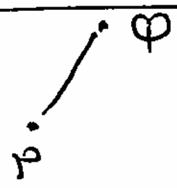
Vector algebra

Invented by Gibbs and Heaviside as an antidote to Hamilton's quaternions, this is an algebraic system for dealing with 3D geometrical problems or, more accurately, Euclidean geometry. My pictures will be 2D but the system is 3D.

Two points P and Q determine a line segment PQ



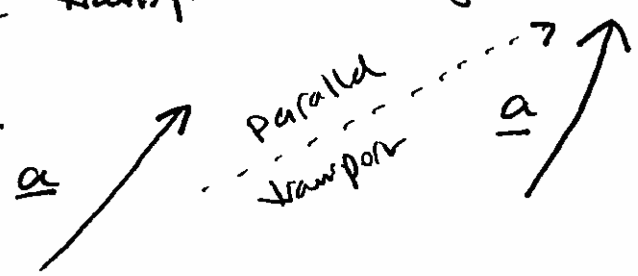
Each line segment can be oriented in one of two ways

Comme ci: $\overrightarrow{PQ}$

or comme ça $\overleftarrow{PQ}$

Two directed line segments are said to determine the same vector if they are parallel, point in the same direction, and have the same length. Vectors will be denoted by bold, l.c. Latin characters **a**, **b**, **c**, . . .

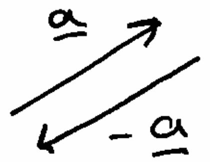
Intuitively vectors carry information about length and angles. View them as directed line segments that can be transported through space parallel to themselves.



The length of a vector $\underline{a}$ is denoted by $\|\underline{a}\|$.

A unit vector is a vector $\underline{a}$ s.t $\|\underline{a}\| = 1$.

Define $-\underline{a}$ to be the same as $\underline{a}$ but oriented in the reverse direction.



The zero vector is determined by any point in space P. Thus $\underline{0}$ is represented by $\overrightarrow{PP}$. Clearly, $\|\underline{0}\| = 0$ but it should be regarded as pointing in all directions.

multiplication by a scalar

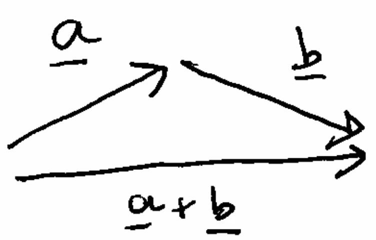
If $\lambda \in \mathbb{R}$, define the vector $\lambda \underline{a}$.

- $0\underline{a} = \underline{0}$; $1\underline{a} = \underline{a}$; $(-1)\underline{a} = -\underline{a}$,

If $\lambda > 0$ then $\lambda\underline{a}$ points in the same direction as $\underline{a}$

If $\lambda < 0$ then $\lambda\underline{a}$ points in the opposite direction to $\underline{a}$.

Vector addition



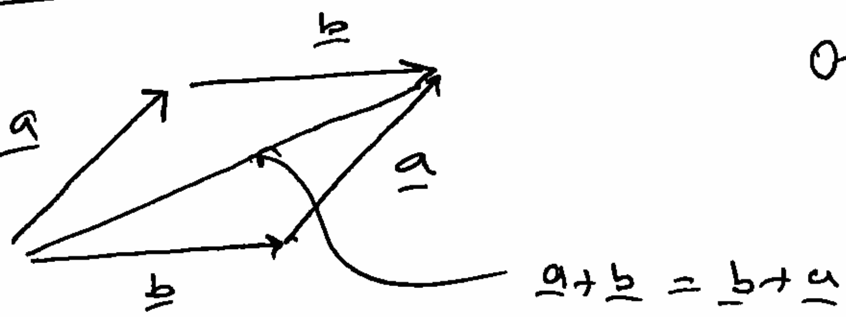
(1) $(\underline{a}+\underline{b})+\underline{c} = \underline{a}+(\underline{b}+\underline{c})$ associativity.

(2) $\underline{a}+\underline{b} = \underline{b}+\underline{a}$. Commutativity

(3) $\underline{a}+\underline{0} = \underline{a} = \underline{0}+\underline{a}$.

