Primitive partial permutation representations of the polycyclic monoids and branching function systems

Mark V. Lawson
Department of Mathematics
and the
Maxwell Institute for Mathematical Sciences
Heriot-Watt University
Riccarton
Edinburgh EH14 4AS
Scotland
M.V.Lawson@ma.hw.ac.uk

February 19, 2009

Abstract

We characterise the maximal proper closed inverse submonoids of the polycyclic inverse monoids, also known as Cuntz inverse semigroups, and so determine all their primitive partial permutation representations. We relate our results to the work of Kawamura on certain kinds of representations of the Cuntz C^*-algebras and to the branching function systems of Bratteli and Jorgensen.

Keywords: polycyclic inverse monoid, Cuntz inverse monoid, branching function systems, partial permutation representations.

2000 *Mathematics Subject Classification*: 20M18, 20M30, 46L05, 20B15.

1 Introduction

This paper is the third in a sequence dealing with the relationship between the polycyclic monoids and the Thompson groups [7, 8]. The polycyclic monoids were introduced by Nivat and Perrot in 1971 [12] and rediscovered by Cuntz in the course of defining what are now termed Cuntz C^*-algebras. For this reason, within C^*-algebra theory, the polycyclic monoids are often called *Cuntz inverse semigroups* [13, 15]. In [8], we described how strong representations of the polycyclic monoid P_n led to the construction of the Thompson group V_n. Such representations have arisen elsewhere, in particular as the branching function systems of [1]. The goal of this paper is to study the structure of the representations by partial permutations of the polycyclic monoids in general with a particular emphasis on the strong representations. Specifically, using the ideas to be found in [9], we classify the primitive representations of the polycyclic monoid on n generators. In the process, we shall give an inverse semigroup interpretation of the work of Kawamura [4, 5], which provided the initial impetus for this work, as well as some results of Bratteli and Jorgensen [1].

We shall call upon standard inverse semigroup theory throughout this paper; see [6], for example, for the rudiments of this theory. All inverse semigroups will have a zero and we shall assume that homomorphisms are homomorphisms that preserve the zero. The product in a semigroup will usually be denoted by concatenation but sometimes we shall use $\cdot$ for emphasis; we shall also use it to denote actions. In an inverse semigroup S we define

$$\mathbf{d}(s) = s^{-1}s \text{ and } \mathbf{r}(s) = ss^{-1}.$$

The natural partial order will be the only partial order considered when we deal with inverse semigroups. If $X \subseteq S$ then $E(X)$ is the set of idempotents in X. An inverse submonoid of S is said to be *wide* if it contains all the idempotents of S.

In this paper, a *representation* of an inverse semigroup by means of partial bijections is a homomorphism $\theta\colon S \to I(X)$ to the symmetric inverse monoid on a set X. If S is a monoid we shall assume that the homomorphism is a monoid homomorphism. A representation of an inverse semigroup in this sense leads to a corresponding notion of an action of the inverse semigroup S on the set X: the associated action is defined by $s \cdot x = \theta(s)(x)$, if defined.[1] For convenience, we shall use the words 'action' and 'representation' interchangeably: if I say the inverse semigroup S acts on a set X then this will imply the existence of an appropriate homomorphism from S to $I(X)$. If S acts on X, I shall often refer to X as a *space* and its elements as *points*. A subset $Y \subseteq X$ closed under the action is called a *subspace*. Disjoint unions of actions are again actions.

An action is said to be *effective* if for each $x \in X$ there exists $s \in S$ such that $\exists s \cdot x$. This is not much of a restriction because every action can be rendered effective by omitting the points which are not acted on by any element. In this paper, all our actions will be assumed to be effective.

The action of an inverse semigroup S on the set X induces an equivalence relation $\sim$ on the set X when we define $x \sim y$ iff $s \cdot x = y$ for some $s \in S$. The action is said to be *transitive* if $\sim$ is $X \times X$. Just as in the theory of permutation representations of groups, every representation of an inverse semigroup is a disjoint union of transitive representations.

[1]The action is therefore a partial function from $S \times X$ to X mapping (s, x) to $s \cdot x$ when $\exists s \cdot x$. We require that $\exists (st) \cdot x$ iff $\exists s \cdot (t \cdot x)$ in which case they are equal and if $\exists e \cdot x$ where e is an idempotent then $e \cdot x = x$.

Let X and Y be spaces. A *morphism* from X to Y is a function $\alpha\colon X \to Y$ such that $\exists s \cdot x$ implies that $\exists s \cdot \alpha(x)$ and $\alpha(s \cdot x) = s \cdot \alpha(x)$. A morphism is said to be *strong* if it satisfies the condition that $\exists s \cdot x \Leftrightarrow \exists s \cdot \alpha(x)$. The terminology 'strong morphism' is taken from [2]. Such morphisms were studied first within inverse semigroup theory in [16] where they were called 'covering morphisms'.

A bijective strong morphism is called an *equivalence*. As with group actions, equivalent actions are the same except for the labelling of the points. The proof of the following is straightforward or can be found in [9].

Lemma 1.1 *The images of strong morphisms are subspaces, and strong morphisms between transitive spaces are surjective.*

■

2 Background

The theory of transitive actions of inverse semigroups was first studied by Boris Schein [17]. An account of this work is described in Section IV.4 of [14], and Section 5.8 of [3]. We describe this work below modified in the obvious way to accommodate our assumption that our inverse semigroups have a zero. Further details can be found in [9].

If $A \subseteq S$ is a subset of an inverse semigroup define

$$A^{\uparrow} = \{s \in S\colon a \leq s \text{ for some } a \in A\}.$$

If $A = A^{\uparrow}$ then A is said to be *closed (upwards)*. We shall be particularly interested in the closed inverse subsemigroups.

Fix a point $x \in X$, and consider the set S_x consisting of all $s \in S$ such that $s \cdot x = x$. We call S_x the *stabiliser* of the point x. If an element s fixes a point then so too will any element above s, and so the set S_x is a closed inverse subsemigroup of S. Observe that stabilisers cannot contain zero. Now let $y \in X$ be any point. By transitivity, there is an element $s \in S$ such that $s \cdot x = y$. Observe that because $s \cdot x$ is defined so too is $s^{-1}s \cdot x$ and that $s^{-1}s \in S_x$. An easy calculation shows that $(sS_x)^{\uparrow}$ is the set of all elements of S which map x to y. Let H be a closed inverse subsemigroup of S that does not contain zero. Define a *left coset* of H to be a set of the form $(sH)^{\uparrow}$ where $s^{-1}s \in H$.

Lemma 2.1

(i) *Two cosets* $(sH)^{\uparrow}$ *and* $(tH)^{\uparrow}$ *are equal iff* $s^{-1}t \in H$.

(ii) *If* $(sH)^{\uparrow} \cap (tH)^{\uparrow} \neq \emptyset$ *then* $(sH)^{\uparrow} = (tH)^{\uparrow}$.

