Topology and geometry: why is a football like a golf ball?

Geometry is about lengths and angles. Here are two problems from geometry.

Question. Two of the angles in a triangle are 90 degrees and 30 degrees. What is the third angle?

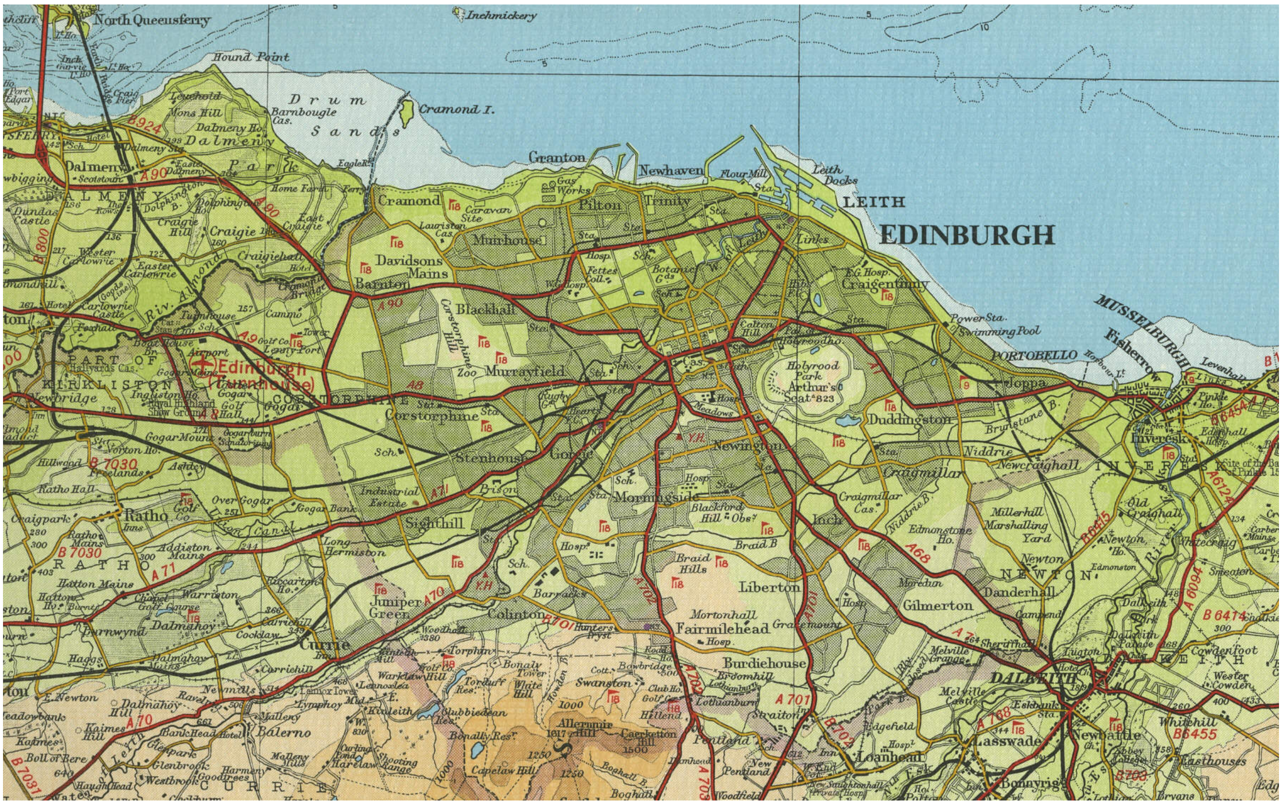
Question. In a right angled triangle the sides forming the right angle have lengths 3 and 4. What is the length of the side opposite the right angle?

Topology is geometry when we forget about lengths and angles.

Is this even a thing?

Yes.

