjoint to Ω and the adjunction restricts to an equivalence of categories between spatial locales and sober spaces.

$\Rightarrow$ Σ preserves limits: e.g., $\Sigma(X \otimes_Z Y) \cong \Sigma(X) \times_{\Sigma(Z)} \Sigma(Y)$.

Example

The Locale of Real Numbers

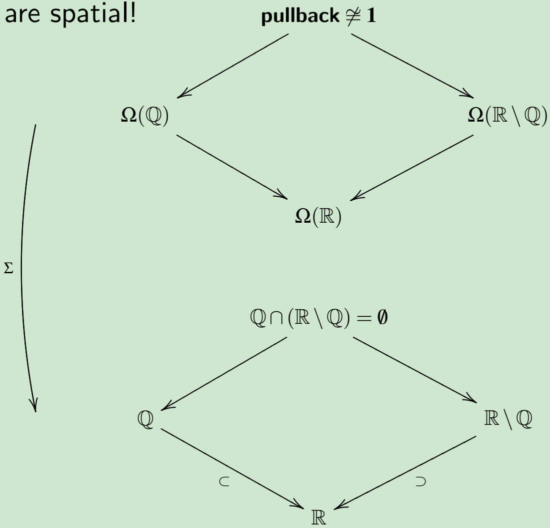
Generators: symbols (q,∞) and (∞,q) with $q \in \mathbb{Q}$

Relations:

$$
\begin{array}{rcl}
1 & = & \bigvee_q (q,\infty) \\
(q',\infty) & \leq & (q,\infty) \quad \text{for all } q < q' \\
(q,\infty) & \leq & \bigvee_{q<q'} (q',\infty) \\
& \vdots & \\
(q,\infty) \wedge (\infty,q) & \leq & 0 \\
1 & \leq & (q,\infty) \vee (\infty,r) \quad \text{for all } q < r
\end{array}
$$

EXAMPLE

Not all locales are spatial!



Example

A **localic group** G with multiplication

$$m : G \otimes G \to G$$

yields a topological group $\Sigma(G)$ with multiplication

$$\Sigma(G) \times \Sigma(G) \xrightarrow{\cong} \Sigma(G \otimes G) \xrightarrow{\Sigma(m)} \Sigma(G) \; .$$

In general the converse, even for sober spaces, is not true because the product of spatial locales may fail to be spatial (e.g. $\Omega(\mathbb{Q}) \otimes \Omega(\mathbb{Q}) \ncong \Omega(\mathbb{Q}^2)$).

But a **locally compact Hausdorff group** is always a localic group because if either S or T is a locally compact space we have

$$\Omega(S \times T) \cong \Omega(S) \otimes \Omega(T) \; .$$

Locales versus topology

Theorem

Localic Tychonoff theorem: $\prod_\alpha X_\alpha$ *is compact if* X_α *is compact for all* α.

(Axiom of choice not needed!)

An important fact that depends on the axiom of choice:

Theorem

Any **coherent locale** *(≅ ideal completion of bounded distributive lattice) is spatial.*

The proof needs Zorn's Lemma in order to show that if two ideals I and J of a bounded distributive lattice D are related by

$$I \not\subset J$$

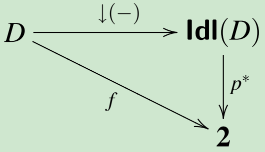
then there exists a prime ideal P of D such that

$$J \subset P \quad \text{and} \quad I \not\subset P\,.$$

EXAMPLE

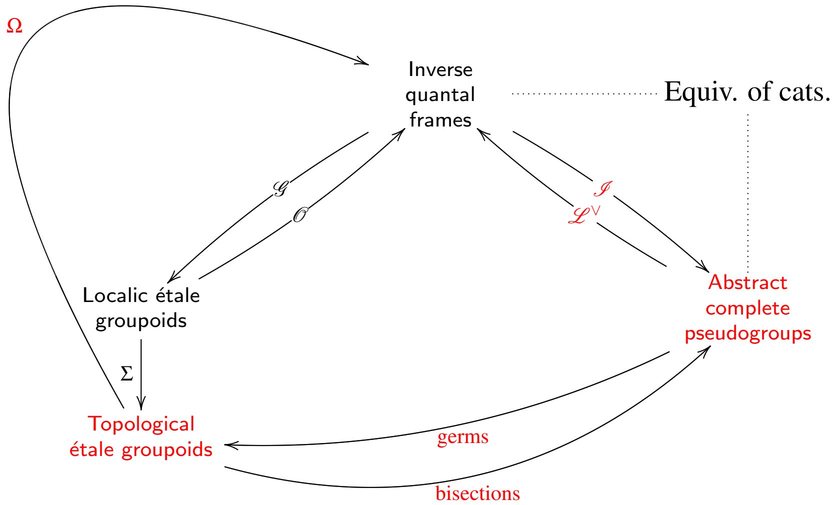
Points as filters:

- Universal property:



- The point p can be identified with $f^{-1}(1)$, a **prime filter** of D.
- If M is a meet-semilattice, $\mathscr{L}(M)$ is a locale whose points can be identified with **filters** of M.
- If S is a complete infinitely distributive inverse semigroup, $\mathscr{L}^{\vee}(S)$ is a locale whose points are the "**compatibly prime filters**" of S ($=$ **germs** $=$ **groupoid arrows**).

ROADMAP



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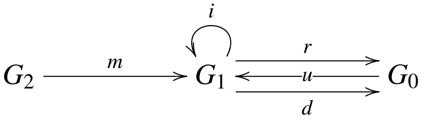
GROUPOIDS and QUANTALES

Groupoids

In a category with "enough" pullbacks, a **groupoid** consists of objects G_0, G_1 and G_2 equipped with structure morphisms as follows,



where $G_2 = G_1 \times_{G_0} G_1$ is the pullback of d and r, satisfying the usual axioms of an internal category (associativity of multiplication, etc.) plus those asserting that i is an inversion operation.

Example

1. **Topological groupoids:** the underlying category is **Top**.
2. **Lie groupoids:** the underlying category is that of smooth manifolds; and d and r are required to be submersions (so that the needed pullbacks exist).
3. **Localic groupoids:** the underlying category is **Loc**.

Let the following be a localic groupoid G:

$$G_2 = G_1 \otimes_{G_0} G_1 \xrightarrow{\qquad m \qquad} \overset{\overset{i}{\circlearrowleft}}{G_1} \begin{array}{c} \xrightarrow{\qquad r \qquad} \\ \xleftarrow{\qquad u \qquad} \\ \xrightarrow[\qquad d \qquad]{} \end{array} G_0$$

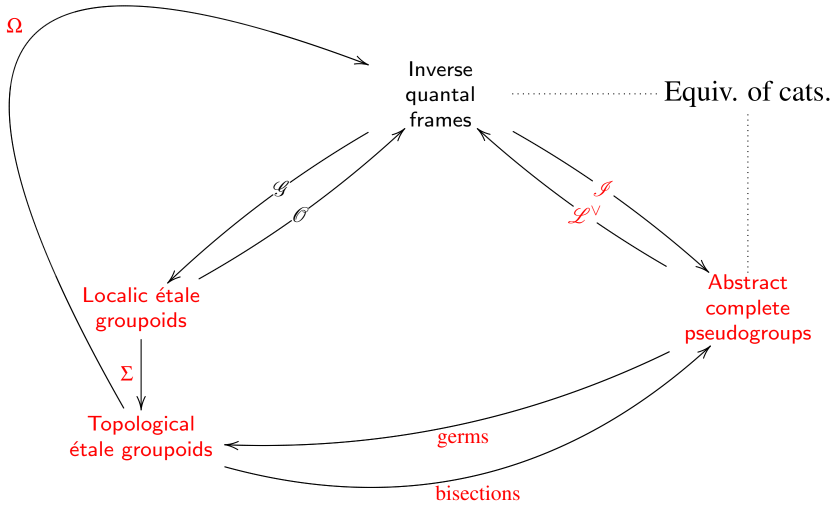
One of the axioms states that $d \circ u = \mathrm{id}_{G_0}$ — this makes u is a sublocale.

G is **étale** if d is a local homeomorphism ($\Rightarrow$ all the maps are local homeomorphisms and u is an open sublocale).

