Complex numbers

Think of real numbers as operators.
Thus 2 means 'double'. etc.

-1 means rotate by 180° anticlockwise

-1×-1 means rotate by 180° anticlockwise twice and so by 360° which is the same as multiplying by 1. Thus $-1 \times -1 = 1$ (geometrically).

Now introduce a new number i with the property that it rotates anticlockwise by 90°. Thus $i i = -1$. or $\boxed{i^2 = -1.}$

$\mathbb{C} = \{a+ib : a, b \in \mathbb{R}\}$

called <u>complex numbers</u>.

$\mathbb{R} \subseteq \mathbb{C}$ since if $a \in \mathbb{R}$ then $a = a + 0i \in \mathbb{C}$.

$a + bi = c + di$ iff $a = c$, $b = d$.

Arithmetic

$$(a+bi) + (c+di) = (a+c) + (b+d)i$$

$$(a+bi) - (c+di) = (a-c) + (b-d)i$$

$$(a+bi)(c+di) = ac + adi + bci + bd\underline{i^2}$$
$$= (ac - bd) + (ad + bc)i$$

Division

$a + bi \neq 0$. What is

$$\frac{1}{a+bi} = ?$$

Need new idea

Let $z = a + bi$.

Then $\bar{z} = a - bi$ called the complex conjugate

$$\bar{z}z = a^2 + b^2 \in \mathbb{R}.$$

Use this to compute

$$\frac{1}{a+bi} = \frac{a-bi}{a^2+b^2} = \frac{a}{a^2+b^2} - \frac{b}{a^2+b^2}\, i$$

multiply numerator and denominator by $a-bi$

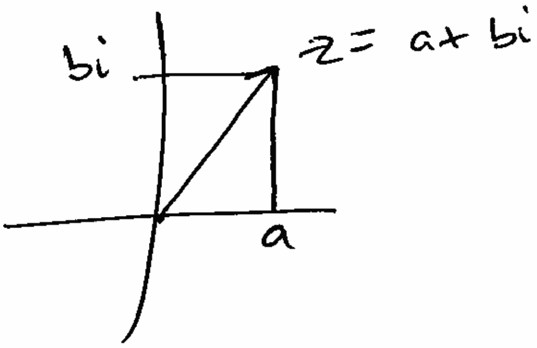
$i^2=-1$ means i is a square root of -1.

Also $-i$ is a square root of -1.

Thus i has two square roots $\pm i$.

Theorem Every complex number ($\neq 0$) has exactly two square roots.

To prove this, useful to employ the following.



$|a+bi| = \sqrt{a^2+b^2}$

the length or **modulus** of z.

(plural of modulus = moduli)

<u>Lemma</u> $u, v \in \mathbb{C}$. $|u||v| = |uv|$.

<u>Proof</u> You already know!

<u>Proof of theorem</u> Let $a+bi \neq 0$.

Need $(x+yi)^2 = a+bi$. from $i^2=-1$

$LHS = x^2 + 2xyi - y^2$

$RHS = a+bi$.

$\therefore \quad a = x^2 - y^2 \quad (1)$

$b = 2xy. \quad (2)$

We can find another equation to make our lives simpler. use our lemma, above

$|(x+yi)^2| = |a+bi|$ Thus

$x^2 + y^2 = \sqrt{a^2+b^2} \quad (3)$

① + ③ gives $2p^2 = a + \sqrt{a^2+b^2}$

We can easily solve this to get two values of p (since $\sqrt{\ }$ means $\pm\sqrt{\ }$).

For each choice of x, get corresponding value of y using equation ②.

This proves the theorem $\blacksquare$

$\boxed{\text{Observe: The roots of } z \text{ are of the form } u, -u.}$