Map of Edinburgh (geometric)



Map of the London Underground (topological)

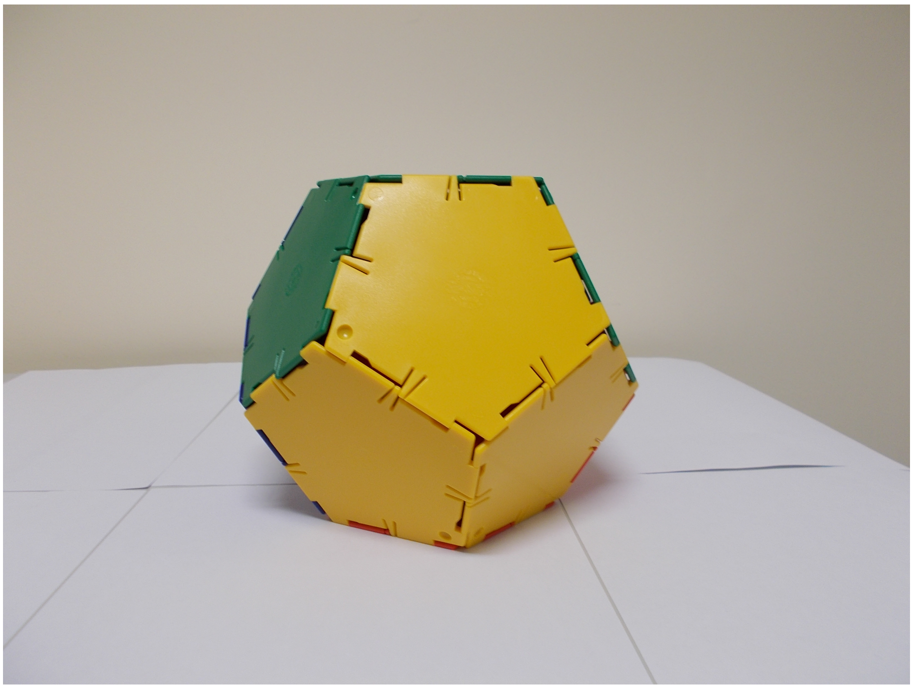
Sometimes *geometry* is the right way to think about a problem and sometimes *topology*.

It depends on the problem.

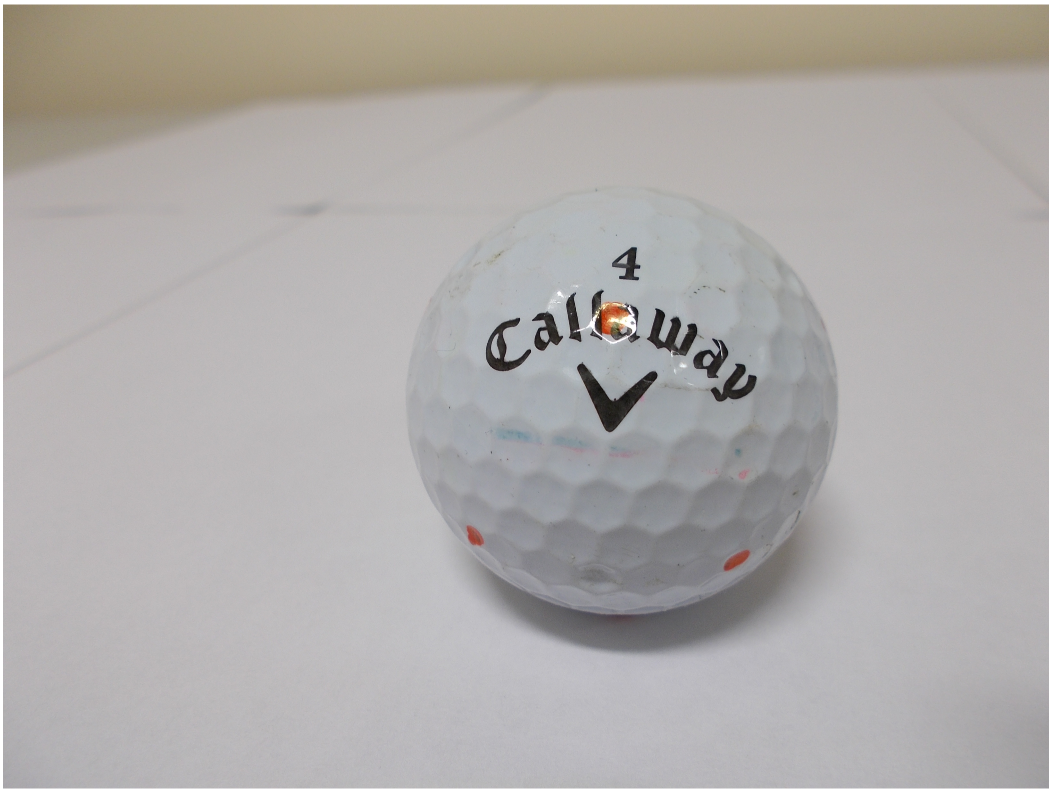
In this talk, I will show you a result that can be understood purely in terms of *topology.*

Question. What do a
dodahedron,
a football
and a Callaway Hex golf ball
have in common?