Since Σ preserves limits we immediately obtain a topological groupoid $\Sigma(G)$ (étale if G is) whose structure maps are as follows:

$$\underbrace{\Sigma(G_1) \times_{\Sigma(G_0)} \Sigma(G_1) \xrightarrow{\;\cong\;} \Sigma(G_2) \xrightarrow{\qquad \Sigma(m) \qquad} \overset{\overset{\Sigma(i)}{\circlearrowleft}}{\Sigma(G_1)}}_{\text{multiplication map}} \begin{array}{c} \xrightarrow{\qquad \Sigma(r) \qquad} \\ \xleftarrow{\qquad \Sigma(u) \qquad} \\ \xrightarrow[\qquad \Sigma(d) \qquad]{} \end{array} \Sigma(G_0)$$

Roadmap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Quantales

By an **involutive quantale** is meant a sup-lattice Q equipped with an associative **multiplication** $(a,b) \mapsto ab$ and an **involution** $a \mapsto a^*$ satisfying:

$$\begin{aligned} a\left(\bigvee_i b_i\right) &= \bigvee_i ab_i \\ \left(\bigvee_i a_i\right)b &= \bigvee_i a_i b \end{aligned} \qquad \begin{aligned} a^{**} &= a \\ (ab)^* &= b^* a^* \\ \left(\bigvee_i a_i\right)^* &= \bigvee_i a_i^* \end{aligned}$$

The name "quantale" was coined by Mulvey in 1983, standing for "quantum locale", in the context of C*-algebras.

A $*$-**homomorphism** of involutive quantales $h : Q \to R$ is a homomorphism of involutive semigroups that preserves $\bigvee$.

Q is **unital** is it has a semigroup unit e.

Example

- $P(S \times S)$ for any set S (unital and involutive).
- $P(G)$ for any group G (unital and involutive).
- $\Omega(G)$ for any locally compact group G (involutive).

EXAMPLE

(Continued)

- $P(C_1)$ for any small category C (unital).
- $\mathrm{Sub}_k(A)$ for any k-algebra A (unital if A is).
- $\mathrm{Max}(A)$ for any C*-algebra A (involutive; unital if A is): $A \cong B \iff \mathrm{Max}(A) \cong \mathrm{Max}(B)$ [Kruml–R 2004].
- $\Omega(G_1)$ for a topological étale groupoid G (unital and involutive).
- $\mathscr{L}(S)$ for an inverse semigroup S (unital and involutive).
- $\mathscr{L}^{\vee}(S)$ for an abstract complete pseudogroup S (unital and involutive).

Quantales

Definition

By an **inverse quantal frame** is meant a locale Q equipped with the additional structure of a unital involutive quantale such that, defining for all $a \in Q$

$$\begin{aligned} \varsigma(a) &= a1 \wedge e \\ \mathscr{I}(Q) &= \{s \in Q \mid ss^* \leq e,\ s^*s \leq e\} \end{aligned}$$

the following three conditions are satisfied for all $a \in Q$:

$$\begin{aligned} \varsigma(a) &\leq aa^* \\ a &\leq \varsigma(a)a \\ \bigvee \mathscr{I}(Q) &= 1 \end{aligned}$$

Quantales

Fact 1: if Q is an inverse quantal frame $\mathscr{I}(Q)$ is an abstract complete pseudogroup.

Fact 2: if Q is an inverse quantal frame we have a surjective homomorphism of unital involutive quantales

$$\begin{array}{rcccc} \varepsilon & : & \mathscr{L}^{\vee}(\mathscr{I}(Q)) & \twoheadrightarrow & Q \\ & & K & \stackrel{\varepsilon}{\mapsto} & \bigvee K \end{array}$$

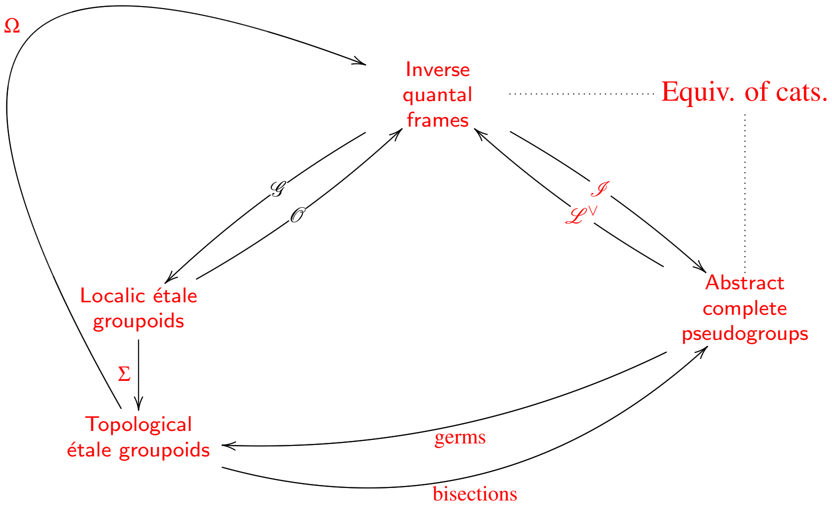
Fact 3: ε is an isomorphism.

