Finishing off Lecture 4

Euler's formula

$$e^x = a_0 + a_1 x + a_2 x^2 + a_3 x^3 \dots$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$$

$$\cos\theta = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots$$

$$e^{i\theta} = 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \dots$$

$$= 1 + i\theta - \frac{\theta^2}{2!} - i\frac{\theta^3}{3!} + \dots$$

Seperate out real and complex parts

$$\boxed{e^{i\theta} = \cos\theta + i\sin\theta}$$

Euler's formula.

Put $\theta = \pi$ to get

Euler's identity $\boxed{e^{i\pi} = -1}$

Every non-zero complex number z can be written in the form $z = re^{i\theta}$ where $r > 0$. Observe that

$$e^{i\theta + 2\pi k} = e^{i\theta} e^{2\pi k} = e^{i\theta}.$$

and that if $u = re^{i\theta}$, $v = se^{i\phi}$ then $uv = rse^{i\theta}e^{i\phi} = rse^{i(\theta+\phi)}$

Lecture 15

Polynomials

$n=1$, linear	$n=4$, quartic
$n=2$, quadratic	$n=5$, quintic
$n=3$, cubic	etc.

degree

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$$

leading term

$a_0, \dots, a_n$ called coefficients

constant term

$\mathbb{R}[x]$ real polys
$\mathbb{C}[x]$ complex polys

$\deg f(x) = n$

If $f(x) = a_0 \ (\neq 0)$ $\deg f(x) = 0$

$\deg f(x)$ not defined if $f(x)$ identically 0.

Lemma If $f(x), g(x)$ non-zero then

$$\deg f(x) g(x) = \deg f(x) + \deg g(x)$$

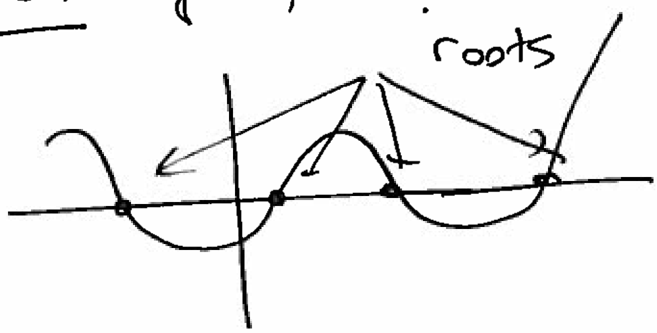
Remainder theorem $f(x) = g(x) q(x) + r(x)$

$\neq 0$

quotient remainder

$\deg r(x) < \deg g(x)$

If $f(a) = 0$ then a is called a <u>root</u> of $f(x)$.



Write $f(x) \mid g(x)$, say $f(x)$ <u>divides</u> $g(x)$ (etc) if $g(x) = f(x)\, q(x)$ for some $q(x)$.

<u>Lemma</u> a is a root of $f(x)$ iff $(x-a) \mid f(x)$ [roots $\Leftrightarrow$ linear factors]

<u>Proof</u> S'pose $(x-a) \mid f(x)$ then $f(x) = q(x)(x-a)$. Thus $f(a) = 0$.

S'pose a is a root and ~~[illegible]~~ $(x-a) \nmid f(x)$

Then $f(x) = q(x)(x-a) + r(x)$

$\deg r(x) < 1$ $\therefore$ $r(x) = b$ a constant

$f(a) = b \neq 0$ #.

First important result

Proposition Let $f(x)$ be a non-constant poly of degree n. Then $f(x)$ has at most n roots.

Proof Let a be a root of $f(x)$

then $f(x) = (x-a) f_1(x)$

where $\deg f_1(x) = n-1$. Repeating this we see that $f(x)$ has at most n roots. ∎

Fundamental theorem of algebra (Gauss)

Every non-constant poly with complex coefficients has at least one root.

Combining these results we get.