Dodecahedron



Football and golf ball



The football and golf ball are constructed from hexagons and pentagons.

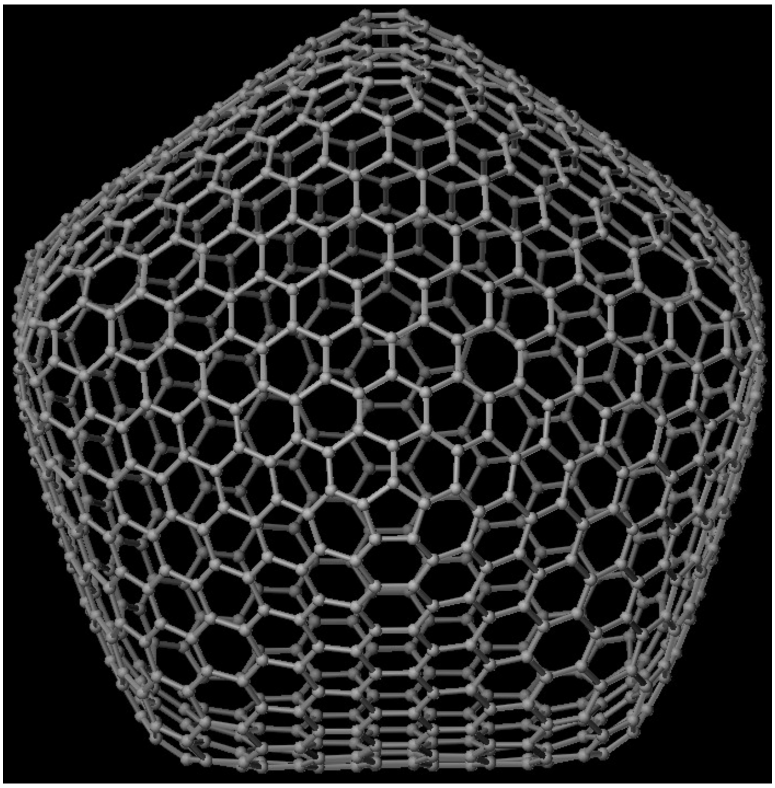
The dodecahedron is constructed only from pentagons.

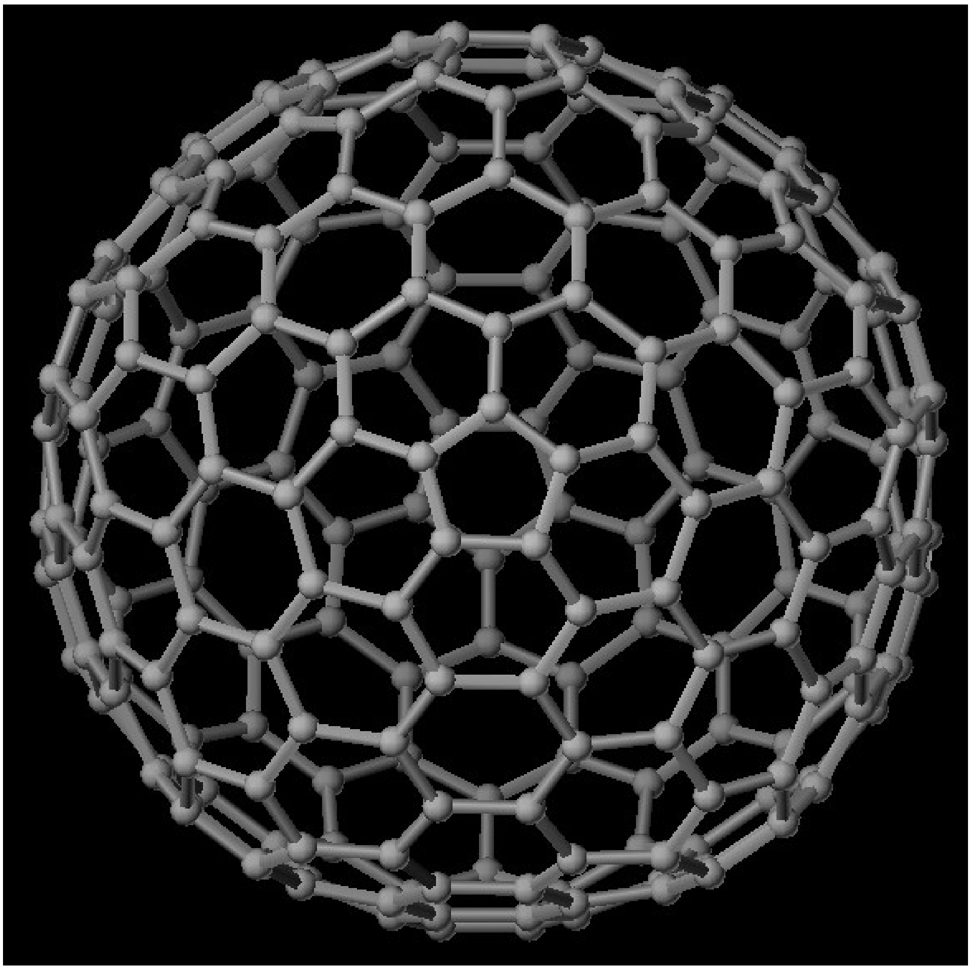
Question. How many pentagons are used in each case?

