Primitive representations of the polycyclic monoids and branching function systems

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Abstract

We generalise the group theoretic notion of a primitive permutation representation to inverse monoids and so obtain a notion of a primitive representation by partial permutations. Such representations are shown to be determined by what we call essentially maximal proper closed inverse submonoids. Such submonoids in the case of the polycyclic inverse monoids (also known as Cuntz inverse semigroups) are characterised and all primitive representations of the polycyclic monoids determined. We relate our results to the work of Kawamura on certain kinds of representations of the Cuntz C*-algebras and to the branching function systems of Bratteli and Jorgensen.

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1 Introduction

This paper is the third in a sequence dealing with the relationship between the polycyclic monoids and the Thompson groups [7, 8]. The polycyclic monoids were introduced by Nivat and Perot in 1971 [13] and rediscovered by Cuntz in the course of defining what are now termed Cuntz C*-algebras. For this reason, within C*-algebra theory, the polycyclic monoids are often called Cuntz inverse semigroups [14, 16]. In [8], we described how what we called the strong representations of the polycyclic monoid $P_n$ led to the construction of the Thompson group $V_n$. Such representations have arisen elsewhere, in particular as the
branching function systems of [1]. The goal of this paper is to study the structure of the representations of the polycyclic monoids in general with a particular emphasis on the strong representations: specifically, we shall extend the notion of primitive representation from groups to inverse monoids and classify the primitive representations of the polycyclic monoid on $n$ generators. In the process, we shall give an inverse semigroup interpretation of the work of Kawamura [3, 4], which provided the initial impetus for this work, as well as some results of Bratteli and Jørgensen [1].

We shall call upon standard inverse semigroup theory throughout this paper; see [6], for example, for the rudiments of this theory. All inverse monoids will have a zero and we shall assume that homomorphisms are monoid homomorphisms that preserve the zero.

The product in a semigroup will usually be denoted by concatenation but sometimes we shall use $\cdot$ for emphasis; we shall also use it to denote actions. In an inverse semigroup $S$ we define

$$d(s) = s^{-1}s \text{ and } r(s) = ss^{-1}.$$ 

The natural partial order will be the only partial order considered when we deal with inverse semigroups. If $A \subseteq S$ is a subset then define

$$[A] = \{ s \in S : a \leq s \text{ for some } a \in A \}.$$ 

To simplify notation, I shall write $[A]$, instead, throughout this paper. If $A = [A]$ then $A$ is said to be closed (upwards). If $X \subseteq S$ then $E(X)$ is the set of idempotents in $X$. An inverse submonoid of $S$ is said to be wide if it contains all the idempotents of $S$. We shall be particularly interested in the closed inverse submonoids. Such an inverse submonoid is said to be proper if it doesn’t contain the zero; the only improper closed inverse submonoid of $S$ is $S$ itself. The intersection of (proper) closed inverse submonoids is a (proper) closed inverse submonoid. The following result is adapted from Lemma 1.9 of Ruyle [17].

**Proposition 1.1** Let $F$ be a closed inverse submonoid of the semilattice of idempotents of the inverse monoid $S$. Define

$$\mathcal{F} = \{ s \in S : s^{-1}Fs \subseteq F, sFs^{-1} \subseteq F \}.$$ 

Then $\mathcal{F}$ is a closed inverse submonoid of $S$ whose semilattice of idempotents is $F$. Furthermore, if $T$ is any closed submonoid of $S$ with semilattice of idempotents $F$ then $T \subseteq \mathcal{F}$.

**Proof** Clearly the set $\mathcal{F}$ is closed under inverses. Let $s, t \in \mathcal{F}$. We calculate

$$(st)^{-1}F(st) = t^{-1}(s^{-1}Fs)t \subseteq t^{-1}Ft \subseteq F$$

and

$$(st)F(st)^{-1} = s(tFt^{-1})s^{-1} \subseteq sFs^{-1} \subseteq F.$$ 

Thus $st \in \mathcal{F}$. It follows that $\mathcal{F}$ is an inverse submonoid of $S$. 

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Let $e \in F$ and $f \in F$. Then by assumption $ef \in F$. But $ef \leq e$ and $F$ is a closed inverse submonoid of the semilattice of idempotents and so $e \in F$. Thus $E(F) = F$.

Let $s \leq t$ where $s \in F$. Then $s = ss^{-1}t = ft$. Let $e \in F$. Then

$$s^{-1}es = t^{-1}feit = t^{-1}eft \leq t^{-1}et.$$ 

Now $s^{-1}es, t^{-1}et$ are idempotents and $s^{-1}es \in F$ thus $t^{-1}et \in F$, because $F$ is a closed inverse submonoid of the semilattice of idempotents. Similarly $tet^{-1} \in F$.

It follows that $t \in F$ and so $F$ is a closed inverse submonoid of $S$.

Finally, let $T$ be a closed inverse submonoid of $S$ such that $E(T) = F$. Let $t \in T$. Then for each $e \in F$ we have that $t^{-1}et, tet^{-1} \in F$. Thus $T \subseteq F$.

A closed inverse submonoid $T$ of $S$ will be said to be saturated if $S = E(S)$. If $H$ is a closed inverse submonoid then we will also write $H$ to mean $E(H)$.

In this paper, a representation of an inverse monoid by means of partial bijections is a monoid homomorphism $\theta: S \to I(X)$ to the symmetric inverse monoid on a set $X$. A representation of an inverse monoid in this sense leads to a corresponding notion of an action of the inverse monoid $S$ on the set $X$: the associated action is defined by $s \cdot x = \theta(s)(x)$, if defined. For convenience, we shall use the words ‘action’ and ‘representation’ interchangeably: if I say the inverse monoid $S$ acts on a set $X$ then this will imply the existence of an appropriate homomorphism from $S$ to $I(X)$. If $S$ acts on $X$ I shall often refer to $X$ as a space and its elements as points. An action is said to be trivial if $\theta: S \to I(X)$ maps every element of $S$ to an idempotent. A subset $Y \subseteq X$ closed under the action is called a subspace. Disjoint unions of actions are again actions.

Let $S$ be an inverse monoid acting on the sets $X$ and $Y$. A bijective function $\alpha: X \to Y$ is said to be an $(S\cdot)$-equivalence from $X$ to $Y$ if $\exists s \cdot x \Leftrightarrow \exists s \cdot \alpha(x)$ and if either side exists we have that $\alpha(s \cdot x) = s \cdot \alpha(x)$. As with group actions, equivalent actions are the same except for the labelling of the points.

The action of an inverse monoid $S$ on the set $X$ induces an equivalence relation $\sim$ on the set $X$ when we define $x \sim y$ iff $s \cdot x = y$ for some $s \in S$. The action is said to be transitive if $\sim$ is $X \times X$. Just as in the theory of permutation representations of groups, every representation of an inverse monoid is a disjoint union of transitive representations. Thus we need only describe transitive representations.

Transitive actions of inverse monoids are characterised by special kinds of inverse submonoids in a way generalising the relationship between transitive group actions and subgroups. Fix a point $x \in X$, and consider the set $S_x$ consisting of all $s \in S$ such that $s \cdot x = x$. We call $S_x$ the stabiliser of the point $x$. If an element $s$ fixes a point then so too will any element above $s$, and so the

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1The action is therefore a partial function from $S \times X$ to $X$ mapping $(s, x)$ to $s \cdot x$ when $\exists s \cdot x$. We require that $\exists (st) \cdot x$ iff $\exists s \cdot (t \cdot x)$ in which case they are equal and if $\exists e \cdot x$ where $e$ is an idempotent then $e \cdot x = x$. 

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set $S_x$ is a closed inverse submonoid of $S$. Stabilisers cannot contain zero and so they are proper closed inverse submonoids. Now let $y \in X$ be any point. By transitivity, there is an element $s \in S$ such that $s \cdot x = y$. Observe that because $s \cdot x$ is defined so too is $s^{-1}s$ and that $s^{-1}s \in S_x$. An easy calculation shows that $[S_x]$ is the set of all elements of $S$ which map $x$ to $y$.

Let $H$ be a proper closed inverse submonoid of $S$. Define a left coset of $H$ to be a set of the form $[sH]$ where $s^{-1}s \in H$. The following are well-known but I include the proofs for the sake of completeness.

**Lemma 1.2**

(i) Two cosets $[sH]$ and $[tH]$ are equal iff $s^{-1}t \in H$.

(ii) If $[sH] \cap [tH] \neq \emptyset$ then $[sH] = [tH]$.

**Proof** (i) Suppose that $[sH] = [tH]$. Then $t \in [sH]$ and so $sh \leq t$ for some $h \in H$. Thus $s^{-1}sh \leq s^{-1}t$. But $s^{-1}sh \in H$ and $H$ is closed and so $s^{-1}t \in H$.

Conversely, suppose that $s^{-1}t \in H$. Then $s^{-1}t = h$ for some $h \in H$ and so $sh = ss^{-1} \leq t$. It follows that $tH \subseteq sH$ and so $[tH] \subseteq [sH]$. The reverse inclusion follows from the fact that $t^{-1}s \in H$ since $H$ is closed under inverses.

(ii) Suppose that $a \in [sH] \cap [tH]$. Then $sh_1 \leq a$ and $th_2 \leq a$ for some $h_1, h_2 \in H$. Thus $s^{-1}sh_1 \leq s^{-1}a$ and $t^{-1}th_2 \leq t^{-1}a$. Hence $s^{-1}a, t^{-1}a \in H$. It follows that $s^{-1}a t^{-1}t \leq s^{-1}t$. This gives the result by (i) above.