■

We denote by S/H the set of all left cosets of H in S. The inverse semigroup S acts on the set S/H when we define

$$a \cdot (sH)^{\uparrow} = (asH)^{\uparrow} \Leftrightarrow \mathbf{d}(as) \in H.$$

This defines a transitive action. If H and K are any closed inverse subsemigroups of S then they determine equivalent actions if and only if there exists $s \in S$ such that

$$sHs^{-1} \subseteq K \text{ and } s^{-1}Ks \subseteq H.$$

This relationship between two closed inverse subsemigroups is called *conjugacy* although it is important to observe that equality need not hold in the definition above.

Lemma 2.2 *If H and K are conjugate as above then $ss^{-1} \in K$ and $s^{-1}s \in H$. Also $(sHs^{-1})^{\uparrow} = K$ and $(s^{-1}Ks)^{\uparrow} = H$.*

The following was motivated by Lemma 2.16 of Ruyle's thesis [16].

Proposition 2.3 *Let S be an inverse semigroup acting transitively on the sets X and Y, and let $x \in X$ and $y \in Y$. Let S_x and S_y be the stabilisers in S of x and y respectively. There is a morphism $\alpha \colon X \to Y$ such that $\alpha(x) = y$ iff $S_x \subseteq S_y$. If such a morphism exists then it is unique.*

Proof We begin by proving uniqueness. Let $\alpha, \beta \colon X \to Y$ be morphisms such that $\alpha(x) = \beta(x) = y$. Let $x' \in X$ be arbitrary. By transitivity there exists $a \in S$ such that $x' = a \cdot x$. By the definition of morphisms we have that $\exists a \cdot \alpha(x)$ and $\exists a \cdot \beta(x)$ and that

$$\alpha(x') = \alpha(a \cdot x) = a \cdot \alpha(x)$$

and

$$\beta(x') = \beta(a \cdot x) = a \cdot \beta(x).$$

But by assumption $\alpha(x) = \beta(x) = y$ and so $\alpha(x') = \beta(x')$. It follows that $\alpha = \beta$.

Let $\alpha \colon X \to Y$ be a morphism such that $\alpha(x) = y$. Let $s \in S_x$. Then $\exists s \cdot x$ and $s \cdot x = x$. By the definition of morphism, it follows that $\exists s \cdot \alpha(x)$ and that $\alpha(s \cdot x) = s \cdot \alpha(x)$. But $s \cdot x = x$ and so $\alpha(x) = s \cdot \alpha(x)$. Hence $s \cdot y = y$. We have therefore proved that $s \in S_y$, and so $S_x \subseteq S_y$.

Suppose now that $S_x \subseteq S_y$. We have to define a morphism $\alpha \colon X \to Y$ such that $\alpha(x) = y$. We start by defining $\alpha(x) = y$. Let $x' \in X$ be any point in X. Then $x' = a \cdot x$ for some $a \in S$. We need to show that $a \cdot y$ exists. Since $a \cdot x$ exists we know that $a^{-1}a \cdot x$ exists and this is equal to x. It follows that $a^{-1}a \in S_x$ and so $a^{-1}a \in S_y$, by assumption. Thus $a^{-1}a \cdot y$ exists and is equal to y. But from the existence of $a^{-1}a \cdot y$ we can deduce the existence of $a \cdot y$. We would therefore like to define $\alpha(x') = a \cdot y$. We have to check that this is well-defined. Suppose that $x' = a \cdot x = b \cdot x$. Then $b^{-1}a \cdot x = x$ and so $b^{-1}a \in S_x$. By assumption, $b^{-1}a \in S_y$ and so $b^{-1}a \cdot y = y$. Thus $bb^{-1}a \cdot y = b \cdot y$ and $bb^{-1}a \cdot y = bb^{-1} \cdot (a \cdot y) = a \cdot y$. Thus $a \cdot y = b \cdot y$. It follows that α is a well-defined function mapping x to y. It remains to show that α is a morphism. Suppose that $s \cdot x'$ is defined. By assumption, there exists $a \in S$ such that $x' = a \cdot x$. By definition $\alpha(x') = a \cdot y$. We have that $s \cdot x' = s \cdot (a \cdot x) = sa \cdot x$. By definition $\alpha(s \cdot x') = sa \cdot y$. But $sa \cdot y = s \cdot (a \cdot y) = s \cdot \alpha(x')$. Hence $\alpha(s \cdot x') = s \cdot \alpha(x')$, as required.

■

The notion of a primitive group action is fundamental. Generalising it to inverse semigroup actions is more problematical. The intuitive idea is that a transitive inverse semigroup action should be ‘primitive’ if it is ‘simple’ in some sense. However, in the case of inverse semigroup actions, there are a number of ways in which the word ‘simple’ can be interpreted. Below we shall describe one interpretation; for a fuller discussion see [9]

An action of S on X is said to be *proper* if for each non-zero $s \in S$ there is $x \in X$ such that $\exists s \cdot x$. In other words, non-zero elements of S are mapped by θ to non-zero elements of $I(X)$. If an action is not proper then the set of elements $s \in S$ which act as the empty function on X forms an ideal in S. A closed inverse subsemigroup H of an inverse semigroup S is said to be *proper* if it satisfies the following two conditions.

(P1) $0 \notin H$.

(P2) For each non-zero $s \in S$ there exists $a \in S$ such that $\mathbf{d}(a), \mathbf{d}(sa) \in H$.

The following is proved in [9].

Proposition 2.4 *Every proper transitive action of the inverse semigroup with zero S is equivalent to the action of S on a space of the form S/H where H is some proper closed inverse subsemigroup of S.*

Groups always act on one-point sets. However, the presence of a zero in the inverse semigroup makes the situation a little more complicated.

Proposition 2.5 *Let S be an inverse semigroup with zero. Then there is a proper representation of S on the one-point set if and only if the zero element of S is adjoined.*

Proof If the zero is adjoined then $H = S \setminus \{0\}$ is a proper closed inverse subsemigroup of S and S/H is a one-point set. Thus S acts properly on a one-point set. Conversely, let S act properly on a one-point set. The action is evidently transitive. Let H be the stabiliser of the unique point. Then H is a proper closed inverse subsemigroup of S. Let $s \in S$ be any non-zero element. Then there exists $a \in S$ such that $\mathbf{d}(a), \mathbf{d}(sa) \in H$. Now $H = (aH)^{\uparrow} = (saH)^{\uparrow}$ since there is only one point. Thus $a, sa \in H$. But then $saa^{-1} \in H$ and $saa^{-1} \leq s$ and so $s \in H$. It follows that $H = S \setminus \{0\}$ and since H is an inverse subsemigroup it follows that the zero is adjoined.

■

Let S be an inverse semigroup with zero in which the zero is not adjoined and let X be a proper transitive space where $|X| > 1$. We say that the action is *primitive* if every morphism from X to another proper transitive space is an equivalence.