Balls made from hexagons and pentagons arise in many situations.

The Eden project
St. Blazey, Cornwall

Buckyballs
Different forms of carbon





What is the mathematics of balls made from hexagons and pentagons?

The 12 Pentagon Theorem

If you want to make a ball out of hexagons (such that each vertex has degree 3) then you must also use pentagons; in fact, you must use ***exactly*** *12 pentagons.*

We shall prove this result by *counting* and by *algebra* but *not by geometry* in the usual sense.

Proving the theorem

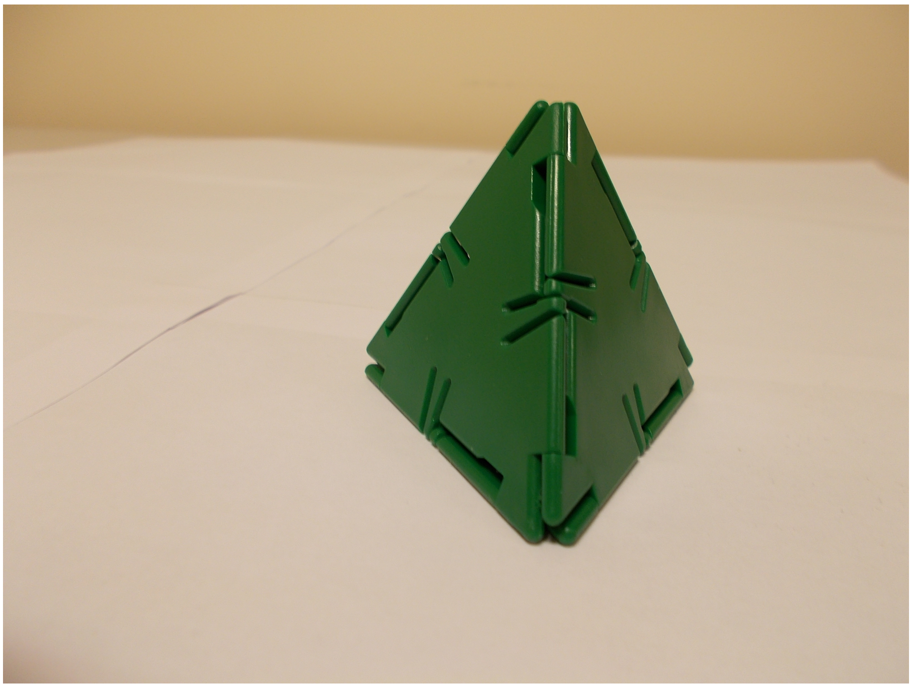
We will need five definitions:

- *faces:* the number of faces is *f.*
- *edges:* the number of edges is *e.*
- *vertices:* the number of vertices *is v.*
- *vertex degree.*This is the number of edges radiating from each vertex.
- *face degree.* This is the number of sides each face has.