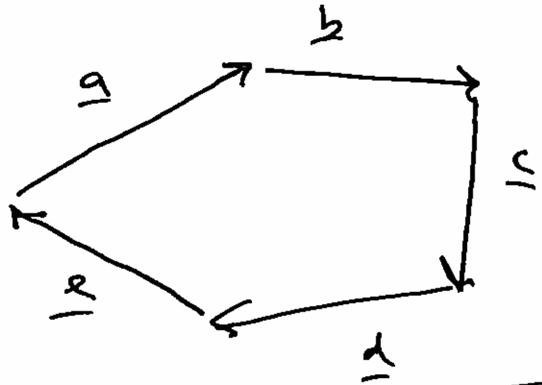
(4) $\underline{a}+(-\underline{a}) = \underline{0} = (-\underline{a})+\underline{a}$

Example $\underline{a}+\underline{b} = \underline{b}+\underline{a}$.



Other properties proved similarly.

Example



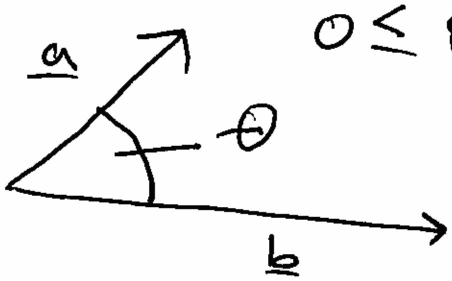
$\underline{a}+\underline{b}+\underline{c}+\underline{d}+\underline{e} = \underline{0}$

If $\underline{a} \neq \underline{0}$ then define $\hat{\underline{a}} = \frac{1}{\|\underline{a}\|}\underline{a}$

Call the <u>normalization of $\underline{a}$</u>. $\|\hat{\underline{a}}\| = 1$

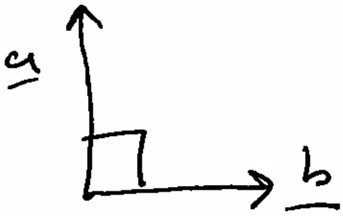
and points in the same direction as $\underline{a}$

<u>Angle between vectors</u>



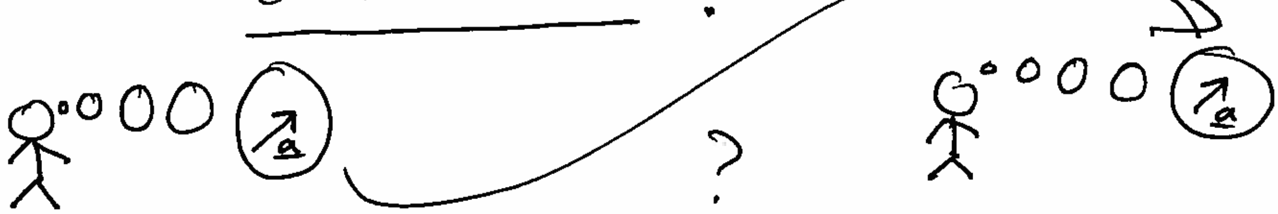
$0 \leq \theta \leq \pi$

If $\theta = \frac{\pi}{2}$,



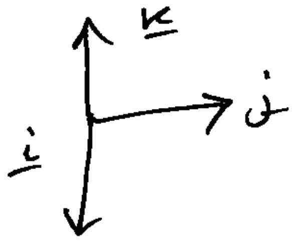
We say that $\underline{a}$ and $\underline{b}$ are <u>orthogonal</u>

Coordinates ?



You — How to communicate information about vectors — Me

Choose three, mutually orthogonal unit vectors



$$\|\underline{i}\| = 1$$
$$\|\underline{j}\| = 1$$
$$\|\underline{k}\| = 1$$

Every vector $\underline{a}$ can be written uniquely as

$$\underline{a} = a_1 \underline{i} + a_2 \underline{j} + a_3 \underline{k}$$

a_1, a_2, a_3 called ~~Componen~~ Coordinates / Components

[Also can be written $\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ but I shall not use this notation].

$$\underline{0} = 0\underline{i} + 0\underline{j} + 0\underline{k}$$

$$\lambda \underline{a} = \lambda a_1 \underline{i} + \lambda a_2 \underline{j} + \lambda a_3 \underline{k}$$

$$\underline{a} + \underline{b} = (a_1 + b_1)\underline{i} + (a_2 + b_2)\underline{j} + (a_3 + b_3)\underline{k}$$

$$\underline{a} = \underline{b} \iff \begin{cases} a_1 = b_1 \\ a_2 = b_2 \\ a_3 = b_3 \end{cases}$$ equating components.

$\boxed{\text{1 vector equation} = \text{3 scalar equations}}$

"Vectors as 'Packaging'"