<u>Theorem</u> Every non-constant poly of degree n with complex coefficients has exactly n roots.

Let $f(x) \in \mathbb{C}[x]$, non-constant, of degree n. Let the roots be $r_1, \ldots, r_n$. Then

$$f(x) = a(x - r_1) \ldots (x - r_n)$$

where $a \in \mathbb{C}$.

We now refine this result when $f(x) \in \mathbb{R}[x]$. (Important in calculus, such as partial fractions).

More properties of the complex conjugate.

- $\overline{u_1 + \dots + u_n} = \overline{u_1} + \dots + \overline{u_n}$
- $\overline{u_1 \dots u_n} = \overline{u_1} \dots \overline{u_n}$
- $u = \overline{u}$ iff $u \in \mathbb{R}$.

<u>Lemma</u> Let $f(x) \in \mathbb{R}[x]$ where

$$f(x) = a_n x^n + \dots + a_1 x + a_0$$

If $f(z) = 0$ then $f(\overline{z}) = 0$.

<u>Proof</u>

$$0 = a_n z^n + \dots a_1 z + a_0$$

$$0 = \overline{a_n z^n + \dots + a_1 z + a_0}$$

$$0 = \overline{a_n}\, \overline{z}^n + \dots + \overline{a_1} \overline{z} + \overline{a_0}$$

$$0 = a_n \overline{z}^n + \dots + a_1 \overline{z} + a_0$$

since a_i are real. ∎

The complex roots of real polys come in complex conjugate pairs.

<u>Lemma</u> $(x-z)(x-\bar{z})$ is a real poly

<u>Proof</u> $(x-z)(x-\bar{z}) = x^2 - x\bar{z} - xz + z\bar{z}$

$$= x^2 - (\bar{z}+z)x + z\bar{z}$$ real poly.

If $z = a+ib$ then

- $z\bar{z} = a^2+b^2$

- $z+\bar{z} = 2a$ $\blacksquare$

~~Real polys with~~

Real quadratics with complex roots are precisely those with $D<0$.

Called <u>irreducible real quadratics</u>

Fundamental theorem of algebra for real polys.

Let $f(x) \in \mathbb{R}[x]$ of degree $n\ (\geqslant 1)$.

Roots of two sorts - real and complex.

- Real roots give real linear factors.
- Complex roots come in complex conjugate pairs and give real irreducible quadratic factors. Thus

$f(x)$ = product of real linear factors and real irreducible quadratic factors.

But how to find roots? Not easy.

Here is an important result.

<u>Theorem</u> (Rational root theorem)

Let $f(x) \in \mathbb{Z}[x]$

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

Let $\frac{p}{q}$ be a rational root of $f(x)$ in its lowest terms. Then

$$q \mid a_n \quad \text{and} \quad p \mid a_0$$

If $a_n = 1$ any rational root is an integer.

<u>Proof</u> Special case

$f(x) = ax^2 + bx + c$

$a, b, c \in \mathbb{Z}$. Let $\frac{p}{q}$ be a rational root where $\gcd(p, q) = 1$.

$$a\frac{p^2}{q^2} + b\frac{p}{q} + c = 0$$

$$ap^2 + bpq + cq^2 = 0$$

- $cq^2 = -ap^2 - bpq \therefore p \mid cq^2$
- $ap^2 = -bpq + cq^2 \therefore q \mid ap^2$

Thus $p \mid c$ and $q \mid a$. $\blacksquare$

We now see how these results may be used in practice.

<u>Example</u> Find all roots of

$$x^4 - 8x^3 + 2\cancel{7}x^2 - 28x + 12$$

$$[\ (x-1)(x-2)^2(x-3)\ ]$$

<u>Example</u> Find all roots of

$$x^4 - 5x^2 - 10x - 6$$

$$= (x+1)(x-3)(x^2+2x+2)$$

$$= (x+1)(x-3)(x+1+i)(x+1-i)$$