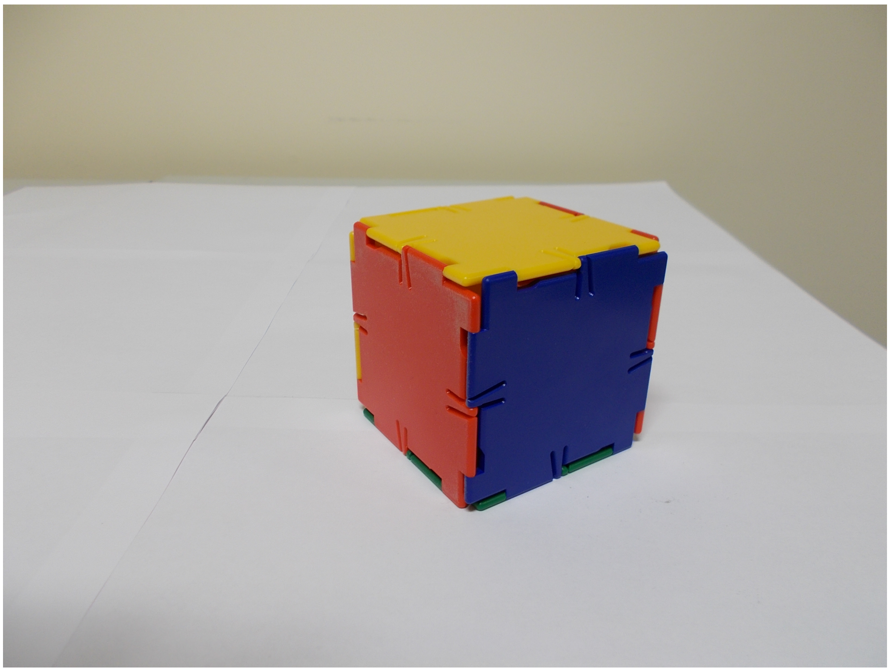
We shall illustrate these ideas by looking at the tetrahedron and the cube.

	f	*e*	*v*	Face degree	Vertex degree
Tetrahedron					
Cube					

Tetrahedron



Cube



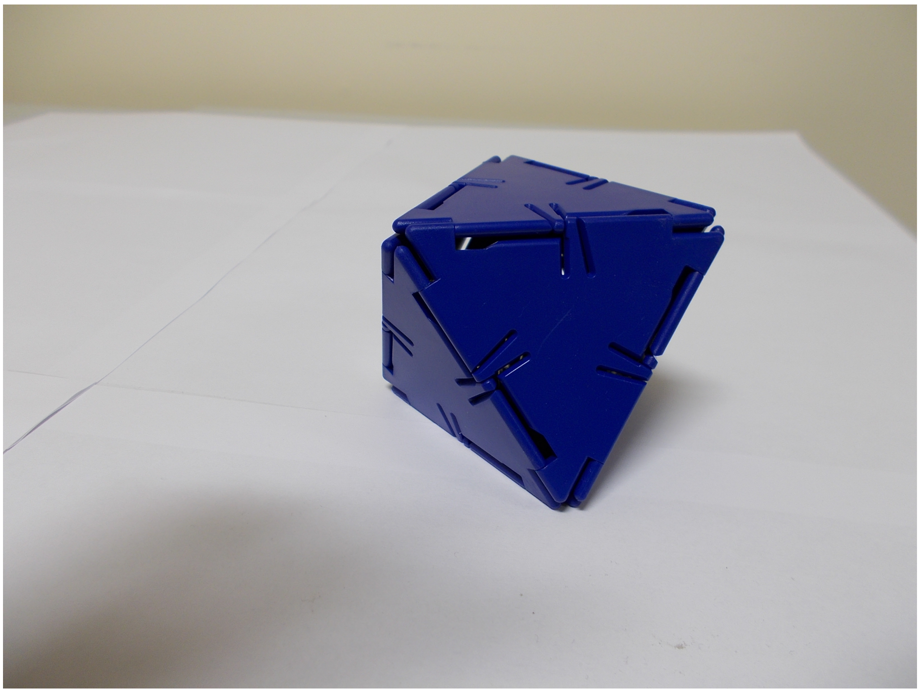
	f	e	v	Face degree	Vertex degree
Tetrahedron	4	6	4	3	3
Cube	6	12	8	4	3