We denote by $S/H$ the set of all left cosets of $H$ in $S$. The inverse monoid $S$ acts on the set $S/H$ when we define

$$a \cdot [sH] = [asH] \Leftrightarrow d(as) \in H.$$ 

It is straightforward to check that this defines a transitive action. It is now a theorem that the transitive space $X$ is equivalent to the left coset space of a stabiliser of a point of $X$. If $H$ and $K$ are any closed inverse submonoids of $S$ then they determine equivalent actions if and only if there exists $s \in S$ such that

$$sHs^{-1} \subseteq K \text{ and } s^{-1}Ks \subseteq H.$$ 

I shall call this relationship between two closed inverse submonoids conjugacy although it is important to observe that equality need not hold in the definition above.

**Lemma 1.3** If $H$ and $K$ are conjugate as above then $ss^{-1} \in K$ and $s^{-1}s \in H$. Also $[sHs^{-1}] = K$ and $[s^{-1}Ks] = H$.

**Proof** Let $e \in H$ be any idempotent. Then $ses^{-1} \in K$. But $ses^{-1} \leq ss^{-1}$ and so $ss^{-1} \in K$. Similarly $s^{-1}s \in H$.

We have that $sHs^{-1} \subseteq K$ and so $[sHs^{-1}] \subseteq K$. Let $k \in K$. Then $s^{-1}ks \in H$ and $s(s^{-1}ks)s^{-1} \in sHs^{-1}$ and $s(s^{-1}ks)s^{-1} \leq k$. Thus $[sHs^{-1}] = K$, as required.
Thus to study transitive actions of an inverse monoid with zero $S$ it is enough to study the proper closed inverse submonoids of $S$ up to conjugacy.

Actions of inverse monoids were first studied by Boris Schein [18] who generalised the theory of transitive group actions to inverse semigroups. An account of this work is described in Section IV.4 of [15], and Section 5.8 of [2]; our overview above was based on this work, but slightly modified because we are studying inverse monoids with zero. Actions of free inverse monoids are the subject of [10], where the authors are able to characterise the closed inverse submonoids of free monoids. Actions of free inverse monoids on finite sets are an essential ingredient in the theory of inverse automata. Robert Ruyle's thesis is a good source for this connection [17].

To conclude this section, I need to deal with a counter-intuitive property of inverse monoid actions. Groups always have trivial transitive actions, but inverse monoids with zero might not have any or might have many inequivalent such actions. To understand why an inverse monoid need not have any trivial transitive actions, let $S$ be an inverse semigroup with zero and suppose that there is a surjective monoid homomorphism $\theta: S \to I(\{\ast\})$ where $\{\ast\}$ is any one-element set. The inverse semigroup $I(\{\ast\})$ is the two-element semilattice $1 > 0$. Thus the kernel of $\theta$ is a congruence containing two congruence classes: one containing the identity and one containing the zero. If a congruence is such that the zero and identity are related then there is exactly one congruence class. It follows that the only inverse monoids with zero that admit trivial transitive actions are those possessing a congruence having exactly two classes. In particular, any congruence-free inverse monoid with more than two elements does not admit trivial transitive actions. The polycyclic monoids we discuss in this paper are congruence-free and consequently the polycyclic monoids do not admit trivial transitive actions. If an inverse monoid does have trivial transitive actions, it may admit many non-equivalent such actions — quite unlike the group case where all trivial transitive actions are equivalent. To show this, we first classify the proper closed inverse submonoids that give rise to trivial transitive actions.

**Lemma 1.4** Let $S$ be an inverse monoid with zero.

(i) Let $H$ be a proper closed inverse submonoid of $S$. Then $|S/H| = 1$ if and only if $d(s) \in H$ implies $s \in H$. Thus $H$ is closed under the $L$-relation in $S$.

(ii) Let $H$ and $K$ be conjugate closed proper inverse submonoids such that $|S/H| = |S/K| = 1$. Then $H = K$.

(iii) Let $H$ be a proper closed inverse submonoid of $S$ such that $|S/H| = 1$. Then $H$ is saturated.

**Proof** (i) Let $H$ be a proper closed inverse submonoid of $S$ such that $|S/H| = 1$. Let $d(s) \in H$. Then $[sH]$ is a left coset of $H$. By assumption $[sH] = H$, where
$H$ is a left coset since $[1H] = H$. Now $s = sd(s) \in sH$ and so $s \in H$, as required. Conversely, suppose that $H$ is a proper closed inverse submonoid such that $d(s) \in H$ implies $s \in H$. The left cosets of $H$ are of the form $[sH]$ where $d(s) \in H$. It follows that $s \in H$ and so $[sH] = H$ giving $|S/H| = 1$.

(ii) By definition there exists $s \in S$ such that $sHs^{-1} \subseteq K$ and $s^{-1}Ks \subseteq H$. In addition, $ss^{-1} \in K$ and $s^{-1}s \in H$. By (i), it follows that $s \in H$ and $s^{-1} \in K$. Hence $s \in H \cap K$. Thus $sHs^{-1} \subseteq H$ and so $[sHs^{-1}] \subseteq H$. But by Lemma 1.3, we have that $[sHs^{-1}] = K$. Thus $K \subseteq H$. A symmetric argument shows that $H \subseteq K$ and so $H = K$, as required.

(iii) Let $H$ be a proper closed inverse submonoid of $S$ satisfying $|S/H| = 1$. Let $s \in S$ such that $s^{-1}E(H)s \subseteq E(H)$ and $sE(H)s^{-1} \subseteq E(H)$. Then in particular $s^{-1}s \in E(H)$ thus by (i) above we have that $s \in H$. It follows that $H = H$, as required.

A proper closed inverse submonoid $H$ of $S$ will be called **trivialising** if it is a union of $\mathcal{L}$-classes of $S$. An inverse monoid with zero need not have any trivialising proper closed inverse submonoids: for example, the polycyclic monoids. However, if it has at least one then it has a smallest, as shown in the next result, whose proof is straightforward.

**Lemma 1.5** Let $S$ be an inverse monoid with zero with at least one trivial transitive action. Then the intersection of all trivialising proper closed inverse submonoids is again a trivialising proper closed inverse submonoid.

We now give some examples of the behaviour of trivial transitive actions.

**Examples 1.6**

(i) Every finite inverse monoid with zero $S$ admits a trivial transitive action. The group of units of $S$ is a proper closed inverse submonoid. In addition, by finiteness, it forms an $\mathcal{L}$-class. Thus the group of units is the minimum trivialising proper closed inverse submonoid. This is also true of any inverse monoid in which $st = 1$ implies that $ts = 1$.

(ii) Finite symmetric inverse monoids have exactly one trivial transitive action. To see why, observe first that the group of units is a trivialising proper closed inverse submonoid and so as in (i) there is at least one trivial transitive action. Now let $H$ be any closed inverse submonoid containing the group of units and at least one other non-zero element, which without loss of generality we can assume is an idempotent, $e$ say. We claim that $H$ must contain an idempotent defined on a subset with $|X| - 1$ elements. However either $e$ itself is such an idempotent or is beneath such an idempotent. Since $H$ is upwardly closed we reach the desired conclusion. We now use the well-known result that the finite symmetric inverse monoid
$I(X)$ is generated by its group of units and any idempotent defined on a subset with $|X| - 1$ elements. Observe that no proper closed inverse submonoid can properly contain the group of units.

(iii) Let $S$ be a 0-bisimple inverse monoid. If the zero is adjoined then there is exactly one trivialising proper closed inverse submonoid: the set of non-zero elements, and so there is exactly one trivial transitive action. If the zero is not adjoined then there are no trivialising proper closed inverse submonoids because the $L$-class of the identity generates the monoid. It follows that there are no trivial transitive actions. This argument provides another proof that the polycyclic inverse monoids do not have any trivial transitive actions.

(iv) We may generalise the above example. Let $S$ be a 0-simple inverse monoid. Let $\theta: S \to I(*)$ be a surjective homomorphism. Then the elements of $S$ mapping to the zero of $I(*)$ form an ideal and the set of elements mapping to the identity of $I(*)$ form a submonoid. It follows that if such a homomorphism exists then the only element mapping to the zero of $I(*)$ is the zero of $S$. However, this implies that if $s$ and $t$ are non-zero elements of $S$ then $st$ is non-zero and so the zero is adjoined. It follows that a 0-simple inverse monoid in which the zero is not adjoined cannot have a trivial transitive action.

(v) In the case of semilattices, every proper closed inverse submonoid is trivialising.

2 Primitive representations of inverse monoids

The goal of this section is to refine the theory of transitive actions described in Section 1. In particular, we shall generalise the theory of primitive group actions to inverse monoids. In group theory there are two equivalent ways of characterising primitive permutation groups. Let $G$ act transitively on the set $X$ where $|X| > 1$. Then the following are equivalent [5]:

(Prim 1) The only $G$-invariant congruences on $X$ are the equality relation and the universal relation.

(Prim 2) The stabiliser $S_x$ of a point $x \in X$ is a maximal subgroup.

A group action satisfying either of these two conditions is said to be primitive. We shall see that in the theory of actions of inverse monoids the relationship between these two approaches is more problematic.

Our first problem is that there are two natural definitions of morphism between $S$-spaces: a weak one and a strong one.

Let $X$ and $Y$ be $S$-spaces. A morphism from $X$ to $Y$ is a function $\alpha: X \to Y$ such that $\exists s \cdot x$ implies that $\exists s \cdot \alpha(x)$ and $\alpha(s \cdot x) = s \cdot \alpha(x)$. A morphism is
said to be a covering if it satisfies the stronger condition that $\exists s \cdot x \Leftrightarrow \exists s \cdot \alpha(x)$.
The terminology ‘covering morphism’ is taken from [17].

Remark Observe that an equivalence is precisely a bijective covering morphism.

Covering morphisms are well-behaved as the following lemma shows.

