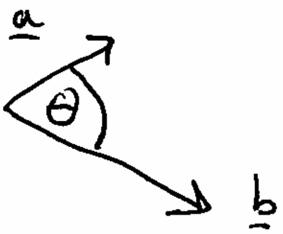
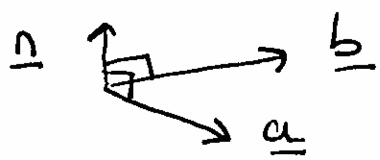
Vector operations

We now introduce three operations on the set of vectors. What these operations mean and how they are used will be explained afterwards.

Definitions (geometry)	Theorems (algebra)
Inner product $\underline{a} \cdot \underline{b} = \lVert \underline{a} \rVert \lVert \underline{b} \rVert \cos\theta$ 	$\underline{a} \cdot \underline{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$
Vector product $\underline{a} \times \underline{b} = \left(\lVert \underline{a} \rVert \lVert \underline{b} \rVert \sin\theta\right) \underline{n}$ $\lVert \underline{n} \rVert = 1$ and 	$\underline{a} \times \underline{b} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$ $= \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \underline{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \underline{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \underline{k}$
Scalar triple product $[\underline{a}, \underline{b}, \underline{c}] = \underline{a} \cdot (\underline{b} \times \underline{c})$	$[\underline{a}, \underline{b}, \underline{c}] = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

Properties of operations

Inner product

(1) $\underline{a} \cdot \underline{b} = \underline{b} \cdot \underline{a}$

(2) $\underline{a} \cdot \underline{a} = \|\underline{a}\|^2$

(3) $\lambda(\underline{a} \cdot \underline{b}) = (\lambda \underline{a}) \cdot \underline{b} = \underline{a} \cdot (\lambda \underline{b})$

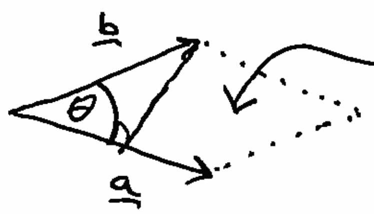
(4) $\underline{a} \cdot (\underline{b} + \underline{c}) = \underline{a} \cdot \underline{b} + \underline{a} \cdot \underline{c}$

Key property If $\underline{a}, \underline{b} \neq \underline{0}$ then $\underline{a} \cdot \underline{b} = 0$ iff $\underline{a}$ and $\underline{b}$ are orthogonal.

Vector product

1. $\underline{a} \times \underline{b} = -\underline{b} \times \underline{a}$ (not commutative).
2. $\lambda(\underline{a} \times \underline{b}) = (\lambda \underline{a}) \times \underline{b} = \underline{a} \times (\lambda \underline{b})$.
3. $\underline{a} \times (\underline{b} + \underline{c}) = \underline{a} \times \underline{b} + \underline{a} \times \underline{c}$.
4. Not associative.

Key property $\underline{a} \times \underline{b}$ is orthogonal to $\underline{a}$ and $\underline{b}$

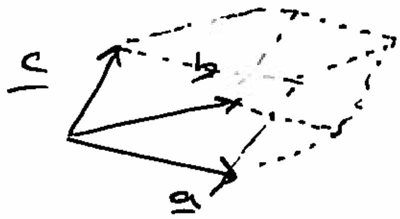


area = $\|\underline{a}\| \|\underline{b}\| \sin\theta$

Scalar triple product

$|[\underline{a}, \underline{b}, \underline{c}]|$ = volume of parallelopiped determined by $\underline{a}, \underline{b}, \underline{c}$.

absolute value



These operations encode important geometric information.

Lengths $\|\underline{a}\| = \sqrt{\underline{a} \cdot \underline{a}}$

$\underline{a} \cdot \underline{a}$ usually written $\underline{a}^2$

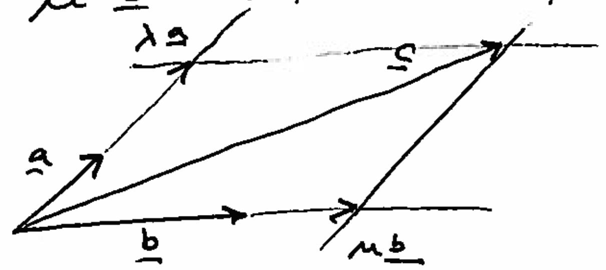
Angles $\cos\theta = \dfrac{\underline{a} \cdot \underline{b}}{\|\underline{a}\| \|\underline{b}\|}$

Areas $|\underline{a} \times \underline{b}|$ is the area of the parallelogram determined by $\underline{a}$ and $\underline{b}$

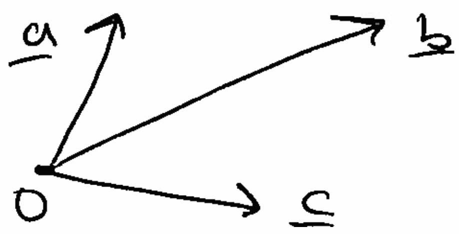
Volumes $|[\underline{a}, \underline{b}, \underline{c}]|$ is the volume of the parallelopiped determined by $\underline{a}, \underline{b}$, and $\underline{c}$

- Two vectors $\underline{a}$ and $\underline{b}$ ($\underline{a} \neq \underline{0}$) are said to be <u>parallel</u> if the directed line segments representing $\underline{a}$ and $\underline{b}$ are parallel. If $\underline{a}$ and $\underline{b}$ are parallel then $\underline{b} = \lambda \underline{a}$ for some λ

- Vectors $\underline{a}$, $\underline{b}$, $\underline{c}$ are <u>coplanar</u> if there is a plane that contains all of them. If $\underline{a}, \underline{b} \neq \underline{0}$ then $\underline{c} = \lambda \underline{a} + \mu \underline{b}$ for some λ, μ.



<u>Position (or bound) vectors</u> The vectors we have used so far are also called <u>free vectors</u> because they are free to move (parallel to themselves). Such vectors cannot be used to determine absolute position. For that, we need another (special) class of vectors. Let O be a <u>fixed point</u> in space. A vector $\underline{a}$ which is restricted to start at O is called a <u>position vector</u>



Observe that position vectors always assume that a fixed point in space has been chosen.

An arbitrary position vector is denoted by $\underline{r}$ ('r' for 'radius vector') in coordinate form

$$\underline{r} = x\underline{i} + y\underline{j} + z\underline{k}$$

We have now defined almost all the ingredients of vector algebra. It remains to show how this system can be used to do geometry.

We shall only have time for a few examples
