The limits of mathematics: known knowns, known unknowns and unknown unknowns

1. Infinite additions

Question Does $1 = 0 \cdot 9999\ldots$?
The ... mean that the 9s don't stop.

Question What is $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \ldots$?

Question What is $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} + \ldots$?

Question What is $1 - 1 + 1 - 1 + 1 - 1 + \ldots$?.

2. Infinite counting

Question You want to convince someone who doesn't speak your language that the number of objects in one pile is the same as the number of objects in another pile. What would you do?

If two sets of things have the same number of objects, we say that they have the *same cardinality*.

I claim that the following sets have the same cardinalities:

- The set of all natural numbers $0, 1, 2, 3, \ldots$ and the set of all even numbers $0, 2, 4, 6, \ldots$.

- The set of all natural numbers $0, 1, 2, 3, \ldots$ and the set of all odd numbers $1, 3, 5, 7, \ldots$.

- The set of all natural numbers $0, 1, 2, 3, \ldots$ and the set of all positive and negative whole numbers.

We define a new number.

It is our first infinite number.

We define the number *aleph nought*, written $\aleph_0$, to be the number of whole numbers.

The letter $\aleph$ is the first letter of the Hebrew alphabet.

The Grand Hilbert Hotel

- This has rooms numbered $1, 2, 3, \ldots$.
- All rooms are occupied.
- A new guest arrives — how can they be accommodated?
- 10 new guests arrive — how can they be accommodated?
- $\aleph_0$ guests arrive — how can they be accommodated?

This tells us that

- $1 + \aleph_0 = \aleph_0$.

- $10 + \aleph_0 = \aleph_0$.

- $\aleph_0 + \aleph_0 = \aleph_0$.

The *real numbers* are the numbers that label the real line.

The interval $(0, 1)$ consists of all real numbers r such that $0 < r < 1$.

I claim that the set of real numbers and the interval $(0, 1)$ have the same cardinality.

Theorem *The interval* $(0, 1)$ *has a bigger cardinality than that of the set of whole numbers.*

We define our second infinite number.

We define the number c, called the *cardinality of the continuum*, to be the cardinality of the real numbers.

We have that $\aleph_0 < c$.

Question Is there an infinite number between $\aleph_0$ and c?

NOBODY KNOWS

Though the question is well understood.

Theorem [The Banach-Tarski Paradox] *A solid, such as an 'apple', can be carved into a finite number of pieces and those pieces reassembled to make a solid, such as the 'moon'.*

3. The game of odds and evens

Here is the ‘game’.

1. Choose a whole number $n \geq 2$.

2. If the number is *even* then divide it by 2.

3. If the number is *odd* multiply it by 3 and add 1.

4. Now repeat (2) or (3) with the number obtained etc.

5. If you ever obtain the number 1 then stop.

Example If we choose 2 we halve it to get 1 and then stop.

Example If we choose 3 we get successively $3 \to 10 \to 5 \to 16 \to 8 \to 4 \to 2 \to 1$ and stop.

Question What do you get if you choose 6?

Question What do you get if you choose 12?

Question What do you get if you choose 27?

Question Does the game of odds and evens always stop?

NOBODY KNOWS

But more is true . . .

It has been said that

> "Mathematics is not ready for such problems".