Lecture 17

We shall show how to solve the equation $x^n - z = 0$ (where $z \in \mathbb{C}$ — Rem $\mathbb{R} \subseteq \mathbb{C}$) explicitly. We have dealt with the case where $n = 2$. So we shall deal here with the case where $n \geq 3$.

Special case $x^n - 1 = 0$

The n roots of this equation are called the <u>n roots of unity</u>. I shall first, by way of an example, show how to solve $x^3 - 1 = 0$

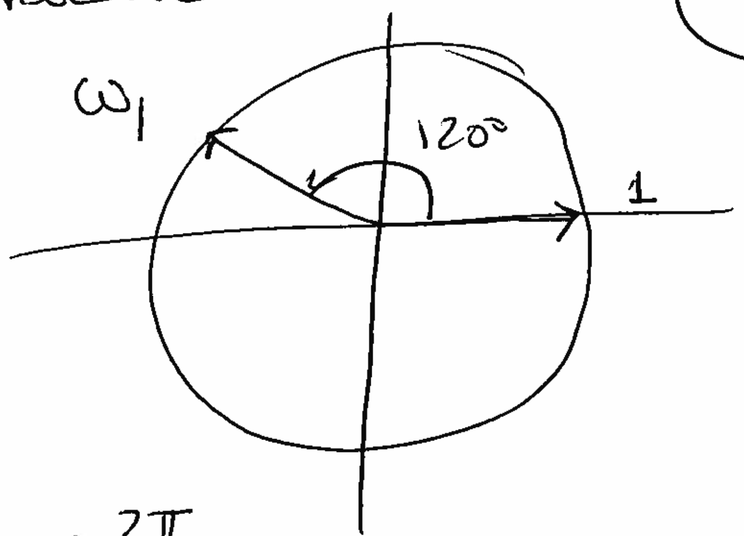
$$\boxed{e^{i\theta} = \cos\theta + i\sin\theta}$$

$\boxed{\omega = \text{omega}, \quad w = \text{double u}}$

Roots of $x^3 - 1 = 0$ — Clearly one root is 1

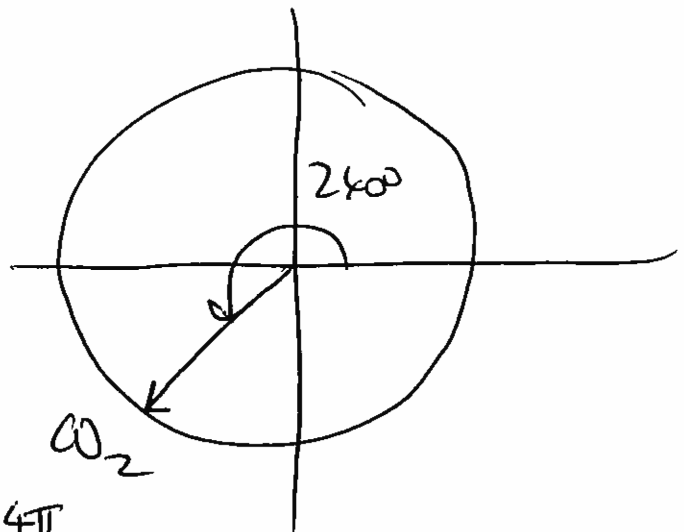
All roots have modulus $= 1$

But there are two others.



$\omega_1 = e^{i\frac{2\pi}{3}}$ is another root because

$$\omega_1^3 = \left(e^{i\frac{2\pi}{3}}\right)^3 = e^{i2\pi} = 1.$$



$\omega_2 = e^{i\frac{4\pi}{3}}$ is another root because

$$\omega_2^3 = \left(e^{i\frac{4\pi}{3}}\right)^3 = e^{4\pi i} = 1.$$

Thus $1, \omega_1, \omega_2$ are the three roots of unity. Observe that $\omega_2 = \omega_1^2$.

Define $\omega = \omega_1$. The roots are

$$1, \omega, \omega^2 \quad (\omega^3 = 1).$$

Theorem The n solutions to $x^n - 1 = 0$ are $1, \omega, \omega^2, \ldots, \omega^{n-1}$ where $\omega = e^{\frac{2\pi}{n} i}$. These form the vertices of a regular n-gon ($=$ polygon with n sides).

General case $x^n - z = 0$

Let $z = re^{i\theta}$ (if $\theta = 0$ use $\theta = 2\pi$)

One n^{th} root is $\boxed{\sqrt[n]{r}\, e^{\frac{\theta}{n} i} = u}$

Where are the other $n-1$ roots?

Let $1, \omega, \ldots, \omega^{n-1}$ be the nth roots of unity. Then the nth roots of z are

$$u, \quad u\omega, \quad \ldots, \quad u\omega^{n-1}$$

$$\sqrt[n]{r}\, e^{i\frac{\theta}{n}}, \quad \sqrt[n]{r}\, e^{i\frac{\theta}{n}} \cdot e^{i\frac{2\pi}{n}} = \sqrt[n]{r}\, e^{i\left(\frac{\theta}{n}+\frac{2\pi}{n}\right)},$$

etc.

We call this the <u>trigonometric form</u> of the roots.

It is sometimes possible to obtain a different, more algebraic, form.

Example

Cube roots of unity	
Trig form	Radical expression
$\cos 0 + i \sin 0$	1
$\cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3}$	$\frac{1}{2}(-1 + i\sqrt{3})$
$\cos \frac{4\pi}{3} + i \sin \frac{4\pi}{3}$	$-\frac{1}{2}(1 + i\sqrt{3})$

Calculated using

θ	$\sin\theta$	$\cos\theta$
$60°$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$

Radical expression = algebraic expression involving roots.

- Trig forms of roots always easy to find.
- Radical forms of roots can always be found (Gauss) but rarely easily.

There are formulae for finding the roots of quadratics, cubics and quartics. Thus their roots can always be described by radical expressions.

<u>Theorem</u> (Evariste Galois (1811 - 1832))

There are no formulae for finding the roots of general quintics or any equation of higher degree. In general, the roots of such equations cannot be expressed by means of radical expressions.

<u>Question after lecture</u> This does not contradict the theorem that a poly equation of degree n has n solutions. These solutions always exist. Instead, it tells us that we cannot expect to find radical expressions for these roots. So, we have to use different methods to estimate the roots.