Proposition 2.6 *Let S be an inverse semigroup with zero in which the zero is not adjoined. Let S act transitively and properly on a space X where $|X| > 1$. Let $x \in X$, and let S_x be its stabiliser. Then the action is primitive if and only if S_x is a maximal proper closed inverse subsemigroup.*

Proof Suppose that the action is primitive. Let $S_x \subseteq H$ where H is a proper closed inverse subsemigroup. Then by Proposition 2.3, there is a morphism $\alpha \colon S/S_x \longrightarrow S/H$ mapping the point S_x to the point H. By assumption α is an equivalence. Applying Proposition 2.3 to the inverse of α we deduce that $S_x = H$.

Suppose that S_x is a maximal proper closed inverse subsemigroup and let $\alpha \colon X \to Y$ be a morphism taking x to y. By Proposition 2.3, we have that $S_x \subseteq S_y$. By assumption, $S_x = S_y$, and so α is an equivalence, as claimed.

■

The zero of the polycyclic monoids is not adjoined and the polycyclic monoids are congruence-free. Thus the proper closed inverse submonoids are just the closed inverse submonoids that do not contain zero. It follows that in order to characterise the primitive actions of such inverse monoids, we shall need to classify their maximal proper closed inverse submonoids up to conjugacy.

3 The polycyclic monoids

In this section, we outline the theory of the polycyclic monoids. Most of the results are well-known although our classification of the wide inverse submonoids of the polycyclic monoids (Theorem 3.3) appears to be new.

Let $n \geq 1$, and put $A_n = \{a_1, \ldots, a_n\}$. A string in A_n^*, the free monoid generated by A_n, will be called *positive*. The empty string is denoted ε. The *length* of the string x is denoted by $|x|$. The free semigroup on A_n is denoted by A_n^+. The set of *right infinite* strings over A_n is denoted by A_n^ω. If x is a finite string then $x^\omega = xxx\ldots$. A right infinite string x is said to be *ultimately periodic* if it is of the form yz^ω where y and z are finite strings and z is non-empty. For the purposes of this paper, an infinite string is *aperiodic* if it is not ultimately periodic. If $u = vw$ are strings, then v is called a *prefix* of u, and a *proper prefix* if w is not the empty string; whereas w is called a *suffix* of u, and a *proper suffix* if v is not the empty string. If x is a string then the notation $\bar{x}$ will be used to denote a prefix of x. A pair of elements of A_n^* is said to be *prefix-comparable* if one is a prefix of the other. If x and y are prefix-comparable we define

$$x \wedge y = \begin{cases} x & \text{if } y \text{ is a prefix of } x \\ y & \text{if } x \text{ is a prefix of } y \end{cases}$$

A string is said to be *primitive* if it is not a proper power of another string. Let x and y be strings. If $x = uv$ and $y = vu$ then x and y are said to be *conjugate*. A *Lyndon word (or string)* is a primitive string that is minimal in its conjugacy class with respect to the lexicographic ordering [10]. For us, Lyndon strings are just a way of choosing a unique representative from conjugacy classes of primitive strings.

The following is proved as Proposition 1.3.4 of [10].

Proposition 3.1 *Let $x, y \in A^+$. Then x and y are conjugate iff $xz = zy$ for some string z.*

■

The *bicyclic monoid* is defined by the following monoid presentation

$$B = P_1 = \langle a_1 \colon a_1^{-1} a_1 = 1 \rangle.$$

The *polycyclic monoid* P_n, where $n \geq 2$, is defined as a monoid with zero by the following presentation

$$P_n = \langle a_1, \ldots, a_n, a_1^{-1}, \ldots, a_n^{-1} \colon a_i^{-1} a_i = 1 \,\text{and}\, a_i^{-1} a_j = 0, i \neq j \rangle.$$

Intuitively, think of $a_1, \ldots, a_n$ as partial bijections of a set X and $a_1^{-1}, \ldots, a_n^{-1}$ as their respective partial inverses. The first relation says that each partial bijection a_i has domain the whole of X and the second says that the ranges of distinct a_i are orthogonal. As a concrete example of P_2, one can take as a_1 and a_2 the two maps that shrink the Cantor set to its lefthand and righthand sides, respectively. Every non-zero element of P_n is of the form yx^{-1} where $x, y \in A_n^*$. Identify the identity with $\varepsilon\varepsilon^{-1}$. The product of two elements yx^{-1} and vu^{-1} is zero unless x and v are prefix-comparable. If they are prefix-comparable then

$$yx^{-1} \cdot vu^{-1} = \begin{cases} yzu^{-1} & \text{if } v = xz \text{ for some string } z \\ y(uz)^{-1} & \text{if } x = vz \text{ for some string } z \end{cases}$$

The non-zero idempotents in P_n are the elements of the form xx^{-1}, where x is positive, and the natural partial order is given by $yx^{-1} \leq vu^{-1}$ iff $y = vp$ and $x = up$ for some positive string p. Observe that if xx^{-1} and yy^{-1} are non-zero idempotents then $xx^{-1} \cdot yy^{-1} \neq 0$ if and only if either $xx^{-1} \leq yy^{-1}$ or $yy^{-1} \leq xx^{-1}$. When non-zero

$$xx^{-1} \cdot yy^{-1} = (x \wedge y)(x \wedge y)^{-1}.$$

An inverse semigroup with zero is said to be E^*-*unitary* if every element above a non-zero idempotent is an idempotent. An inverse semigroup is said to be E-*unitary* if every element above an idempotent is an idempotent. The bicyclic monoid is E-unitary and the polycyclic monoids are E^*-unitary.

There is a function λ from the non-zero elements of P_n to the additive group $\mathbb{Z}$ defined by $\lambda(uv^{-1}) = |u| - |v|$. If $yx^{-1} \cdot vu^{-1} \neq 0$ then direct verification shows that

$$\lambda(yx^{-1} \cdot vu^{-1}) = \lambda(yx^{-1}) + \lambda(vu^{-1}).$$

Lemma 3.2 *In a polycyclic monoid if $uv^{-1} \leq xy^{-1}, wz^{-1}$ then xy^{-1} and wz^{-1} are comparable.*

Proof It follows that $u = xp = wq$ and $v = yp = zq$ for some strings p and q. If $|p| = |q|$ then $xy^{-1} = wz^{-1}$, if $|p| < |q|$ then $xy^{-1} \leq wz^{-1}$, and if $|q| < |p|$ then $wz^{-1} \leq xy^{-1}$.

■

Meakin and Sapir [11] proved there was a bijection between the congruences on the free monoid on n generators A^* and the positively self-conjugate wide inverse submonoids of the polylcyclic monoid on n generators P_n. We generalise this result as follows.