Lemma 2.1 Let $S$ be an inverse monoid acting on both $X$ and $Y$, and let $\alpha: X \to Y$ be a covering morphism.

(i) The image of $\alpha$ is a subspace.

(ii) If $X$ and $Y$ are transitive $S$-spaces then $\alpha$ is surjective.

Proof (i) Put $Y = \alpha(X)$. Let $y \in Y$ and $s \in S$ such that $s \cdot y$ is defined. By assumption, $y = \alpha(x)$. Thus $s \cdot \alpha(x)$. Since $\alpha$ is a covering morphism, we know that $\exists s \cdot x$. But then $\alpha(s \cdot x) = s \cdot y$ and so $s \cdot y \in Y$. Hence the image is a subspace.

(ii) Let $x \in X$ and $y = \alpha(x) \in Y$. Let $y' \in Y$. By transitivity there is $s \in S$ such that $s \cdot y = y'$. Thus $\exists s \cdot \alpha(y)$. Because $\alpha$ is a covering morphism we have that $\exists s \cdot x$. But $\alpha(s \cdot x) = s \cdot \alpha x = y'$. Thus $\alpha$ is surjective.

Neither of the above results is true for morphisms, as the following example shows.

Example 2.2 Let $S$ be the inverse submonoid of $I(\{1, 2\})$ consisting all all elements except the transposition and let $X = \{1, 2\}$. Then $S$ acts transitively on $X$ via the natural action. The monoid $S$ also acts on the one element set $\{\ast\}$ by defining the following action: the identity of $S$ fixes $\ast$ whereas for all other elements the action is not defined on $\ast$: in other words the transitive trivial action determined by the group of units. In this way, we have a trivial transitive action of $S$ on $\{\ast\}$. Let $\alpha: \{\ast\} \to X$ be the function mapping $\ast$ to 1. Then $\alpha$ is a morphism but it is not surjective and its image is not a subspace.

Continuing with this theme, for covering morphisms we can prove a first isomorphism theorem. To do this we need a suitable notion of congruence. Let the inverse monoid $S$ act on the set $X$. An $(S)$-covering congruence on $X$ is an equivalence relation $\sim$ on the set $X$ such that if $x \sim y$ then $\exists s \cdot x \Leftrightarrow \exists s \cdot y$ and if the actions are defined we have that $s \cdot x \sim s \cdot y$.

Lemma 2.3

(i) Let $\alpha: X \to Y$ be a covering morphism. Then the kernel of $\alpha$, $\sim$, is a covering congruence.
(ii) Let $\sim$ be a covering congruence on $X$. Denote the $\sim$-class containing the element $x$ by $[x]$ (notation warning). Define $s \cdot [x] = [s \cdot x]$ if $\exists s \cdot x$. Then this defines an action $S$ on the set of $\sim$-congruence classes $X/\sim$ and the natural map $\nu: X \to X/\sim$ is a covering morphism.

(iii) Let $\alpha: X \to Y$ be a covering morphism, let its kernel by $\sim$ and let $\nu: X \to X/\sim$ be the associated natural map. Then there is a unique injective covering morphism $\beta: X/\sim \to Y$ such that $\beta\nu = \alpha$.

Proof (i) Suppose that $x \sim y$ and that $s \cdot x$ is defined. By definition $\alpha(x) = \alpha(y)$ and $\exists s \cdot \alpha(x)$. Thus $\exists s \cdot \alpha(y)$ and so $\exists s \cdot y$. By symmetry we have therefore proved that $\sim$ is a covering congruence.

(ii) We check first that we do have an action. Suppose that $\exists e \cdot [x]$ where $e$ is an idempotent. Then $e \cdot [x] = [e \cdot x] = [x]$. Next, suppose that $\exists (st) \cdot [x]$. Then $(st) \cdot [x] = [(st) \cdot x] = [s \cdot (t \cdot x)] = s \cdot (t \cdot [x])$. The reverse direction is proved similarly. Thus $X/\sim$ is an $S$-space. It remains to check that $\nu: X \to X/\sim$ is a covering morphism. However, this is immediate from the definition: $\exists s \cdot [x]$ iff $\exists s \cdot x$.

(iii) This is now straightforward.

The above lemma implies, as always with a first isomorphism theorem in algebra, that if $\alpha: X \to Y$ and $\beta: X \to Z$ are both surjective covering morphisms having the same kernel then there is an equivalence from $X$ to $Y$ making the diagram of maps commute. Thus $\alpha$ and $\beta$ are essentially the same. The following example shows that we cannot hope to prove a first isomorphism theorem for morphisms.

Example 2.4 Consider the semilattice $S$ given by $1 > e, 1 > f, e, f > 0$. Put $I = \{1\}, H = \{1, e\}$ and $K = \{1, f\}$. Then $I$, $H$ and $K$ are trivialising proper closed inverse submonoids of $S$. There are morphisms from $S/I$ to each of $S/H$ and $S/K$ and necessarily these two morphisms have the same kernel and both are surjective. However the action of $S$ on $S/H$ is not equivalent to the action of $S$ on $S/K$ because for this to happen $H$ and $K$ would have to be conjugate and so equal by Lemma 1.4.

Remark It is worth looking in a little more detail at what goes wrong with the first isomorphism theorem for morphisms. It is easy to check that the kernels of morphisms with domain the $S$-space $X$ have the following property: if $x \sim y$ and $\exists s \cdot x$ and $\exists s \cdot y$ then $s \cdot x \sim s \cdot y$. For the purposes of this remark we shall say that an equivalence relation on $X$ is a weak congruence if it satisfies this condition. Let $\sim$ be an arbitrary weak congruence on $X$, and denote the $\sim$-equivalence class containing $x$ by $[x]$. We attempt to define an action of $S$ on the quotient $X/\sim$. Define $s \cdot [x] = [s \cdot x']$ if there exists $x' \sim x$ such that $\exists s \cdot x'$. If $\exists (st) \cdot [x]$ then it is certainly true that $\exists s \cdot (t \cdot [x])$, however the converse
is not true in general. As a result, if we define $\theta: S \to I(X/ \sim)$ by $\theta(s)([x])$ is defined iff $\exists s \cdot x'$ for some $x' \sim x$, in which case $\theta(s)([x]) = [s \cdot x']$, then $\theta$ is a prehomomorphism [6] rather than a homomorphism because we only have that $\theta(st) \subseteq \theta(s)\theta(t)$. However, for free monoids $S$ this problem disappears because every homomorphism from $S$ to $I(X)$ is determined by the values of the homomorphism on a set of free inverse generators for $S$: these values can be defined as above and then extended uniquely to $S$ yielding a homomorphism. This explains why the congruences associated with morphisms can be used in the study of inverse automata [11].

After these negative results, we do have the following positive result motivated by Lemma 2.16 of Ruyle’s thesis [17].

**Proposition 2.5** Let $S$ be an inverse monoid acting transitively on the sets $X$ and $Y$, and let $x \in X$ and $y \in Y$. Let $S_x$ and $S_y$ be the stabilisers in $S$ of $x$ and $y$ respectively.

(i) There is a morphism $\alpha: X \to Y$ such that $\alpha(x) = y$ iff $S_x \subseteq S_y$. If such a morphism exists then it is unique.

(ii) There is a covering morphism $\alpha: X \to Y$ such that $\alpha(x) = y$ iff $S_x \subseteq S_y$ and $E(S_x) = E(S_y)$. If such a morphism exists then it is unique.

**Proof** (i) We begin by proving uniqueness. Let $\alpha, \beta: X \to Y$ be morphisms such that $\alpha(x) = \beta(x) = y$. Let $x' \in X$ be arbitrary. By transitivity there exists $a \in S$ such that $x' = a \cdot x$. By the definition of morphisms we have that $\exists a \cdot \alpha(x)$ and $\exists a \cdot \beta(x)$ and that

$$\alpha(x') = \alpha(a \cdot x) = a \cdot \alpha(x)$$

and

$$\beta(x') = \beta(a \cdot x) = a \cdot \beta(x).$$

But by assumption $\alpha(x) = \beta(x) = y$ and so $\alpha(x') = \beta(x')$. It follows that $\alpha = \beta$.

Let $\alpha: X \to Y$ be a morphism such that $\alpha(x) = y$. Let $s \in S_x$. Then $\exists s \cdot x$ and $s \cdot x = x$. By the definition of morphism, it follows that $\exists s \cdot \alpha(x)$ and that $\alpha(s \cdot x) = s \cdot \alpha(x)$. But $s \cdot x = x$ and so $\alpha(x) = s \cdot \alpha(x)$. Hence $s \cdot y = y$. We have therefore proved that $s \in S_y$, and so $S_x \subseteq S_y$.

Suppose now that $S_x \subseteq S_y$. We have to define a morphism $\alpha: X \to Y$ such that $\alpha(x) = y$. We start by defining $\alpha(x) = y$. Let $x' \in X$ be any point in $X$. Then $x' = a \cdot x$ for some $a \in S$. We need to show that $a \cdot y$ exists. Since $a \cdot x$ exists we know that $a^{-1}a \cdot x$ exists and this is equal to $x$. It follows that $a^{-1}a \in S_x$ and so $a^{-1}a \in S_y$, by assumption. Thus $a^{-1}a \cdot y$ exists and is equal to $y$. But from the existence of $a^{-1}a \cdot y$ we can deduce the existence of $a \cdot y$. We would therefore like to define $\alpha(x') = a \cdot y$. We have to check that this is well-defined. Suppose that $x' = a \cdot x = b \cdot x$. Then $b^{-1}a \cdot x = x$ and so $b^{-1}a \in S_x$. By assumption, $b^{-1}a \in S_y$ and so $b^{-1}a \cdot y = y$. Thus $bb^{-1}a \cdot y = b \cdot y$.