Number	**Greek word (in English)**
4	tetra
8	octa
12	dodeca
20	icosa

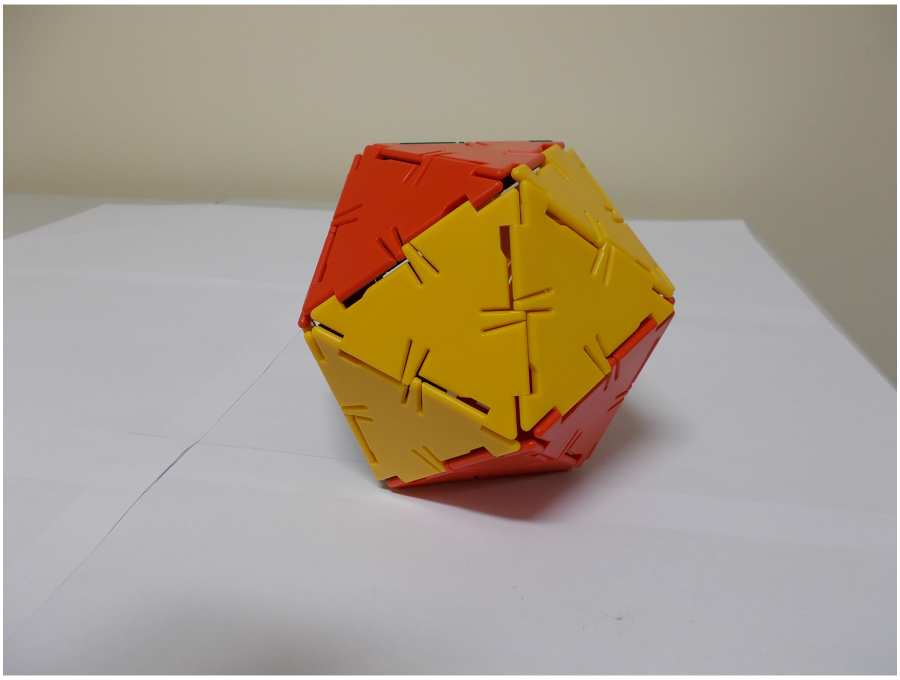
Complete the last column of this table.

	f	e	v	$f - e + v$
Tetrahedron	4	6	4	
Cube	6	12	8	
Octahedron	8	12	6	
Dodecahedron	12	30	20	
Icosahedron	20	30	12	

Octahedron



Icosahedron



	f	e	v	f – e - v
Football	32	90	60	
Hex Golf ball	344	1026	684	

Results

These calculations suggest, but don't prove, the following. The proof is not hard but would need another lecture.

Euler's Theorem

For connected planar graphs

$$f - e + v = 2.$$

The sum of the face degrees = $2e$.

Proof. Each edge lies between exactly two faces.

Example. Let's check this result with the cube. Each face has four sides and so has face degree 4. There are 6 faces. Thus the sum of the face degrees is 24 (= 4 times 6). The number of edges is 12, and 2 times 12 is 24.

The sum of the vertex degrees = $2e$.

Proof. Each edge joins exactly two vertices.

Example. Let's check this result with the tetrahedron. Each vertex has degree 3. There are 4 vertices and so the sum of the vertex degrees is 12 (= 3 times 4). The number of edges is 6, and 2 times 6 is 12.

We need one assumption. The dodecahedron, the football and the golf ball all satisfy the following extra condition.

Assumption

We only consider balls composed of hexagons (possibly none) and pentagons (at least one) where each vertex has degree 3.

Dodecahedra, footballs and golf balls:
the calculations

Let the number of pentagons be p and let the number of hexagons be h.

$$\text{Number of faces: } p + h = f \tag{1}$$

$$\text{Euler's formula: } (p + h) - e + v = 2 \tag{2}$$

$$\text{Sum of face degrees: } 5p + 6h = 2e \tag{3}$$

$$\text{Sum of vertex degrees: } 3v = 2e \tag{4}$$

Divide both sides of Equation (4) by 3 to get

$$v = \frac{2}{3}e. \tag{5}$$

Substitute this value for v into Equation (2) to get

$$(p + h) - e + \frac{2}{3}e = 2. \tag{6}$$

This simplifies to

$$(p + h) - \frac{1}{3}e = 2. \tag{7}$$

Multiply this equation by 6, which clears out the fraction and puts a 12 on the right hand side. We get

$$(6p + 6h) - 2e = 12. \tag{8}$$

We now use Equation (3) to replace the $2e$ in Equation (8) by $5p + 6h$ to get

$$(6p + 6h) - (5p + 6h) = 12. \tag{9}$$

This gives

$$p = 12.$$

PROJECT

It is a theorem that if you have 12 pentagons and *any* number of hexagons (except 1) then you can construct a ball.

Example If you have 12 pentagons and 0 hexagons, you get a dodecahedron.

Example If you have 12 pentagons and 20 hexagons, you get a football.

Show how to make balls with 12 pentagons and n hexagons where $2 \leq n \leq 19$.