Theorem 3.3 *There is a bijection between right congruences on A^* and wide inverse submonoids of P_n.*

Proof Let ρ be a right congruence on A^*. Let

$$P_\rho = \{xy^{-1} \in P_n \colon x\rho y\} \cup \{0\}.$$

We prove that P_ρ is a wide inverse subsemigroup of P_n. The nonzero idempotents of P_n are the elements of the form xx^{-1}. Since ρ is an equivalence relation for every string x we have that $x\rho x$. Thus P_ρ contains all the idempotents of P_n. If $xy^{-1} \in P_\rho$ then $x\rho y$ and so $y\rho x$, since ρ is an equivalence relation, and it follows that $yx^{-1} \in P_\rho$. Thus P_ρ is closed under inverses. It remains to show that P_ρ is closed under products. Let $xy^{-1}, wz^{-1} \in P_\rho$ Suppose that $xy^{-1} \cdot wz^{-1} \neq 0$. There are two possibilities. Suppose that $w = yp$. Then $xy^{-1} \cdot wz^{-1} = (xp)z^{-1}$. Now $x\rho y$ and $yp = w\rho z$. By assumption, ρ is a right congruence and so $xp\rho yp$. By transitivity, we have that $xp\rho z$ and so $(xp)z^{-1} \in P_\rho$, as required. Now suppose that $y = wp$. Then $xy^{-1} \cdot wz^{-1} = x(zp)^{-1}$. Now $x\rho y = wp$ and $w\rho z$. By assumption, ρ is a right congruence and so $wp\rho zp$. By transitivity, $x\rho zp$ and so $x(zp)^{-1} \in P_\rho$. Thus we have proved that P_ρ is a wide inverse submonoid of P_n.

We now go in the other direction. Let S be a wide inverse submonoid of P_n. Define a relation $\rho = \rho_S$ on A^* by

$$x\rho y \Leftrightarrow xy^{-1} \in S.$$

We shall prove that ρ is a right congruence on A^*. Let $x \in A^*$. Then S is a wide inverse subsemigroup and so $xx^{-1} \in S$. It follows that $x\rho x$ and so ρ is reflexive. Suppose that $x\rho y$. Then $xy^{-1} \in S$. But S is an inverse subsemigroup and so closed under inverses. Thus $yx^{-1} \in S$. Hence $y\rho x$. It follows that ρ is symmetric. Suppose that $x\rho y$ and $y\rho z$. Then $xy^{-1}, yz^{-1} \in S$. Since S is closed under products we have that $xz^{-1} \in S$. Thus $x\rho z$. It follows that ρ is transitive. Finally suppose that $x\rho y$ and $z \in A^*$ is arbitrary. By assumption $xy^{-1} \in S$. Because S is a wide inverse submonoid of P_n it is an order ideal. Observe that $xz(yz)^{-1} \leq xy^{-1}$. Thus $xz(yz)^{-1} \in S$. It follows that $xz\rho yz$. We have therefore proved that ρ is a right congruence, as claimed.

■

We specialise the above result to the case of the bicyclic monoid P_1. The theory we describe here is part of the theory of inverse ω-semigroups [14], where an inverse monoid is said to be an *ω-semigroup* if its idempotents form a chain order isomorphic to the natural numbers with respect to the dual ordering. The free monoid associated with the bicyclic monoid is just $\mathbb{N}$ under addition. Because this monoid is commutative, right congruences on $\mathbb{N}$ are just congruences. The congruences on $\mathbb{N}$ can be easily classified and this will enable us to classify all wide inverse submonoids of P_1. We follow Howie [3]. A monogenic monoid is determined by two natural numbers: the *index* $m \geq 0$ and the *period* $r \geq 1$. Define the relation $\equiv_r^m$ on $\mathbb{N}$ as follows: $a \equiv_r^m b$ iff $0 \leq a, b < m$ and $a = b$, or $a, b \geq m$ and $a \equiv b \pmod r$.

We now describe the wide inverse submonoid of P_1 that corresponds to $\equiv_r^m$. I'll denote the generator of the free monoid on one generator by a. There are elements $a^p a^{-q}$ where $p, q \geq m$ and $p \equiv q \pmod r$ together with the idempotents $a^i a^{-i}$ where $0 \leq i < m$. There's no agreed notation for this inverse submonoid, but if we denote the bicyclic monoid by B then it is natural to denote this submonoid by B_r^m. The inverse monoids B_r^m are precisely the fundamental inverse ω-monoids. If $m = 0$ then we denote this monoid by B_r (which agrees with Howie [3]).

Lemma 3.4 *The inverse monoid B_r^m is simple if and only if $m = 0$.*

Proof Suppose that B_r^m is simple and that $m > 0$ and $1 \leq i \leq m$. Consider the idempotent $a^i a^{-i}$ and the idempotent $a^j a^{-j}$ where $j \geq r$. Then we must have $a^i a^{-i} \leq a^j a^{-j}$, which is impossible.

Now suppose that $m = 0$. Let $a^i a^{-i}$ and $a^j a^{-j}$ be arbitrary idempotents. Let $k \geq j$ and congruent to i modulo r. Then $a^i a^{-i} \mathcal{D} a^k a^{-k} \leq a^j a^{-j}$. Thus B_r is simple.

■

4 Representations of the polycyclic monoids

The goal of this section is to describe all the proper closed inverse submonoids of the polycyclic monoids up to conjugacy and, in particular, determine the maximal proper closed inverse submonoids. By Proposition 2.6, this will enable us to describe all the primitive representations of the polycyclic monoids. It turns out that a strong constraint on the proper closed inverse submonoids is that they cannot contain zero.

We begin by constructing a family of proper closed inverse submonoids of P_n.

Lemma 4.1 *Let x and p be strings such that p is non-empty and where x and p have no non-trivial suffix in common. The smallest closed inverse submonoid of P_n containing the element $x(xp)^{-1}$ is*

$$P_n^{x,p} = \{xp^r\bar{p}(xp^s\bar{p})^{-1} \colon r, s \geq 0, \bar{p} \text{ is a prefix of } p\} \cup \{\bar{x}\bar{x}^{-1} \colon \bar{x} \text{ is a prefix of } x\}.$$

The idempotents of this semigroup are the elements of the form yy^{-1} where y is a prefix of the string xp^{ω}.

Proof It is clear that $P_n^{x,p}$ is upwardly closed under the natural partial order and closed under inversion. We prove that it is closed under multiplication. Observe first that we can show that the elements

$$\{xp^r\bar{p}(xp^s\bar{p})^{-1} \colon r, s \geq 0, \bar{p} \text{ a prefix of } p\}$$

form an inverse subsemigroup (isomorphic to $B_{|p|}$) by means of a routine verification. The products

$$xp^r\bar{p}(xp^s\bar{p})^{-1} \cdot \bar{x}\bar{x}^{-1}$$

give nothing new. It follows that $P_n^{x,p}$ is a closed inverse submonoid of P_n containing the element $x(xp)^{-1}$.

We now prove that it is the smallest closed inverse submonoid containing $x(xp)^{-1}$. Let H be a closed inverse submonoid containing $x(xp)^{-1}$. Then H contains xx^{-1} and so contains all idempotents $\bar{x}\bar{x}^{-1}$ where $\bar{x}$ is a prefix of x. By taking powers we can get all elements of the form $x(xp^r)^{-1}$ and so by products all elements of the form $xp^r(xp^s)^{-1}$. Because of upward closure under the natural partial order we must also have all elements of the form $xp^r\bar{p}(xp^s\bar{p})^{-1}$ where $\bar{p}$ is a prefix of p. We have thus shown that $P_n^{x,p} \subseteq H$.