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and \(bb^{-1}a \cdot y = bb^{-1} \cdot (a \cdot y) = a \cdot y\). Thus \(a \cdot y = b \cdot y\). It follows that \(\alpha\) is a well-defined function mapping \(x\) to \(y\). It remains to show that \(\alpha\) is a morphism. Suppose that \(s \cdot x'\) is defined. By assumption, there exists \(a \in S\) such that \(x' = a \cdot x\). By definition \(\alpha(x') = a \cdot y\). We have that \(s \cdot x' = s \cdot (a \cdot x) = sa \cdot x\). Hence \(\alpha(s \cdot x') = s \cdot \alpha(x')\), as required.

(ii) Uniqueness is inherited from (i).

Let \(\alpha: X \to Y\) be a covering morphism such that \(\alpha(x) = y\). In particular, \(\alpha\) is a morphism and so \(\alpha\) is a covering and so \(\exists x \cdot a\). Clearly \(e \in E(S_x)\). It follows that \(E(S_x) = E(S_y)\).

Suppose now that \(S_x \subseteq S_y\) and \(E(S_x) = E(S_y)\). By (i), there is a morphism \(\alpha: X \to Y\) such that \(\alpha(x) = y\). It remains to show that it is a covering. Let \(\alpha(x') = y'\) and suppose that \(\exists s \cdot x'\). We shall prove that \(\exists s \cdot x'\). Observe that \(\exists s^{-1} \cdot y'\) and that it is enough to prove that \(\exists s^{-1} \cdot x'\). Let \(x' = u \cdot x\), which exists since we are assuming that our action is transitive. Then because \(\alpha\) is a morphism we have that \(y' = u \cdot y\). Observe that \(u^{-1}(s^{-1}s)u = y = y\) and so \(u^{-1}(s^{-1}s)u \in E(S_y)\). It follows by our assumption that \(u^{-1}(s^{-1}s)u \in E(S_x)\) and so \(u^{-1}(s^{-1}s)u \cdot x = x\). It readily follows that \(\exists s^{-1} \cdot x'\), and so \(\exists x \cdot a\), as required.

\[\square\]

**Proposition 2.6** Let \(S\) be an inverse monoid acting transitively on a space \(X\) where \(|X| > 1\). Let \(x \in X\), and \(S_x\) its stabiliser. Then every covering morphism with domain \(X\) is an equivalence if and only if \(S_x\) is saturated.

**Proof** Suppose that the only covering congruence on \(X\) is the equality relation. Let \(H\) be a proper closed inverse submonoid of \(S\) such that \(S_x \subseteq H\) and \(E(S_x) = E(H)\). The inverse monoid \(S\) acts transitively on \(S/H\). The stabiliser of the point \(y = H \in S/H\) is the closed inverse submonoid \(H\), and by construction \(S_x \subseteq S_y = H\). Thus by Proposition 2.5(ii) there is a covering morphism \(\alpha: X \to S/H\) such that \(\alpha(x) = y\). But by assumption the kernel of \(\alpha\) is the equality relation. If the kernel is the equality relation then \(\alpha\) is a bijective covering morphism. By applying Proposition 2.5(ii) to the inverse of \(\alpha\) we deduce that \(S_x = H\). Thus \(S_y\) is saturated.

Suppose that \(S_x\) is saturated. Let \(\sim\) be a covering congruence on \(X\). Then \(S\) acts on \(X/\sim\) and \(\nu: X \to X/\sim\) is a covering morphism. Put \(y = \nu(x)\). Then by Proposition 2.5(ii), we have that \(S_x \subseteq S_y\) and \(E(S_x) = E(S_y)\). By assumption, \(S_x = S_y\), because \(S_x\) is saturated. Thus \(\nu\) is an equivalence and so \(\sim\) is the equality relation.

\[\square\]

We now focus on the category of transitive \(S\)-spaces and the morphisms between them. The isomorphisms in this category are precisely the equivalences (invertible morphisms). There is an argument that we should focus only on the surjective morphisms (but this would require us to strengthen what we mean...
by containment of one closed inverse submonoid in another) but I shall consider arbitrary such morphisms.

**Proposition 2.7** Let $S$ be an inverse monoid acting transitively on a space $X$ where $|X| > 1$. Let $x \in X$, and $S_x$ its stabiliser. Then the only morphisms from $X$ to transitive $S$-spaces are equivalences if and only if the stabiliser $S_x$ is a maximal proper closed inverse submonoid of $S$.

**Proof** Suppose that the only morphisms with domain $X$ are equivalences. Let $H$ be a proper closed inverse submonoid of $S$ such that $S_x \subseteq H$. The inverse monoid $S$ acts transitively on $S/H$. The stabiliser of the point $y = H \in S/H$ is the closed inverse submonoid $H$, and by construction $S_x \subseteq S_y = H$. Thus by Proposition 2.5(i) there is a morphism $\alpha: X \to S/H$ such that $\alpha(x) = y$. By assumption $\alpha$ is an equivalence, and so by applying Proposition 2.5(i) to the inverse of $\alpha$ we deduce that $H \subseteq S_x$ and so $H = S_x$. Thus $S_x$ is a maximal proper closed inverse submonoid.

Suppose that $S_x$ is a maximal proper closed inverse submonoid of $S$. Let $\alpha: X \to Y$ be a morphism to a transitive space $Y$. Let $\alpha(x) = y$. Then by Proposition 2.5(i), $S_x \subseteq S_y$. By assumption $S_x = S_y$, and so $\alpha$ is an equivalence.

Let $X$ be a transitive $S$-space where $|X| > 1$. We say that the action is *primitive* if every morphism from $X$ to a transitive $S$-space is either an equivalence or maps to a trivial transitive space.

We say that a proper closed inverse submonoid $H$ of $S$ which is not trivialising is *essentially maximal* if the only proper closed inverse submonoids containing $H$ are themselves trivialising.

**Theorem 2.8** Let $S$ be an inverse monoid acting transitively on a space $X$ where $|X| > 1$. Let $x \in X$, and $S_x$ its stabiliser. Then the action is primitive if and only if $S_x$ is essentially maximal.

**Proof** Suppose that the action is primitive. Let $S_x \subseteq H$ where $H$ is a proper closed inverse submonoid. Then by Proposition 2.5(i), there is a morphism $\alpha: S/S_x \to S/H$ mapping the point $S_x$ to the point $H$. There are now two possibilities. If $\alpha$ is an equivalence then by applying Proposition 2.5(i) to the inverse of $\alpha$ we deduce that $S_x = H$. The only other possibility is that $\alpha$ is a morphism to a trivial transitive action and so $H$ is a trivialising proprer closed inverse submonoid. We deduce that $S_x$ is essentially maximal.

Suppose that $S_x$ is essentially maximal and let $\alpha: X \to Y$ be a morphism taking $x$ to $y$. By Proposition 2.5(i), we have that $S_x \subseteq S_y$. By assumption, either $S_x = S_y$, in which case $\alpha$ is an equivalence, or $S_y$ is trivialising in which case $Y$ is a trivial transitive space.
It is worth stepping back a little to see what the problems are in generalising primitivity from groups to inverse monoids. The most natural generalisation of (Prim 1) would be based on the notion of a covering congruence but Lemma 2.3 tells us that such congruences are the kernels of covering morphisms and such morphisms are too restrictive from our point of view. On the other hand, the morphisms we are interested in cannot be described by their kernels, or ‘weak congruences’ to use the terminology of the Remark following Example 2.4. However, we can use morphisms to get a generalisation of (Prim 2), and it is this which we have chosen to base the theory of primitive actions upon. The usefulness of this concept remains to be seen, but some evidence that it is useful, and arises ‘in nature’, is provided by the remainder of this paper.

**Remark** For those inverse monoids with zero that admit no trivial transitive actions we can replace ‘essentially maximal’ by ‘maximal’. It is this special case that we shall need in the remainder of this paper.

### 3 The polycyclic monoids

In this section, we outline the theory of the polycyclic monoids. Most of the results are well-known although our classification of the wide inverse submonoids of the polycyclic monoids (Theorem 3.3) appears to be new.

Let $n \geq 1$, and put $A_n = \{a_1, \ldots, a_n\}$. A string in $A_n^*$, the free monoid generated by $A_n$, will be called *positive*. The empty string is denoted $\varepsilon$. The free semigroup on $A_n$ is denoted by $A_n^+$. The set of *right infinite* strings over $A_n$ is denoted by $A_n^\omega$. If $x$ is a finite string then $x^\omega = xxx \ldots$. A right infinite string $x$ is said to be *ultimately periodic* if it is of the form $yz^\omega$ where $y$ and $z$ are finite strings and $z$ is non-empty. For the purposes of this paper, an infinite string is *aperiodic* if it is not ultimately periodic. If $u = vw$ are strings, then $v$ is called a *prefix* of $u$, and a *proper prefix* if $w$ is not the empty string; whereas $w$ is called a *suffix* of $u$, and a *proper suffix* if $v$ is not the empty string. If $x$ is a string then the notation $\bar{x}$ will be used to denote a prefix of $x$. A pair of elements of $A_n^*$ is said to be *prefix-comparable* if one is a prefix of the other. If $x$ and $y$ are prefix-comparable we define

$$x \wedge y = \begin{cases} x & \text{if } y \text{ is a prefix of } x \\ y & \text{if } x \text{ is a prefix of } y \end{cases}$$

A string is said to be *primitive* if it is not a proper power of another string. Let $x$ and $y$ be strings. If $x = uv$ and $y = vu$ then $x$ and $y$ are said to be *conjugate*. A Lyndon word (or string) is a primitive string that is minimal in its conjugacy class with respect to the lexicographic ordering [9]. For us, Lyndon strings are just a way of choosing a unique representative from conjugacy classes of primitive strings.

