$\mathbb{R}[x]$ polys with real coefficients
$\mathbb{C}[x]$ ——— complex ———

Lecture 16

Let $f(x)$ be a poly of degree n with complex coefficients. Then $f(x)$ has n roots $a_1, \ldots, a_n$ and δ

$$f(x) = a(x-a_1)\ldots\ldots(x-a_n)$$

Example

$$x^4 - 8x^3 + \cancel{\quad}\,23x^2 - 28x + 12 = f(x)$$

has roots 1, 2 (twice) and 3.

Thus $f(x) = (x-1)(x-2)^2(x-3)$

Example

$$x^4 - 5x^2 - 10x - 6 = (x+1)(x-3)(x+1+i)(x+1-i)$$

has roots $-1, 3, -(1+i), -(1-i)$

A poly is called real if it has real coefficients.

Problem How to use the above result when poly is real to avoid complex factors?

More properties of the complex conjugate.

- $\overline{u_1 + \dots + u_n} = \overline{u_1} + \dots + \overline{u_n}$
- $\overline{u_1 \dots u_n} = \overline{u_1} \dots \overline{u_n}$
- $u = \overline{u}$ iff $u \in \mathbb{R}$.

<u>Lemma</u> Let $f(x) \in \mathbb{R}[x]$ where

$$f(x) = a_n x^n + \dots + a_1 x + a_0$$

If $f(z) = 0$ then $f(\overline{z}) = 0$.

<u>Proof</u>

$$0 = a_n z^n + \dots a_1 z + a_0$$

$$0 = \overline{a_n z^n + \dots + a_1 z + a_0}$$

$$0 = \overline{a_n}\, \overline{z}^n + \dots + \overline{a_1}\, \overline{z} + \overline{a_0}$$

$$0 = a_n \overline{z}^n + \dots + a_1 \overline{z} + a_0$$

since a_i are real. ∎

The complex roots of real polys come in complex conjugate pairs.

<u>Lemma</u> $(x-z)(x-\bar{z})$ is a real poly

<u>Proof</u> $(x-z)(x-\bar{z}) = x^2 - x\bar{z} - xz + z\bar{z}$

$$= x^2 - (\underline{\bar{z}+z})x + \underline{z\bar{z}} \quad \text{real poly.}$$

If $z = a+ib$ then

- $z\bar{z} = a^2+b^2$
- $z+\bar{z} = 2a$ $\blacksquare$

Real quadratics with complex roots are precisely those with $D<0$.

Called <u>irreducible real quadratics</u>

Fundamental theorem of algebra for real polys

Let $f(x) \in \mathbb{R}[x]$ be a real poly of degree $n (\geq 1)$. Roots are of two sorts – real and (really) complex.

- Real roots give real linear factors.
- Complex roots come in complex conjugate pairs and give real irreducible quadratic factors. Thus

$f(x)$ = product of real linear factors and real irreducible quadratic factors.

Example $x^4 - 5x^2 - 10x - 6$

$$= (x+1)(x-3)(x+1+i)(x+1-i)$$

complex conjugates

$$= (x+1)(x-3)(x^2+2x+2)$$

now all factors are real

Problem How to write a real poly as a product of real linear and real irreducible quadratic polys?

Example Let

$$f(x) = x^4 - 4x^3 + 7x^2 - 6x + 2.$$

Write $f(x)$ as a product of real linear and real irreducible quadratic factors.

To start we use the rational root theorem. The above poly is monic. This states that the only possible rational roots are integers and divide the constant term 2. Thus potential rational roots $= \pm 1, \pm 2$

We calculate

$f(1)$, $f(-1)$, $f(2)$, $f(-2)$

We find only $f(1) = 0$.

Thus $x-1$ is a factor.

$$f(x) = (x-1)\underbrace{(x^3-3x^2+4x-2)}$$

Repeat above procedure to get

that 1 is a root.

$$f(x) = (x-1)(x-1)(x^2-2x+2)$$

$$x^2 - 2x + 2 = (x-1)^2 + 1 = 0$$

$$\therefore \quad (x-1)^2 = -1$$

$$x - 1 = \pm i$$

$$\therefore \quad x = 1+i, \quad 1-i$$

Complex Conjugates.

$\therefore$ Roots = 1 (twice), $1+i$, $1-i$

Factorization $(x-1)(x-1)(x^2 - 2x + 2)$

Irred. quadratic.

Factorizing a real polynomial

- If a is a root then $(x-a)$ is a factor

* If monic + coefficients are integers then only rational roots must divide constant term. *

- If $a+bi$ is a root then $a-bi$ is a root.

Important in partial fractions and integrating rational functions (A rational function looks like $\frac{p(x)}{q(x)}$ where $p(x), q(x)$ are real polys)