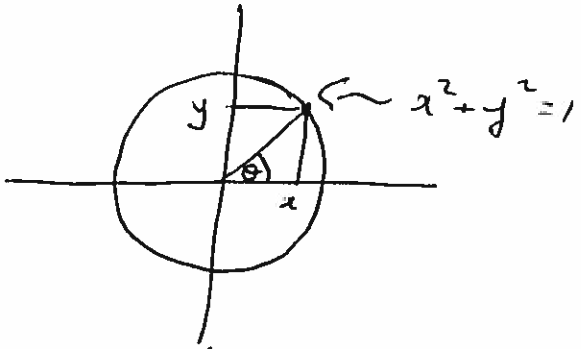
Lecture 2

The theme of the relationship between algebra and geometry continued.

$$x^2 + y^2 = 1 \quad (*)$$



This equation can be solved to get

$$x = \cos\theta, \quad y = \sin\theta$$

where θ is called the parameter. In this way, we get the parametric equation of the circle $\theta \longmapsto (\cos\theta, \sin\theta)$ where $0 \le \theta < 2\pi$.

What about finding rational points on the circle? In Q9, Ex 2.3, we derive the following: the rational points on the unit circle are $(-1, 0)$ and all points of the form

$$\left(\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2} \right)$$

where t is rational.

Check

$$\frac{(1-t^2)^2 + (2t)^2}{(1+t^2)^2} = \frac{1-2t^2+t^4+4t^2}{(1+t^2)^2} = 1$$

You can use this result to find a formula for Pythagorean triples (See Q9, Ex 5.3 — but it needs new ideas discussed in Chapter 5).

Theme Different kinds of numbers. We need some notation to describe them.

A *set* is a collection of objects that we wish to regard as a whole. The objects in a set are called its *elements*; the elements of a set are demarcated by $\{\ \}$ and the elements are listed separated by commas.

Example $\{\alpha, 1, \square, *\}$ is a set having four elements. We denote sets by capital ~~elements~~ letters. So, we might define $A = \{\alpha, 1, \square, *\}$

If A is a set and a is an element of A we write $a \in A$; if a is not an element of A we write $a \notin A$.

Two sets are <u>equal</u> when they contain exactly the same elements.

In describing a set, the <u>order</u> of the elements is ignored. So $\{a,b\} = \{b,a\}$. In addition, repetitions are ignored so

$$\{a,a,b\} = \{a,b\}$$

I shall say more about sets later.

<u>Important sets of numbers</u>

also on

$\mathbb{N} = \{0,1,2,\ldots\}$ <u>natural numbers</u>

$\mathbb{Z} = \{\ldots,-2,-1,0,1,2,\ldots\}$ <u>integers</u>

$\mathbb{Q}$ all nos that can be written $\frac{p}{q}$ where $q \neq 0$ and $p,q \in \mathbb{Z}$. <u>Rational numbers</u>

$\mathbb{R}$ all numbers that can be written as decimals possibly infinite <u>real numbers</u>.

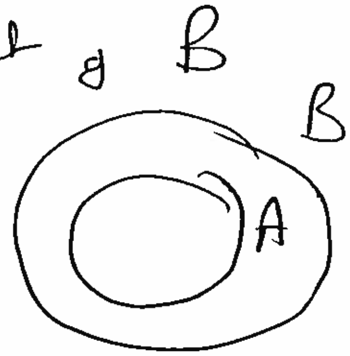
Later, we shall also study the complex numbers $\mathbb{C}$.

Let A and B be sets.

If every element of A is also an element of B

we say that A is a subset of B

and write $A \subseteq B$



$\mathbb{N} \subseteq \mathbb{Z} \subseteq \boxed{\mathbb{Q} \subseteq \mathbb{R}} \subseteq \mathbb{C}$

$\mathbb{Q}$. Every rational numbers goes under

many aliases $\quad \frac{1}{2} = \frac{2}{4} = \frac{-2}{-4} = \dots$

$\mathbb{R}$. $\quad \pi = 3.14159265358\dots$

Green

$\frac{1}{2} = 0.5$ or $0.5000\dots$

If $A \subseteq B$ and $A \neq B$ we say

A is a <u>proper subset of</u> B and

write $A \subset B$

In fact, $\mathbb{N} \subset \mathbb{Z} \subset \boxed{\mathbb{Q} \subseteq \mathbb{R}} \subset \mathbb{C}$

Is this a proper inclusion?

The answer is yes, (done).

We prove that $\mathbb{Q} \neq \mathbb{R}$ by proving

that $\sqrt{2} \notin \mathbb{Q}$.

Need some preliminaries.

A nat no $a \in \mathbb{Z}$ is said to be **even** if $a = 2m$ for some $m \in \mathbb{Z}$.

A nat no. $a \in \mathbb{Z}$ is said to be **odd** if $a = 2m+1$ for some $m \in \mathbb{Z}$.

A nat no. is odd or even but not both.

Lemma

(1) The square of an even no is even.

(2) The square of an odd no is odd.

Proof. (1) Let a be even. Then $a = 2m$ for some $m \in \mathbb{Z}$. Squaring a we get

$$a^2 = (2m)^2 = 2^2 m^2 = 4m^2 = 2(2m^2).$$

So a^2 is even.

(2) Let a be odd.

Then $a = 2m+1$ for some $m \in \mathbb{Z}$.

Squaring a we get

$$a^2 = 4m^2 + 4m + 1$$

$$= 2(2m^2 + 2m) + 1$$

Thus a^2 is odd. $\blacksquare$ ← end of proof.

We now make the following important deduction from the Lemma above.

<u>Lemma</u> Let $a \in \mathbb{Z}$. If a^2 is even then a is even.

<u>Proof.</u> a is either odd or even.

If a was odd then a^2 would be odd and we are told that a^2 is even. Thus a cannot be odd and so must be even. $\blacksquare$

Proof that $\sqrt{2} \notin \mathbb{Q}$

Suppose that $\sqrt{2} = \frac{a}{b}$ where $a, b \in \mathbb{N}$ and $b \neq 0$. We can assume that a and b have no common divisor (*)

Squaring both sides, we get

$$2 = \frac{a^2}{b^2}$$

and so $2b^2 = a^2$.

Thus a^2 is even and so a is even.

We may write $a = 2m$ for some $m \in \mathbb{N}$.

Thus $2b^2 = (2m)^2 = 4m^2$.

Divide both sides by 2 to get

$$b^2 = 2m^2.$$

It follows that b is even. Thus a and b are both even contradicting (*) ■