The following is proved as Proposition 1.3.4 of [9].
**Proposition 3.1** Let \( x, y \in A^+ \). Then \( x \) and \( y \) are conjugate iff \( xz = yz \) for some string \( z \).

The **bicyclic monoid** is defined by the following monoid presentation

\[
B = P_1 = \langle a_1 : a_1^{-1}a_1 = 1 \rangle.
\]

The **polycyclic monoid** \( P_n \), where \( n \geq 2 \), is defined as a monoid with zero by the following presentation

\[
P_n = \langle a_1, \ldots, a_n, a_1^{-1}, \ldots, a_n^{-1} : a_i^{-1}a_i = 1 \text{ and } a_i^{-1}a_j = 0, i \neq j \rangle.
\]

Intuitively, think of \( a_1, \ldots, a_n \) as partial bijections of a set \( X \) and \( a_1^{-1}, \ldots, a_n^{-1} \) as their respective partial inverses. The first relation says that each partial bijection \( a_i \) has domain the whole of \( X \) and the second says that the ranges of distinct \( a_i \) are orthogonal. As a concrete example of \( P_2 \), one can take as \( a_1 \) and \( a_2 \) the two maps that shrink the Cantor set to its lefthand and righthand sides, respectively. Every non-zero element of \( P_n \) is of the form \( xy^{-1} \) where \( x, y \in A_*^+ \). Identify the identity with \( \varepsilon \). The product of two elements \( xy^{-1} \) and \( vu^{-1} \) is zero unless \( x \) and \( v \) are prefix-comparable. If they are prefix-comparable then

\[
yx^{-1} \cdot vu^{-1} = \begin{cases} 
  yzu^{-1} \cdot y(uz)^{-1} & \text{if } v = xz \text{ for some string } z \\
  yv^{-1} & \text{if } v = vz \text{ for some string } z
\end{cases}
\]

The non-zero idempotents in \( P_n \) are the elements of the form \( xx^{-1} \), where \( x \) is positive, and the natural partial order is given by \( xx^{-1} \leq vu^{-1} \) iff \( (y, x) = (v, u)p \) for some positive string \( p \). Observe that if \( xx^{-1} \) and \( yy^{-1} \) are non-zero idempotents then \( xx^{-1} \cdot yy^{-1} \neq 0 \) if and only if either \( xx^{-1} \leq yy^{-1} \) or \( yy^{-1} \leq xx^{-1} \). When non-zero

\[
xx^{-1} \cdot yy^{-1} = (x \land y)(x \land y)^{-1}.
\]

An inverse semigroup with zero is said to be **\( E^* \)-unitary** if every element above a non-zero idempotent is an idempotent. An inverse semigroup is said to be **\( E \)-unitary** if every element above an idempotent is an idempotent. The bicyclic monoid is \( E \)-unitary and the polycyclic monoids are \( E^* \)-unitary.

There is a function \( \lambda \) from the non-zero elements of \( P_n \) to the additive group \( \mathbb{Z} \) defined by \( \lambda(uv^{-1}) = |u| - |v| \). If \( xy^{-1} \cdot vu^{-1} \neq 0 \) then direct verification shows that

\[
\lambda(yx^{-1} \cdot vu^{-1}) = \lambda(yx^{-1}) + \lambda(vu^{-1}).
\]

**Lemma 3.2** In a polycyclic monoid if \( uv^{-1} \leq xy^{-1} \), \( wz^{-1} \) then \( xy^{-1} \) and \( wz^{-1} \) are comparable.

**Proof** It follows that \( u = xp = wq \) and \( v = yp = zq \) for some strings \( p \) and \( q \). If \( |p| = |q| \) then \( xy^{-1} = wz^{-1} \), if \( |p| < |q| \) then \( xy^{-1} \leq wz^{-1} \), and if \( |q| < |p| \) then \( wz^{-1} \leq xy^{-1} \).
Meakin and Sapir [12] proved there was a bijection between the congruences on the free monoid on \( n \) generators \( A^* \) and the positively self-conjugate wide inverse submonoids of the polycyclic monoid on \( n \) generators \( P_n \). We generalise this result as follows.

**Theorem 3.3** There is a bijection between right congruences on \( A^* \) and wide inverse submonoids of \( P_n \).

**Proof** Let \( \rho \) be a right congruence on \( A^* \). Let

\[
P_\rho = \{xy^{-1} \in P_n : x\rho y \} \cup \{0\}.
\]

We prove that \( P_\rho \) is a wide inverse subsemigroup of \( P_n \). The nonzero idempotents of \( P_n \) are the elements of the form \( xx^{-1} \). Since \( \rho \) is an equivalence relation for every string \( x \) we have that \( x\rho x \). Thus \( P_\rho \) contains all the idempotents of \( P_n \). If \( xy^{-1} \in P_\rho \) then \( x\rho y \) and so \( y\rho x \), since \( \rho \) is an equivalence relation, and it follows that \( yx^{-1} \in P_\rho \). Thus \( P_\rho \) is closed under inverses. It remains to show that \( P_\rho \) is closed under products. Suppose that \( xy^{-1}, wz^{-1} \in P_\rho \). There are two possibilities. Suppose that \( w = yp \). Then \( xy^{-1} \cdot wz^{-1} = (xp)z^{-1} \).

Now \( x\rho y \) and \( yp = wpz \). By assumption, \( \rho \) is a right congruence and so \( xxpyp \). By transitivity, we have that \( x\rho y \) and so \( (xy)z^{-1} \in P_\rho \), as required. Now suppose that \( y = wp \). Then \( xy^{-1} \cdot wz^{-1} = x(zp)^{-1} \). Now \( x\rho y \) and \( wpz \). By assumption, \( \rho \) is a right congruence and so \( wppzp \). By transitivity, \( xzp \) and so \( x(zp)^{-1} \in P_\rho \). Thus we have proved that \( P_\rho \) is a wide inverse submonoid of \( P_n \).

We now go in the other direction. Let \( S \) be a wide inverse submonoid of \( P_n \). Define a relation \( \rho = \rho_S \) on \( A^* \) by

\[
x\rho y \iff xy^{-1} \in S.
\]

We shall prove that \( \rho \) is a right congruence on \( A^* \). Let \( x \in A^* \). Then \( S \) is a wide inverse subsemigroup and so \( xx^{-1} \in S \). It follows that \( x\rho x \) and so \( \rho \) is reflexive. Suppose that \( x\rho y \). Then \( xy^{-1} \in S \). But \( S \) is an inverse subsemigroup and so closed under inverses. Thus \( yx^{-1} \in S \). Hence \( y\rho x \). It follows that \( \rho \) is symmetric. Suppose that \( x\rho y \) and \( ypz \). Then \( xy^{-1} \cdot yz^{-1} \in S \). Since \( S \) is closed under products we have that \( xz^{-1} \in \rho \). Thus \( x\rho z \). It follows that \( \rho \) is transitive. Finally suppose that \( x\rho y \) and \( z \in A^* \) is arbitrary. By assumption \( xy^{-1} \in S \). Because \( S \) is a wide inverse submonoid of \( P_n \) it is an order ideal. Observe that \( x(z(yz)^{-1}) \leq xy^{-1} \). Thus \( xz(yz)^{-1} \in S \). It follows that \( xz\rho yz \). We have therefore proved that \( \rho \) is a right congruence, as claimed.

We specialise the above result to the case of the bicyclic monoid \( P_1 \). The theory we describe here is part of the theory of inverse \( \omega \)-semigroups [15], where an inverse monoid is said to be an \( \omega \)-semigroup if its idempotents form a chain order isomorphic to the natural numbers with respect to the dual ordering. The
free monoid associated with the bicyclic monoid is just \( \mathbb{N} \) under addition. Because this monoid is commutative, right congruences on \( \mathbb{N} \) are just congruences. The congruences on \( \mathbb{N} \) can be easily classified and this will enable us to classify all wide inverse submonoids of \( P_1 \). We follow Howie [2]. A monogenic monoid is determined by two natural numbers: the index \( m \geq 0 \) and the period \( r \geq 0 \).

Define the relation \( \equiv^m_r \) on \( \mathbb{N} \) as follows: \( a \equiv^m_r b \iff 0 \leq a, b < m \) and \( a = b \), or \( a, b \geq m \) and \( a \equiv b \pmod{r} \).

We now describe the wide inverse submonoid of \( P_1 \) that corresponds to \( \equiv^m_r \). I’ll denote the generator of the free monoid on one generator by \( a \). There are idempotents \( a^i a^{-i} \) where \( 0 \leq i < m \). The remaining elements are of the form \( a^p a^{-q} \) where \( p, q \geq m \) and \( p \equiv q \pmod{r} \). There’s no agreed notation for this inverse submonoid, but if we denote the bicyclic monoid by \( B \) then it is natural to denote this submonoid by \( B^m_r \). The inverse monoids \( B^m_r \) are precisely the fundamental inverse \( \omega \)-monoids. If \( m = 0 \) then we denote this monoid by \( B_r \) (which agrees with Howie [2]).

**Lemma 3.4** The inverse monoid \( B^m_r \) is simple if and only if \( m = 0 \).

**Proof** Suppose that \( B^m_r \) is simple and that \( m > 0 \) and \( 1 \leq i \leq m \). Consider the idempotent \( a^i a^{-i} \) and the idempotent \( a^j a^{-j} \) where \( j \geq r \). Then we must have \( a^i a^{-i} \leq a^j a^{-j} \), which is impossible.

Now suppose that \( m = 0 \). Let \( a^i a^{-i} \) and \( a^j a^{-j} \) be arbitrary idempotents. Let \( k \geq j \) and congruent to \( i \) modulo \( r \). Then \( a^i a^{-i} D a^k a^{-k} \leq a^j a^{-j} \). Thus \( B_r \) is simple.