■

The closed inverse submonoids $P_n^{x,p}$ will play an important role in what follows.

Notation We write P_n^p instead of $P_n^{\varepsilon,p}$.

Lemma 4.2 *We have that $P_n^{x,p} \subseteq P_n^{y,q}$ if and only if $x = y$ and $p = q^s$ for some $s \geq 0$ with equality iff $x = y$ and $p = q$.*

Proof Suppose that $P_n^{x,p} \subseteq P_n^{y,q}$. Since $x(xp)^{-1} \in P_n^{y,q}$, we have that $x(xp)^{-1} = yq^r\bar{q}(yq^s\bar{q})^{-1}$. Thus

$$x = yq^r\bar{q} \text{ and } xp = yq^s\bar{q}.$$

From the first equality we deduce that $|x| \geq |y|$. There are now two cases to consider.

Case 1. Suppose that $|x| = |y|$. Then $x = y$. It follows that $r = 0$ and $\bar{q} = \varepsilon$. Hence $p = q^s$ for some $s \geq 0$.

Case 2. Suppose that $|x| > |y|$ thus either $r > 0$ or $\bar{q}$ is not the empty string. Substituting and cancelling, we get that

$$q^r\bar{q}p = q^s\bar{q}.$$

Observe that $|p| < |q|$ cannot occur because by comparing lengths we have that $r|q| + |\bar{q}| + |p| = s|q| + |\bar{q}|$ and so $r|q| + |p| = s|q|$. It follows that $|p|$ is congruent to 0 modulo $|q|$. However $|p| < |q|$ and so p must be the empty string. But this is excluded by our assumptions on x and p. Thus $|p| \geq |q|$. It follows that $\bar{q}$ is a suffix of both x and p. Hence by our assumption $\bar{q} = \varepsilon$. Thus $x = yq^r$ and $q^r p = q^s$ and $r > 0$. If $r = s$ then p is the empty string which contradicts our assumptions. If $s < r$ then $q^{r-s}p = \varepsilon$ which is impossible. It follows that $s > r$ and so $p = q^{s-r}$ where $s - r \geq 1$. However now x and p have q at least as a common suffix, which contradicts our assumption. It follows that this case cannot occur.

To prove the converse, suppose that $x = y$ and $p = q^r$. Then $x(xp)^{-1} = y(yq^r)^{-1} \in P_n^{y,q}$. However $P_n^{x,p}$ is the smallest closed inverse submonoid containing $x(xp)^{-1}$ by Lemma 4.1 and so $P_n^{x,p} \subseteq P_n^{y,q}$.

■

Theorem 4.3 *Each proper closed inverse submonoid of P_n belongs to exactly one of the following classes:*

1. Finite chain type*: it consists of a finite chain of idempotents.*
2. Infinite chain type*: it consists of an infinite chain of idempotents.*
3. Cycle type*: it is of the form $P_n^{x,p}$ where $p \neq \varepsilon$ and where x and p have no non-trivial suffix in common. If $x = \varepsilon$ we say that P_n^p is of* pure cycle type*.*

Proof Let H be a proper closed inverse submonoid of P_n. Any two idempotents $xx^{-1}, yy^{-1} \in H$ must be comparable for if not then their product would be zero. It follows that $E(H)$ is a linearly ordered set. There are now two possibilities: this linearly ordered set is either finite or infinite. If it is finite then it is a finite chain of groups by Proposition 5.2.13 of [6]. However, the polycyclic monoids are combinatorial and so the only groups available are the trivial ones. It follows that if $E(H)$ is finite then $H = E(H)$ is just a finite chain of idempotents and so is the closed inverse subsemigroup determined by its smallest idempotent. This accounts for both finite and infinite chain types.

In the light of the above, we need now only describe those closed inverse submonoids that contain non-idempotent elements. It follows that H is an inverse ω-monoid and since it is contained in the polycyclic monoid which is combinatorial it must be fundamental. Hence H must be abstractly isomorphic to a monoid of the form B_r^m. However, we need to discover how that monoid is embedded in P_n. We shall prove that in fact $H = P_n^{x,p}$ for some strings x and p.

We shall use the function λ defined in Section 3. Because H doesn't contain zero, the restriction of λ to H defines a monoid homomorphism from H to the additive group $\mathbb{Z}$. Because H is inverse its image will be inverse and so, since the image lives in a group, its image will be a subgroup of $\mathbb{Z}$. If $xy^{-1} \in H$ then $xx^{-1}, yy^{-1} \in H$ and so they must be comparable. It follows that either $x = yp$ or $y = xp$ for some string p. If the former then $ypy^{-1} \in H$ and if the latter then $x(xp)^{-1} \in H$ and so by inverting $xpx^{-1} \in H$. It follows that since we are assuming that H contains non-idempotent elements the image of λ is a non-trivial subgroup of $\mathbb{Z}$. In particular, $\lambda(xy^{-1}) = |p| \in \mathbb{Z}$, with the notation above. Thus the image of λ restricted to H is of the form $n\mathbb{Z}$ where $n \geq 1$. We may therefore find a maximal (with respect to the natural partial order) non-idempotent element yx^{-1} in H such that $\lambda(yx^{-1})$ is positive and as small as possible. We may therefore write $y = xp$ for some non-empty string p. Thus the image of λ restricted to H is the subgroup $|p|\mathbb{Z}$. The element $x(xp)^{-1} \in H$ is also maximal, and so we know that x and xp have no non-trivial suffix in common. The conditions of Lemma 4.1 hold and so $P_n^{x,p} \subseteq H$.

Let $uv^{-1} \in H$. We shall prove that $uv^{-1} \in P_n^{x,p}$. Suppose first that $u = v$. Choose r large enough so that $|xp^r| > |u|$. Since $uu^{-1}, xp^r(xp^r)^{-1} \in H$ we know that they are comparable. Thus u is a prefix of xp^r by length considerations. It follows that $uu^{-1} \in P_n^{x,p}$. Suppose now that uv^{-1} is not an idempotent and, by taking inverses, if necessary, we can assume that $\lambda(uv^{-1}) > 0$. Because $uu^{-1}, vv^{-1} \in P_n^{x,p}$, by our first result, we can write $uv^{-1} = xp^r\overline{p}(xp^s\overline{\overline{p}})^{-1}$. We will be done if we can show that $\overline{p} = \overline{\overline{p}}$. Multiplying uv^{-1} on the left by $x\overline{p}(xp^r\overline{p})^{-1}$ we get that $x\overline{p}(xp^s\overline{\overline{p}})^{-1} \in H$. Multiplying this element on the right by $xp^s\overline{\overline{p}}(x\overline{\overline{p}})^{-1}$ we get that $x\overline{p}(x\overline{\overline{p}})^{-1} \in H$. By taking inverses if necessary we can assume that $|\overline{\overline{p}}| \leq |\overline{p}|$. Then $0 \leq \lambda(x\overline{p}(x\overline{\overline{p}})^{-1}) < |p|$. It follows that $\overline{p} = \overline{\overline{p}}$ and so $uv^{-1} \in P_n^{x,p}$, as required.