Fact 4: $S \cong \mathscr{I}(\mathscr{L}^{\vee}(S))$ for every abstract complete pseudogroup S.

Theorem (R 2007)

The category of inverse quantal frames (with homomorphisms of unital involutive quantales as arrows) is equivalent to the category of abstract complete pseudogroups (with monoid homomorphisms that preserve joins of compatible subsets).

Roadmap



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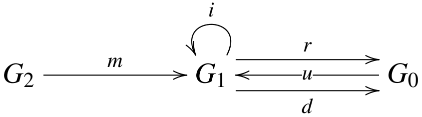
Let Q be an inverse quantal frame. Its **groupoid** $\mathscr{G}(Q)$ is the localic groupoid



defined by, for all $a \in Q$ and all $b \in \downarrow(e)$:

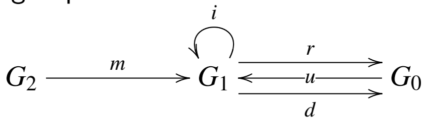
- $G_1 = Q$ (as locales)
- $G_0 = \downarrow(e)$
- $i^*(a) = a^*$
- $u^*(a) = a \wedge e$
- $d^*(b) = b1$ (right adjoint to $\varsigma : Q \to \downarrow(e)$)
- $r^*(b) = 1b$
- $m^*(a) = \bigvee_{xy \leq a} x \otimes y$ $\left(= \bigvee_{s \in \mathscr{I}(Q)} s \otimes s^* a\right.$ [R 2012])

THEOREM (R 2007)

$\mathscr{G}(Q)$ is a localic étale groupoid.

The quantale of a localic étale groupoid

Let G be the localic groupoid



The inverse quantal frame $Q = \mathscr{O}(G)$ is defined by:

- $Q = G_1$ (as locales)
- $a^* = i^*(a)$ for all $a \in Q$
- $ab = m_!(a \otimes b)$ for all $a, b \in Q$, where $m_!$ is the left adjoint of m^* ($m_!$ exists because G is étale)

Theorem (R 2007)

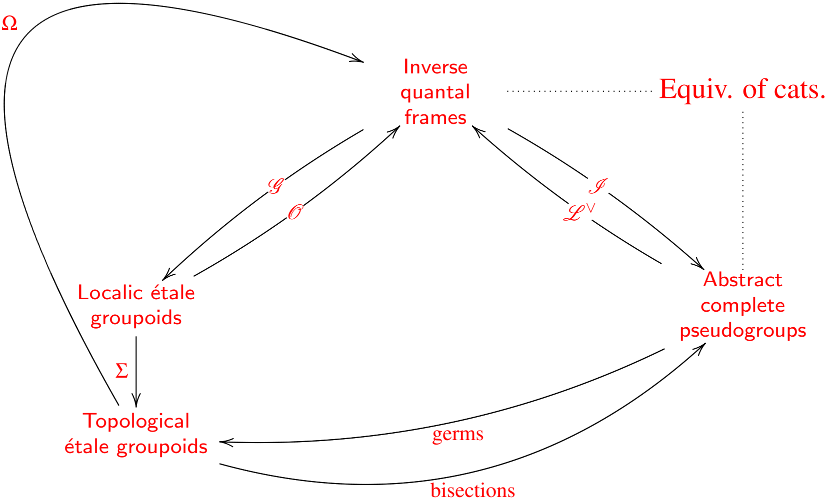
$\mathscr{O}(Q)$ is an inverse quantal frame.

Theorem

G étale groupoid $\Rightarrow$ $G \cong \mathscr{G}(\mathscr{O}(G))$

Q inverse quantal frame $\Rightarrow$ $Q \cong \mathscr{O}(\mathscr{G}(Q))$

ROADMAP



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Conclusion

- Open groupoids [Protin–R 2012]
- Groupoid sheaves as modules on inverse quantal frames [R 2012]
- Topos theoretic applications
- Functoriality [Kudryavtseva, Lawson, Lenz]
- Applications to logic [Marcelino–R 2008]
- Applications to C*-algebras stemming from [Kruml–R 2004], [Mulvey–Pelletier 2001]

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