\( \blacksquare \)

## 4 Representations of the polycyclic monoids

The goal of this section is to describe all the proper closed inverse submonoids of the polycyclic monoids up to conjugacy and, in particular, determine the maximal proper closed inverse submonoids. By Theorem 2.8 and the Remark following, this will enable us to describe all the primitive representations of the polycyclic monoids. It turns out that a strong constraint on the proper closed inverse submonoids is that they cannot contain zero.

We begin by constructing a family of proper closed inverse submonoids of \( P_n \).

**Lemma 4.1** Let \( x \) and \( p \) be strings such that \( p \) is non-empty and \( x \) and \( xp \) have no non-trivial suffix in common. The smallest closed inverse submonoid of \( P_n \) containing the element \( x(xp)^{-1} \) is

\[
P_{x,p} = \{ xp^r \bar{p}(xp^s \bar{p})^{-1} \colon r, s \geq 0, \bar{p} \text{ is a prefix of } p \} \cup \{ \bar{x} \bar{x}^{-1} \colon \bar{x} \text{ is a prefix of } x \}.
\]

The idempotents of this semigroup are the elements of the form \( yy^{-1} \) where \( y \) is a prefix of the string \( xp^r \).
Proof It is clear that $P^x_p$ is upwardly closed under the natural partial order and closed under inversion. We prove that it is closed under multiplication. Observe first that we can show that the elements

$$\{xp^r\bar{p}(xp^s\bar{p})^{-1} : r, s \geq 0, \bar{p} \text{ a prefix of } p\}$$

form an inverse subsemigroup (isomorphic to $B_{|p|}$) by means of a routine verification. The products

$$xp^r\bar{p}(xp^s\bar{p})^{-1} \cdot \bar{x}x^{-1}$$

give nothing new. It follows that $P^x_p$ is a closed inverse submonoid of $P_n$ containing the element $x(xp)^{-1}$.

We now prove that it is the smallest closed inverse submonoid containing $x(xp)^{-1}$. Let $H$ be a closed inverse submonoid containing $x(xp)^{-1}$. Then $H$ contains $xx^{-1}$ and so contains all idempotents $\bar{x}x^{-1}$ where $\bar{x}$ is a prefix of $x$. By taking powers we can get all elements of the form $x(xp)^{-1}$ and so by products all elements of the form $xp^r(xp^s)^{-1}$. Because of upward closure under the natural partial order we must also have all elements of the form $xp^r\bar{p}(xp^s\bar{p})^{-1}$ where $\bar{p}$ is a prefix of $p$. We have thus shown that $P^x_p \subseteq H$.

The closed inverse submonoids $P^x_p$ will play an important role in what follows.

Notation We write $P^p_n$ instead of $P^x_p$.

Lemma 4.2 We have that $P^x_p \subseteq P^y_q$ if and only if $x = y$ and $p = q^s$ for some $s \geq 0$ with equality iff $x = y$ and $p = q$.

Proof Suppose that $P^x_p \subseteq P^y_q$. Since $x(xp)^{-1} \in P^y_q$, we have that $x(xp)^{-1} = yy^r\bar{q}(yy^s\bar{q})^{-1}$. Thus

$$x = yy^r\bar{q} \text{ and } xp = yy^s\bar{q}.$$ 

From the first equality we deduce that $|x| \geq |y|$. There are now two cases to consider.

Case 1. Suppose that $|x| = |y|$. Then $x = y$. It follows that $r = 0$ and $\bar{q} = \varepsilon$. Hence $p = q^s$ for some $s \geq 0$.

Case 2. Suppose that $|x| > |y|$ thus either $r > 0$ or $\bar{q}$ is not the empty string. Substituting and cancelling, we get that

$$q^r\bar{q}p = q^s\bar{q}.$$ 

Observe that $|p| < |q|$ cannot occur because by comparing lengths we have that $r|q| + |\bar{q}| + |p| = s|q| + |\bar{q}|$ and so $r|q| + |p| = s|q|$. It follows that $|p|$ is congruent to 0 modulo $|q|$. However $|p| < |q|$ and so $p$ must be the empty string. But this is excluded by our assumptions on $x$ and $p$. Thus $|p| \geq |q|$. It follows that $\bar{q}$ is a suffix of both $x$ and $p$. Hence by our assumption $\bar{q} = \varepsilon$. Thus $x = yy^r$.
and \( q'p = q^s \) and \( r > 0 \). If \( r = s \) then \( p \) is the empty string which contradicts our assumptions. If \( s < r \) then \( q'^{-r}p = \varepsilon \) which is impossible. It follows that \( s > r \) and so \( p = q'^{-s} \) where \( s - r \geq 1 \). However now \( x \) and \( p \) have \( q \) at least as a common suffix, which contradicts our assumption. It follows that this case cannot occur.

To prove the converse, suppose that \( x = y \) and \( p = q' \). Then \( x(xp)^{-1} = y(yy')^{-1} \in P_n^{q,q'} \). However \( P_n^{x,p} \) is the smallest closed inverse submonoid containing \( x(xp)^{-1} \) by Lemma 4.1 and so \( P_n^{x,p} \subseteq P_n^{y,q'} \).

\[ \Box \]

**Theorem 4.3** Each proper closed inverse submonoid of \( P_n \) belongs to exactly one of the following classes:

1. Finite chain type: it consists of a finite chain of idempotents.
2. Infinite chain type: it consists of an infinite chain of idempotents.
3. Cycle type: it is of the form \( P_n^{x,p} \) where \( p \neq \varepsilon \) and \( x \) and \( xp \) have no non-trivial suffix in common. If \( x = \varepsilon \) we say that \( P_n^p \) is of pure cycle type.

**Proof** Let \( H \) be a proper closed inverse submonoid of \( P_n \). Any two idempotents \( xx^{-1}, yy^{-1} \in H \) must be comparable for if not then their product would be zero. It follows that \( E(H) \) is a linearly ordered set. There are now two possibilities: this linearly ordered set is either finite or infinite. If it is finite then it is a finite chain of groups by Proposition 5.2.13 of [6]. However, the polycyclic monoids are combinatorial and so the only groups available are the trivial ones. It follows that if \( E(H) \) is finite then \( H = E(H) \) is just a finite chain of idempotents and so is the closed inverse subsemigroup determined by its smallest idempotent. This accounts for both finite and infinite chain types.

In the light of the above, we need now only describe those closed inverse submonoids that contain non-idempotent elements. It follows that \( H \) is an inverse \( \omega \)-monoid and since it is contained in the polycyclic monoid which is combinatorial it must be fundamental. Hence \( H \) must be abstractly isomorphic to a monoid of the form \( B_n^\omega \). However, we need to discover how that monoid is embedded in \( P_n \). We shall prove that in fact \( H = P_n^{x,p} \) for some strings \( x \) and \( p \).

We shall use the function \( \lambda \) defined in Section 3. Because \( H \) doesn’t contain zero, the restriction of \( \lambda \) to \( H \) defines a monoid homomorphism from \( H \) to the additive group \( \mathbb{Z} \). Because \( H \) is inverse its image will be inverse and so, since the image lives in a group, its image will be a subgroup of \( \mathbb{Z} \). If \( xy^{-1} \in H \) then \( xx^{-1}, yy^{-1} \in H \) and so must be comparable. It follows that either \( x = yp \) or \( y = xp \) for some string \( p \). If the former then \( ypy^{-1} \in H \) and if the latter then \( x(xp)^{-1} \in H \) and so by inverting \( xp(x)^{-1} \in H \). It follows that since we are assuming that \( H \) contains non-idempotent elements the image of \( \lambda \) is a non-trivial subgroup of \( \mathbb{Z} \). In particular, \( \lambda(xy^{-1}) = |p| \in \mathbb{Z} \), with the notation above. Thus the image of \( \lambda \) restricted to \( H \) is of the form \( n\mathbb{Z} \) where \( n \geq 1 \).
Theorem 4.4

Let $uv^{-1} \in H$. We shall prove that $uv^{-1} \in P_n^{x,p}$. Suppose first that $u = v$. Choose $r$ large enough so that $|xp^r| > |u|$. Since $uu^{-1}, xp^r(xp^r)^{-1} \in H$ we know that they are comparable. Thus $u$ is a prefix of $xp^r$ by length considerations. It follows that $uu^{-1} \in P_n^{x,p}$. Suppose now that $uv^{-1}$ is not an idempotent and, by taking inverses if necessary we can assume that $\lambda(uv^{-1}) \neq 0$. Because $uu^{-1},uv^{-1} \in P_n^{x,p}$, by our first result, we can write $uv^{-1} = xp^r(p(xp^r)^{-1})$. We will be done if we can show that $\overline{p} = \overline{p}$. Multiplying $uv^{-1}$ on the left by $xp^r(p(xp^r)^{-1})$ we get that $xp^r(xp^r)^{-1} \in H$. Multiplying this element on the right by $xp^r(p(xp^r)^{-1})$ we get that $xp^r(p(xp^r)^{-1}) \in H$. By taking inverses if necessary we can assume that $|\overline{p}| \leq |\overline{p}|$. Then $0 \leq \lambda(xp^r(xp^r)^{-1}) < |p|$. It follows that $\overline{p} = \overline{p}$ and so $uv^{-1} \in P_n^{x,p}$, as required.