■

Let H be a proper closed inverse submonoid. If $E(H)$ is finite then the *type* of H is the finite string w with the property that the idempotents of H are precisely those elements of the form uu^{-1} where u is a prefix of w. If $E(H)$ is infinite then the *type* of H is the infinite string w with the property that the idempotents of H are precisely those elements of the form uu^{-1} where u is a prefix of w. We say that H is *ultimately periodic* if its type is an ultimately periodic infinite string, and we say that it is *aperiodic* if its type is an infinite string which is not ultimately periodic.

We shall now set about classifying the proper closed inverse submonoids of P_n up to conjugacy.

Theorem 4.4

1. *Let H be a proper closed inverse submonoid of S of finite chain type. Then all closed inverse submonoids conjugate to it are of finite chain type, and all submonoids of finite chain type are conjugate.*

2. *Let H be a proper closed inverse submonoid of infinite chain type. The only closed inverse submonoids conjugate to H are also of infinite chain type. Two closed inverse submonoids of infinite chain type are conjugate if and only if there are idempotents $vv^{-1} \in H$ and $uu^{-1} \in K$ such that for all strings p we have that $vp(vp)^{-1} \in H$ iff $up(up)^{-1} \in K$. It follows that they are conjugate iff their types differ in only a finite number of places.*

3. *Let H be a proper closed inverse submonoid of cycle type. The only closed inverse submonoids conjugate to H are also of cycle type. Furthermore $P_n^{x,p}$ is conjugate to $P_n^{y,q}$ if and only if p and q are conjugate strings.*

Proof 1. Let H be a proper closed inverse submonoid consisting entirely of idempotents. Let K be conjugate to H. By assumption, there exists an element s such that $sHs^{-1} \subseteq K$ and $s^{-1}Ks \subseteq H$. Let $k \in K$. Then $s^{-1}ks \in H$. By assumption $s^{-1}ks = e$, an idempotent, necessarily nonzero. It follows that $ses^{-1} = ss^{-1}kss^{-1} \leq k$. Now ses^{-1} is an idempotent, necessarily nonzero. Thus k is an idempotent because P_n is E^*-unitary. Hence K consists entirely of idempotents.

Suppose now that H consists of only a finite number of idempotents. We prove that K contains only a finite number of idempotents. The set sHs^{-1} also consists of a finite set of idempotents and $(sHs^{-1})^{\uparrow} = K$. Because H has a smallest idempotent so too does sHs^{-1}. It follows that K has only finitely many idempotents.

We now prove that any two proper closed inverse submonoids of finite chain type are conjugate. Let $H = (uu^{-1})^{\uparrow}$ and $K = (vv^{-1})^{\uparrow}$. We show that $vu^{-1}Huv^{-1} = \{vv^{-1}\}$. By direct calculation, if $\bar{u}$ is a prefix of u then $vu^{-1} \cdot \bar{u}\bar{u}^{-1} \cdot uv^{-1} = vv^{-1}$. It follows that $vu^{-1}Huv^{-1} \subseteq K$ and $uv^{-1}Kvu^{-1} \subseteq H$. Thus H and K are conjugate.

2. The proof of the first claim follows from the proof of (1) above. Let H and K both be of infinite chain type such that there are idempotents $vv^{-1} \in H$ and $uu^{-1} \in K$ such that for all strings p we have that $vp(vp)^{-1} \in H$ iff $up(up)^{-1} \in K$. We prove that $uv^{-1}Hvu^{-1} \subseteq K$. Consider first an idempotent of the form $\bar{v}\bar{v}^{-1}$. Direct calculation shows that $uv^{-1} \cdot \bar{v}\bar{v}^{-1} \cdot vu^{-1}$ is equal to uu^{-1}. The elements below vv^{-1} in H are of the form $vp(vp)^{-1}$ for some p, and $uv^{-1} \cdot vp(vp)^{-1} \cdot vu^{-1} = up(up)^{-1}$. It follows that $uv^{-1}Hvu^{-1} \subseteq K$. It readily follows now that if our condition is satisfied then H and K are conjugate.

Suppose that H and K are conjugate. Then there is an element uv^{-1} such that $uv^{-1}Hvu^{-1} \subseteq K$ and $vu^{-1}Kuv^{-1} \subseteq H$. In addition, $vv^{-1} \in H$ and $uu^{-1} \in K$. It is easy to check that $vp(vp)^{-1} \in H$ implies that $up(up)^{-1} \in K$ and vice-versa.

3. Suppose that $uv^{-1}P_n^{x,p}vu^{-1} \subseteq P_n^{y,q}$ and $vu^{-1}P_n^{y,q}uv^{-1} \subseteq P_n^{x,p}$. Then $vv^{-1} \in P_n^{x,p}$ and $uu^{-1} \in P_n^{y,q}$. Thus $v = xp^r\bar{p}$ and $u = yq^s\bar{q}$. We know that

$$uv^{-1} \cdot x(xp)^{-1} \cdot vu^{-1} \in P_n^{y,q}.$$

Carrying out the calculations and using Proposition 3.1, we find that p is conjugate to a power of q. A similar argument shows that q is conjugate to a power of p. It follows that p and q are conjugate.

Suppose now that p and q are conjugate. Then $p = cd$ and $q = dc$ for some strings c and d. We shall show that

$$y(xc)^{-1} \cdot P_n^{x,p} \cdot xcy^{-1} \subseteq P_n^{y,q}.$$

First, we have that

$$y(xc)^{-1} \cdot \bar{x}\bar{x}^{-1} \cdot xcy^{-1} = yy^{-1}.$$

Now let $r, s \geq 1$ then

$$y(xc)^{-1} \cdot xp^r \bar{p}(xp^s \bar{p})^{-1} \cdot xcy^{-1} = yq^{r-1}(d\bar{p})(yq^{s-1}(d\bar{p}))^{-1}.$$

The string $d\bar{p}$ is equal to $\bar{q}$ or $q\bar{q}$ for some $\bar{q}$. Thus this string belongs to $P_n^{y,q}$. The cases $r = 0$ or $s = 0$ also lead to strings that belong to $P_n^{y,q}$, and we have proved our claim. It now readily follows that $P_n^{x,p}$ and $P_n^{y,q}$ are conjugate.

■

We can now classify the proper closed inverse submonoids corresponding to primitive actions.

Theorem 4.5 *Every proper closed inverse submonoid of the polycyclic monoid P_n which corresponds to a primitive action is conjugate to a closed inverse submonoid of one of the following two types:*

1. Aperiodic type *with two such being conjugate when their types differ in only a finite number of places.*
2. Pure cycle type P_n^p *where p is a primitive string.*

Proof From the theory developed in Section 2, we have first to locate the maximal proper closed inverse submonoids. The proper closed inverse submonoids are described by Theorem 4.3. Those of finite chain type cannot be maximal because they can be embedded in proper closed inverse submonoids of infinite chain type. If H is of infinite chain type xp^ω then $H \subset P_n^{x,p}$. Thus the maximal proper closed inverse submonoids of infinite chain type are those of aperiodic type.