Let $H$ be a proper closed inverse submonoid. If $E(H)$ is finite then the type of $H$ is the finite string $w$ with the property that the idempotents of $H$ are precisely those elements of the form $uu^{-1}$ where $u$ is a prefix of $w$. If $E(H)$ is infinite then the type of $H$ is the infinite string $w$ with the property that the idempotents of $H$ are precisely those elements of the form $uu^{-1}$ where $u$ is a prefix of $w$. We say that $H$ is ultimately periodic if its type is an ultimately periodic infinite string, and we say that it is aperiodic if its type is an infinite string which is not ultimately periodic.

We shall now set about classifying the proper closed inverse submonoids of $P_n$ up to conjugacy.

Theorem 4.4

1. Let $H$ be a proper closed inverse submonoid of $S$ of finite chain type. Then all closed inverse submonoids conjugate to it are of finite chain type, and all submonoids of finite chain type are conjugate.

2. Let $H$ be a proper closed inverse submonoid of infinite chain type. The only closed inverse submonoids conjugate to $H$ are also of infinite chain type. Two closed inverse submonoids of infinite chain type are conjugate if and only if there are idempotents $vv^{-1} \in H$ and $uu^{-1} \in K$ such that for all strings $p$ we have that $vp(vp)^{-1} \in H$ iff $wp(wp)^{-1} \in K$. It follows that they are conjugate iff their types differ in only a finite number of places.

3. Let $H$ be a proper closed inverse submonoid of cycle type. The only closed inverse submonoids conjugate to $H$ are also of cycle type. Furthermore $P_n^{x,p}$ is conjugate to $P_n^{x,q}$ if and only if $p$ and $q$ are conjugate strings.

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Proof 1. Let $H$ be a proper closed inverse submonoid consisting entirely of idempotents. Let $K$ be conjugate to $H$. By assumption, there exists an element $s$ such that $sHs^{-1} \subseteq K$ and $s^{-1}Ks \subseteq H$. Let $k \in K$. Then $s^{-1}ks \in H$. By assumption $s^{-1}ks = e$, an idempotent, necessarily nonzero. It follows that $ses^{-1} = ss^{-1}kss^{-1} \leq k$. Now $ses^{-1}$ is an idempotent, necessarily nonzero. Thus $k$ is an idempotent because $P_n$ is $E^*$-unitary. Hence $K$ consists entirely of idempotents.

Suppose now that $H$ consists of only a finite number of idempotents. We prove that $K$ contains only a finite number of idempotents. The set $sHs^{-1}$ also consists of a finite set of idempotents and $[sHs^{-1}] = K$. Because $H$ has a smallest idempotent so too does $sHs^{-1}$. It follows that $K$ has only finitely many idempotents.

We now prove that any two proper closed inverse submonoids of finite chain type are conjugate. Let $H = [uw^{-1}]$ and $K = [vv^{-1}]$. We show that $vv^{-1}Huw^{-1} = \{vv^{-1}\}$. By direct calculation, if $\bar{u}$ is a prefix of $u$ then $vv^{-1} \cdot \bar{u} = vv^{-1}$. It follows that $vv^{-1}Huw^{-1} \subseteq K$ and $uw^{-1}Kvu^{-1} \subseteq H$. Thus $H$ and $K$ are conjugate.

2. The proof of the first claim follows from the proof of (1) above. Let $H$ and $K$ both be of infinite chain type such that there are idempotents $uv^{-1} \in H$ and $uw^{-1} \in K$ such that for all strings $p$ we have that $up(vp)^{-1} \in H$ iff $up(up)^{-1} \in K$. We prove that $uv^{-1}Hvu^{-1} \subseteq K$. Consider first an idempotent of the form $\bar{v}v^{-1}$. Direct calculation shows that $uv^{-1} \cdot \bar{v}v^{-1} \cdot vuv^{-1} = uv^{-1}$. The elements below $uv^{-1}$ in $H$ are of the form $vp(vp)^{-1}$ for some $p$, and $uv^{-1} \cdot vp(vp)^{-1} \cdot vv^{-1} = up(up)^{-1}$. It follows that $uv^{-1}Hvu^{-1} \subseteq K$. It readily follows now that if our condition is satisfied then $H$ and $K$ are conjugate.

Suppose that $H$ and $K$ are conjugate. Then there is an element $uv^{-1} \in H$ such that $uv^{-1}Hvu^{-1} \subseteq K$ and $vu^{-1}Kvu^{-1} \subseteq H$. In addition, $uv^{-1} \in H$ and $uw^{-1} \in K$. It is easy to check that $uv^{-1}Hvu^{-1} \subseteq K$ and vice-versa.

3. Suppose that $uv^{-1}P_n^{x,p}vu^{-1} \subseteq P_n^{y,q}$ and $vu^{-1}P_n^{y,q}uv^{-1} \subseteq P_n^{x,p}$. Then $uv^{-1} \in P_n^{x,p}$ and $uw^{-1} \in P_n^{y,q}$. Thus $v = xp^\ast \bar{p}$ and $u = yq^\ast \bar{q}$. We know that $uv^{-1} = x(xp)^{-1} \cdot vv^{-1} \in P_n^{y,q}$.

Carrying out the calculations and using Proposition 3.1, we find that $p$ is conjugate to a power of $q$. A similar argument shows that $q$ is conjugate to a power of $p$. It follows that $p$ and $q$ are conjugate.

Suppose now that $p$ and $q$ are conjugate. Then $p = cd$ and $q = dc$ for some strings $c$ and $d$. We shall show that $y(xc)^{-1} \cdot P_n^{x,p} \cdot xcy^{-1} \subseteq P_n^{y,q}$.

First, we have that $y(xc)^{-1} \cdot xcx^{-1} = yy^{-1}$.

Now let $r, s \geq 1$ then

$y(xc)^{-1} \cdot xp^\ast \bar{p}(xp^s\bar{p})^{-1} \cdot xcy^{-1} = yq^r(yq^s(d\bar{p})(yq^s^{-1}(d\bar{p}))^{-1}$. 

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The string \( d\bar{p} \) is equal to \( \bar{q} \) or \( q\bar{q} \) for some \( \bar{q} \). Thus this string belongs to \( P_{n}^{q,\bar{q}} \). The cases \( r = 0 \) or \( s = 0 \) also lead to strings that belong to \( P_{n}^{q,\bar{q}} \), and we have proved our claim. It now readily follows that \( P_{n}^{x,p} \) and \( P_{n}^{y,\bar{q}} \) are conjugate.

We can now classify the proper closed inverse submonoids corresponding to primitive actions.

**Theorem 4.5** Every proper closed inverse submonoid of the polycyclic monoid \( P_{n} \) which corresponds to a primitive action is conjugate to a closed inverse submonoid of one of the following two types:

1. Aperiodic type with two such being conjugate when their types differ in only a finite number of places.
2. Pure cycle type \( P_{n}^{p} \) where \( p \) is a primitive string.

**Proof** From the theory developed in Section 2, we have first to locate the maximal proper closed inverse submonoids. The proper closed inverse submonoids are described by Theorem 4.3. Those of finite chain type cannot be maximal because they can be embedded in proper closed inverse submonoids of infinite chain type. If \( H \) is of infinite chain type \( x\bar{p}^{\omega} \) then \( H \subset P_{n}^{x,p} \). Thus the maximal proper closed inverse submonoids of infinite chain type are those of aperiodic type.

If \( H \) is of cycle type \( P_{n}^{x,p} \) and \( p \) is not a primitive string then \( P_{n}^{x,p} \) is not maximal by Lemma 4.2. It follows that the maximal proper inverse submonoids of cycle type are precisely those of the form \( P_{n}^{x,p} \) where \( p \) is a primitive string.

We now use Theorem 4.4 to classify the maximal proper inverse submonoids up to conjugacy. Those of cycle type can be represented by \( P_{n}^{p} \) where \( p \) is a primitive string. Two proper closed inverse submonoids of aperiodic type are conjugate iff their corresponding infinite strings differ in only a finite number of places.

Thus the conjugacy classes of the maximal proper closed inverse submonoids of pure cycle type are classified using Lyndon strings.

We now describe the actions associated with each of the conjugacy classes of proper closed inverse submonoids of \( P_{n} \). We begin with the simplest case first.

Define an action of \( P_{n} \) on \( A_{n}^{*} \) as follows:

\[
xy^{-1} \cdot u = \begin{cases} 
    xp & \text{if } u = yp \text{ for some string } p \\
    \text{undefined} & \text{otherwise}
\end{cases}
\]

We call this the *natural action of \( P_{n} \) on \( A_{n}^{*} \). Observe that this action is transitive because \( xy^{-1} \cdot y = x \) for any two strings \( x \) and \( y \). The stabiliser of the point \( x \) consists of all the idempotents of the form \( \bar{x}x^{-1} \). The following is now immediate.
Proposition 4.6 The action corresponding to a proper closed inverse submonoid of finite chain type is the natural action of \( P_n \) on \( A_n^* \).

Define an action of \( P_n \) on \( A_n^\omega \) as follows:

\[
xy^{-1} \cdot u = \begin{cases} \ x p & \text{if } u = yp \text{ for some infinite string } p \\ \text{undefined} & \text{otherwise} \end{cases}
\]

We call this the natural action of \( P_n \) on \( A_n^\omega \). This action is no longer transitive but we shall study the orbits of the action each of which gives rise to a transitive action of \( P_n \). Observe that if \( xp^\omega \) is an ultimately periodic string we can assume that \( x \) and \( p \) have no suffix in common because if they did we could write \( x = \bar{x}y \) and \( p = \bar{p}y \), where \( y \) is as long as possible, and then \( xp^\omega = \bar{x}(yp)^\omega \) with \( \bar{x} \) and \( \bar{y} \) having no non-trivial suffix in common. Next we can assume that \( p \) is primitive because if \( p = q^r \) then \( xp^\omega = xq^r \).

The proof of the following is straightforward.

Proposition 4.7 With respect to the natural action of \( P_n \) on \( A_n^\omega \) we have the following.

1. The ultimately periodic string \( xp^\omega \) where \( p \) is primitive and \( x \) and \( p \) have non-trivial suffix in common has the stabiliser \( P_n^{x,p} \).
2. The infinite aperiodic string \( x \) has the stabiliser the closed inverse submonoid of chain type \( x \).

It follows that the natural action of the polycyclic monoid on the set of infinite strings is the disjoint union of each of the primitive transitive representations of the polycyclic monoid with each such representation occurring exactly once.

5 Strong representations and branching function systems

We now single out a special class of representations of the polycyclic monoids. A representation of \( P_n \) on a set \( X \) is said to be strong if

\[
X = a_1 \cdot X \cup \ldots \cup a_n \cdot X.
\]

Such actions played an important role in our paper [8] and were introduced in [1]. Not all actions are strong: for example, the natural action of \( P_n \) on \( A_n^* \) is not strong because the empty string is not in the set \( A_n^+ = (a_1 + \ldots + a_n)A^* \)
However, the natural action of $P_n$ on $A_ω^*$ is strong because every point is an infinite string and so begins with some letter of $A_n$; that is $A_ω^* = (a_1 + \ldots + a_n)A_ω^*$. The disjoint union of strong actions is strong.

**Remark** Let $θ: S \to I(X)$ be a representation of the inverse monoid $S$. We shall say that the action is *strong* if

$$X = \bigcup_{e \in E(S), e \neq 1} \text{dom}(θ(e)).$$

In the polycyclic monoid $P_n$, the idempotents $a_i a_i^{-1}$ for $1 \leq i \leq n$ are the maximal non-identity idempotents. Thus a representation of a polycyclic monoid will be strong in this new sense precisely when it is strong in the original sense.

**Lemma 5.1** The only transitive action of $P_n$ which is not strong is its natural action on $A_ω^*$.

**Proof** This can obviously be proved following our classification of closed inverse submonoids. However, we give an alternative proof that explains what is going on. Let $P_n$ act transitively on the set $X$. Let $x \in X$ and let its stabiliser in $P_n$ be $S_x$. All we shall assume about $S_x$ is that it has infinitely many idempotents linearly ordered. Let $y \in X$ be arbitrary. Then there is a string $uv^{-1}$ such that $uv^{-1} \cdot x = y$. If $u \neq ε$ then there is nothing to prove so we shall assume that $u = ε$. It follows that the elements of $P_n$ mapping $x$ to $y$ are precisely those in $[v^{-1}H]$. We now use the fact that the idempotents in $H$ are linearly ordered and that there are infinitely many of them. Choose $ww^{-1} \in H$ such that $|w| > |v|$. Then $v^{-1} \cdot ww^{-1}$ is defined and so $w = vp$ for some non-empty string $p$. It follows that $v^{-1} \cdot ww^{-1} = pw^{-1}$. Hence $pw^{-1} \cdot x = y$ and again $y$ belongs to $(a_1 + \ldots + a_n)X$. Thus the action of $P_n$ on $X$ is strong. 

**Lemma 5.2** Let $P_n$ be an action on $X$ which is not strong. Put $X' = X \setminus (a_1 + \ldots + a_n) \cdot X$. If $x, y \in X'$ are distinct then neither is in the orbit of the other. Thus the subspaces $P_n x$ and $P_n y$ are disjoint. In addition, the stabiliser of $x$ is the empty string.

**Proof** Suppose that $uv^{-1} \cdot x = y$. If $u \neq ε$ then $y$ is in the image of the action of the first letter of $u$. Thus $u = ε$. But then $x = v \cdot y$ which implies that $x$ is in the image of the first letter of $v$. Hence $v = ε$ and so $x = y$.

Suppose now that $uv^{-1} \cdot x = x$. Then the argument above shows that $u = v = ε$.

Thus in an action of $P_n$ on the set $X$ which is not strong, each point of $X$ which is not in $(a_1 + \ldots + a_n) \cdot X$ has an orbit which is equivalent to the natural action.
Lemma 5.3 Let $P_n$ act strongly on the set $X$. Let $x \in X$. Then for each natural number $m$ there is a unique string $u$ of length $m$ and a point $y$ such that $x = u \cdot y$ (which is equivalent to saying that $\exists u^{-1} \cdot x$). Suppose that $u^{-1} \cdot x$ and $v^{-1} \cdot x$ are defined and $|u| \geq |v|$ then $v$ is a prefix of $u$.

Proof Because the action is strong, there is a $b_1 \in A_n$ and a point $x_1$ such that $x = b_1 \cdot x_1$. The same argument can be applied to $x_1$. Thus induction supplies the existence of a string $u$ and a point $y$. Suppose that $x = u \cdot y = v \cdot z$ and $u$ and $v$ have the same length. Let the first letter of $u$ be $a$. Then $a^{-1} \cdot x$ is defined. Thus $a^{-1} v$ is defined and is non-zero. It follows that the first letter of $v$ is also $a$. By induction it follows that $u = v$, and so $y = z$.

We have that $x = u \cdot y$ iff $\exists u^{-1} \cdot x = y$ which proves the alternative characterisation stated in the brackets.

Suppose that $\exists u^{-1} \cdot x$ and $\exists v^{-1} \cdot x$ and that $|u| > |v|$. Both $u u^{-1} \cdot x$ and $v v^{-1} \cdot x$ are defined and so $u$ and $v$ are comparable, and from our assumption on their respective lengths we have that $v$ is a prefix of $u$.

Morphisms between strong representations behave well.

Lemma 5.4 Let $P_n$ act strongly on both $X$ and $Y$. Then every morphism $\alpha: X \to Y$ is a covering morphism.

Proof We have to prove that $\exists \alpha^{-1} \cdot \alpha(x)$ implies that $\exists \alpha^{-1} \cdot x$. Because $w \cdot x$ is defined for all positive strings $w$ it is enough to prove the result when $u = \varepsilon$. Suppose that $\alpha^{-1} \cdot \alpha(x)$. By Lemma 5.3 there is a string $u$ of length $|v|$ and a point $y$ such that $x = u \cdot y$. Thus $\alpha(x) = u \cdot y$. It follows that $\alpha^{-1} u$ is non-zero and so, since they have the same length, they must be equal. Thus $x = v \cdot y$ and so, in particular, $\exists v^{-1} \cdot x$, as claimed.

The following is just our version of the coding map of [1].

Lemma 5.5 For every strong action of $P_n$ on $X$ there is a covering morphism $\sigma$ to the natural action of $P_n$ on infinite strings such that every finite prefix of $\sigma(x)$ of length $m$ is the unique string $u$ of length $m$ such that $u^{-1} \cdot x$ is defined.

Proof The map $\sigma$ is well-defined by Lema 5.3. It remains to show that $\sigma$ is a morphism. Suppose that $w z^{-1} \cdot x$ is defined. Then $z$ is a prefix of $\sigma(x)$ and so $\sigma(x) = z z'$ where $z'$ is infinite. From the definition of the natural action on infinite strings we have that $w z^{-1} \cdot \sigma(x)$ is defined and is equal to $w z'$. But $\sigma(w z^{-1} \cdot x)$ is also equal to $w z'$. Thus $\sigma(w z^{-1} \cdot x) = w z^{-1} \cdot \sigma(x)$.
We call the map $\sigma$ the *coding morphism*. If the coding morphism is injective then Bratteli and Jorgensen [1] define the action to be *multiplicity-free*. It follows by Lemmas 2.1 and 5.4 that a multiplicity-free strong action is equivalent to a subspace of the natural action of $P_n$ on infinite strings. Thus by Proposition 4.7, a strong action is multiplicity-free if and only if it is a disjoint union of primitive strong actions each of which occurs at most once, thus providing a completely algebraic characterisation of this notion.

Strong representations of $P_n$ determine and are determined by ‘$n$-ary branching function systems’ which we now define. A branching function system is a set $X$ equipped with $n$ injective functions $f_i: X \to X$ for $i = 1, \ldots, n$ such that the images of the functions form a partition of the set $X$ [1]. Thus branching function systems are special kinds of unary algebras (in the sense of universal algebra). The equivalence between strong representations and branching function systems is easy to establish. Given a strong action of $P_n$ on $X$ its restriction to $A_n$ and so to $A_∗$ gives rise to a branching function system. Conversely, given a branching function system on $X$, we have a function from $A_n$ to $I(X)$. This can be extended to a monoid homomorphism of $A_∗$ to $I(X)$ using the fact that $A_*$ is the free monoid on $A_n$. This homomorphism can be extended to a homomorphism of $P_n$ to $I(X)$ using the relations implicit in the definition of a branching function system.

There is an equivalent way of expressing the data making up a branching function system. Let $(X, f_1, \ldots, f_n)$ be a system. Define
\[
\bigsqcup_{i=1}^{n} X = \bigcup_{i=1}^{n} \{i\} \times X
\]
to be the disjoint union of $n$ copies of $X$ and define a function $\alpha: X \to \bigsqcup_{i=1}^{n} X$ by $\alpha(x) = (i, y)$ where $f_i(y) = x$. It is easy to check that $\alpha$ is a bijection. Conversely, every bijection $\alpha$ from $X$ to $\bigsqcup_{i=1}^{n} X$ defines a branching function system by putting $f_i(y) = \alpha^{-1}(i, y)$. See Section 9.3 of [6] for more on this approach to branching function systems.

Thus to study strong representations of the polycyclic monoid on $n$ generators, it is enough to study $n$-ary branching function systems. It is now easy to see why Kawamura’s results [3, 4] can be derived from ours: in fact, his papers inspired us. His approach via branching function systems is much more straightforward than ours. However, the results of this paper put his results in a broader context, and the methods could be applied elsewhere.

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References