If H is of cycle type $P_n^{x,p}$ and p is not a primitive string then $P_n^{x,p}$ is not maximal by Lemma 4.2. It follows that the maximal proper inverse submonoids of cycle type are precisely those of the form $P_n^{x,p}$ where p is a primitive string.

We now use Theorem 4.4 to classify the maximal proper inverse submonoids up to conjugacy. Those of cycle type can be represented by P_n^p where p is a primitive string. Two proper closed inverse submonoids of aperiodic type are conjugate iff their corresponding infinite strings differ in only a finite number of places.

■

Thus the conjugacy classes of the maximal proper closed inverse submonoids of pure cycle type are classified using Lyndon strings.

We now describe the actions associated with each of the conjugacy classes of proper closed inverse submonoids of P_n. We begin with the simplest case first.

Define an action of P_n on A_n^* as follows:

$$xy^{-1} \cdot u = \begin{cases} xp & \text{if } u = yp \text{ for some string } p \\ \text{undefined} & \text{otherwise} \end{cases}$$

We call this the *natural action of P_n on A_n^**. Observe that this action is transitive because $xy^{-1} \cdot y = x$ for any two strings x and y. The stabiliser of the point x consists of all the idempotents of the form $\bar{x}\bar{x}^{-1}$. The following is now immediate.

Proposition 4.6 *The action corresponding to a proper closed inverse submonoid of finite chain type is the natural action of P_n on A_n^*.*

■

Define an action of P_n on A_n^ω as follows:

$$xy^{-1} \cdot u = \begin{cases} xp & \text{if } u = yp \text{ for some infinite string } p \\ \text{undefined} & \text{otherwise} \end{cases}$$

We call this the *natural action of P_n on A_n^ω*. This action is no longer transitive but we shall study the orbits of the action each of which gives rise to a transitive action of P_n. Observe that if xp^ω is an ultimately periodic string we can assume that x and p have no suffix in common because if they did we could write $x = \bar{x}y$ and $p = \bar{p}y$, where y is as long as possible, and then $xp^\omega = \bar{x}(y\bar{p})^\omega$ with $\bar{x}$ and $y\bar{p}$ having no non-trivial suffix in common. Next we can assume that p is primitive because if $p = q^s$ then $xp^\omega = xq^\omega$.

The proof of the following is straightforward.

Proposition 4.7 *With respect to the natural action of P_n on A_n^ω we have the following.*

1. *The ultimately periodic string xp^ω, where p is primitive and x and p have non non-trivial suffix in common, has the stabiliser $P_n^{x,p}$.*

2. *The infinite aperiodic string x has the stabiliser the closed inverse submonoid of chain type x.*

■

It follows that the natural action of the polycyclic monoid on the set of infinite strings is the disjoint union of each of the primitive transitive representations of the polycyclic monoid with each such representation occuring exactly once.

5 Strong representations and branching function systems

We now single out a special class of representations of the polycyclic monoids. A representation of P_n on a set X is said to be *strong* if

$$X = a_1 \cdot X \cup \ldots \cup a_n \cdot X.$$

Such actions played an important role in our paper [8] and were introduced in [1]. Not all actions are strong: for example, the natural action of P_n on A_n^* is not strong because the empty string is not in the set $A_n^+ = (a_1 + \ldots + a_n)A^*$. However, the natural action of P_n on A_n^ω is strong because every point is an infinite string and so begins with some letter of A_n; that is

$$A_n^\omega = (a_1 + \ldots + a_n)A_n^\omega.$$

The disjoint union of strong actions is strong.

Proposition 5.1 *The only transitive action of P_n which is not strong is its natural action on A_n^*.*

Proof This can obviously be proved following our classification of closed inverse submonoids. However, we give an alternative proof that explains what is going on. Let P_n act transitively on the set X. Let $x \in X$ and let its stabiliser in P_n be S_x. All we shall assume about S_x is that it has infinitely many idempotents linearly ordered. Let $y \in X$ be arbitrary. Then there is a string uv^{-1} such that $uv^{-1} \cdot x = y$. If $u \neq \varepsilon$ then there is nothing to prove so we shall assume that $u = \varepsilon$. It follows that the elements of P_n mapping x to y are precisely those in $[v^{-1}H]$. We now use the fact that the idempotents in H are linearly ordered and that there are infinitely many of them. Choose $ww^{-1} \in H$ such that $|w| > |v|$. Then $v^{-1} \cdot ww^{-1}$ is defined and so $w = vp$ for some non-empty string p. It follows that $v^{-1} \cdot ww^{-1} = pw^{-1}$. Hence $pw^{-1} \cdot x = y$ and again y belongs to $(a_1 + \ldots + a_n)X$. Thus the action of P_n on X is strong.

■

Proposition 5.2 *Let P_n act on X. If the action is not strong then either X is just a disjoint union of copies of the natural action of P_n on A_n^* or it can be written $X = X^f \cup X^\infty$, a disjoint union of subspaces, where the action of P_n on X^∞ is strong and the action of P_n on X^f is again a disjoint union of copies of the natural action of P_n on A_n^*.*

Proof Define

$$X^\infty = \bigcap_{i=1}^{\infty} A_n^i X.$$

This is just the set of all points x of X such that for each integer $m \geq 0$ there exists a string u of length m such that $u^{-1} \cdot x$ is defined. In other words, it consists of all points whose stabilisers are infinite. If X^∞ is non-empty, then it is a subpace of X and by construction P_n acts strongly on it.

Put $X^f = X \setminus X^\infty$. If non-empty, this consists of those points x in X such that for some integer $m \geq 0$ there are no strings u of length m such that $u^{-1} \cdot x$ is defined. In other words, it consists of all points whose stabilisers are finite. The action of P_n on each orbit of X^f is therefore equivalent to its natural action on A_n^*.

■

The above two results deal with actions which are not strong. We now turn to those which are. The following lemma just spells out the technique that lay behind the proof of the above proposition.

Lemma 5.3 *Let P_n act strongly on the set X. Let $x \in X$. Then for each natural number m there is a unique string u of length m and a point y such that $x = u \cdot y$ (which is equivalent to saying that $\exists u^{-1} \cdot x$). Suppose that $u^{-1} \cdot x$ and $v^{-1} \cdot x$ are defined and $|u| \geq |v|$ then v is a prefix of u.*

Proof Because the action is strong, there is a $b_1 \in A_n$ and a point x_1 such that $x = b_1 \cdot x_1$. The same argument can be applied to x_1. Thus induction supplies the existence of a string u and a point y. Suppose that $x = u \cdot y = v \cdot z$ and u and v have the same length. Let the first letter of u be a. Then $a^{-1} \cdot x$ is defined. Thus $a^{-1}v$ is defined and is non-zero. It follows that the first letter of v is also a. By induction it follows that $u = v$, and so $y = z$.

We have that $x = u \cdot y$ iff $\exists u^{-1} \cdot x = y$ which proves the alternative characterisation stated in the brackets.

Suppose that $\exists u^{-1} \cdot x$ and $\exists v^{-1} \cdot x$ and that $|u| > |v|$. Both $uu^{-1} \cdot x$ and $vv^{-1} \cdot x$ are defined and so u and v are comparable, and from our assumption on their respective lengths we have that v is a prefix of u.

■

Morphisms between strong representations behave well.

Lemma 5.4 *Let P_n act strongly on both X and Y. Then every morphism $\alpha\colon X \to Y$ is strong.*

Proof We have to prove that $\exists uv^{-1} \cdot \alpha(x)$ implies that $\exists uv^{-1} \cdot x$. Because $w \cdot x$ is defined for all positive strings w it is enough to prove the result when $u = \varepsilon$. Suppose that $v^{-1} \cdot \alpha(x)$. By Lemma 5.3 there is a string u of length $|v|$ and a point y such that $x = u \cdot y$. Thus $\alpha(x) = u \cdot y$. It follows that $v^{-1}u$ is non-zero and so, since u and v have the same length, they must be equal. Thus $x = v \cdot y$ and so, in particular, $\exists v^{-1} \cdot x$, as claimed.

■

The following is just our version of the coding map of [1].

Proposition 5.5 *Let P_n act strongly on X. Then there is a strong morphism*

$$\sigma\colon X \to A_n^{\omega}$$

such that every finite prefix of $\sigma(x)$ of length m is the unique string u of length m such that $u^{-1} \cdot x$ is defined.

Proof The map σ is well-defined by Lema 5.3. It remains to show that σ is a morphism. Suppose that $wz^{-1} \cdot x$ is defined. Then z is a prefix of $\sigma(x)$ and so $\sigma(x) = zz'$ where z' is infinite. From the definition of the natural action on infinite strings we have that $wz^{-1} \cdot \sigma(x)$ is defined and is equal to wz'. But $\sigma(wz^{-1} \cdot x)$ is also equal to wz'. Thus $\sigma(wz^{-1} \cdot x) = wz^{-1} \cdot \sigma(x)$.

■

We call the map σ the *coding morphism*. If the coding morphism is injective then Bratteli and Jorgensen [1] define the action to be *multiplicity-free*. It follows by Lemma 1.1 and Proposition 5.5 that a multiplicity-free strong action is equivalent to a subspace of the natural action of P_n on infinite strings. Thus by Proposition 4.7, a strong action is multiplicity-free if and only if it is a disjoint union of primitive strong actions each of which occurs at most once, thus providing a completely algebraic characterisation of this notion.

Strong representations of P_n determine and are determined by 'n-ary branching function systems' which we now define. A *branching function system* is a set X equipped with n injective functions $f_i \colon X \to X$ for $i = 1, \ldots, n$ such that the images of the functions form a partition of the set X [1]. Thus branching function systems are special kinds of unary algebras (in the sense of universal algebra). The equivalence between strong representations and branching function systems is easy to establish. Given a strong action of P_n on X its restriction to A_n^* and so to A_n gives rise to a branching function system. Conversely, given a branching function system on X, we have a function from A_n to $I(X)$. This can be extended to a monoid homomorphism of A_n^* to $I(X)$ using the fact that A_n^* is the free monoid on A_n. This homomorphism can be extended to a homomorphism of P_n to $I(X)$ using the relations implicit in the definition of a branching function system.

There is an equivalent way of expressing the data making up a branching function system. Let $(X, f_1, \ldots, f_n)$ be a system. Define

$$\bigsqcup_{i=1}^{n} X = \bigcup_{i=1}^{n} \{i\} \times X$$

to be the disjoint union of n copies of X and define a function $\alpha \colon X \to \bigsqcup_{i=1}^{n} X$ by $\alpha(x) = (i, y)$ where $f_i(y) = x$. It is easy to check that α is a bijection. Conversely, every bijection α from X to $\bigsqcup_{i=1}^{n} X$ defines a branching function system by putting $f_i(y) = \alpha^{-1}(i, y)$. See Section 9.3 of [6] for more on this approach to branching function systems.

Thus to study strong representations of the polycyclic monoid on n generators, it is enough to study n-ary branching function systems. It is now easy to see why Kawamura's results [4, 5] can be derived from ours: in fact, his papers inspired us. His approach via branching function systems is more straightforward than ours, but the results of this paper put his results in a broader context, and the methods could be applied elsewhere.

Acknowledgements I am grateful to Des FitzGerald and K. Kawamura for comments on a draft of this paper, and Stuart Margolis, R. Morgenstern and G. I. Zhitomirskiy for discussions on what constitutes a primitive partial permutation representation of an inverse semigroup.

References

[1] O. Bratteli, P. E. T. Jorgensen, *Iterated function systems and permutation representations of the Cuntz algebra*, Memoirs of the A.M.S. No. 663, 1999.

[2] G. Grätzer, *Universal algebra*, D. Van Nostrand Company, inc., 1968.

[3] J. M. Howie, *Fundamentals of semigroup theory*, Clarendon Press, Oxford, 1995.

[4] K. Kawamura, Polynomial endomorphisms of the Cuntz algebras arising from permutations I. General theory, *Lett. Math. Phys.* **71** (2005), 149–158.

[5] K. Kawamura, Polynomial endomorphisms of the Cuntz algebras arising from permutations II. Branching laws of endomorphisms, Preprint RIMS-1433 (2003).

[6] M. V. Lawson, *Inverse semigroups*, World-Scientific, 1998.

[7] M. V. Lawson, Orthogonal completions of the polycyclic monoids, *Comms in Alg.* **35** (2007), 1651–1660.

[8] M. V. Lawson, The polycyclic monoids P_n and the Thompson groups $V_{n,1}$, *Comms in Alg.* **35** (2007), 4068–4087.

[9] M. V. Lawson, S. Margolis, R. Morgenstern, G. I. Zhitomirskiy, Primitive partial permutation representations of inverse semigroups, in preparation.

[10] M. Lothaire, *Combinatorics on words*, CUP, 1997.

[11] J. Meakin, M. Sapir, Congruences on free monoids and submonoids of polycyclic monoids, *J. Austral. Math. Soc.(Series A)* **54** (1993), 236–253.

[12] M. Nivat, J.-F. Perrot, Une généralisation du monoïde bicyclique, *Comptes Rendus de l'Académie des Sciences de Paris* **271** (1970), 824–827.

[13] A. L. T. Paterson, *Groupoids, inverse semigroups, and their operator algebras*, Birkhäuser, 1999.

[14] M. Petrich, *Inverse semigroups*, John Wiley & Sons, 1984.

[15] J. Renault, *A groupoid approach to C^*-algebras*, Springer-Verlag, 1980.

[16] R. L. Ruyle, *Pseudovarieties of inverse monoids*, PhD Thesis, University of Nebraska-Lincoln, 1997.

[17] B. M. Schein, Representations of generalized groups, *Izv. Vyssh. Uchebn. Zaved. Matematika* **3** (1962), 164–176 (in Russian).