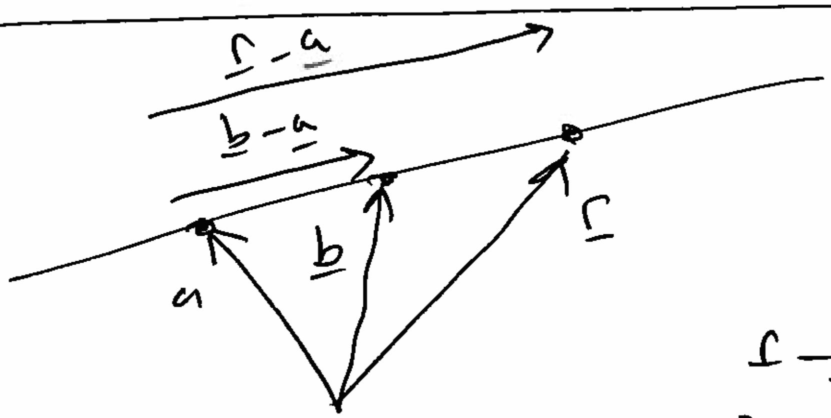
Equations of lines and planes

A <u>line</u> is det'd by

(1) Two distinct points and so by two distinct position vectors.

or

(2) A point and a direction and so by a position vector and a free vector.



$\underline{r}-\underline{a}$ is parallel to $\underline{b}-\underline{a}$ $(\neq \underline{0})$ so

$\underline{r}-\underline{a} = \lambda(\underline{b}-\underline{a})$

for $\lambda \in \mathbb{R}$

$$\boxed{\underline{r} = \underline{a} + \lambda(\underline{b}-a)}$$

λ ← parameter

is the <u>parametric</u> vector equation of the line.

Observe

$\underline{c} = \underline{b}-\underline{a}$

is a free vector parallel to the line.

We now derive the scalar equations of the line. Equate components. Thus

$$\left.\begin{aligned} x &= a_1 + \lambda c_1 \\ y &= a_2 + \lambda c_2 \\ z &= a_3 + \lambda c_3 \end{aligned}\right\} \text{ parametric scalar equations.}$$

Suppose $c_1, c_2, c_3 \neq 0$. Then

$$\lambda = \frac{x - a_1}{c_1} = \frac{y - a_2}{c_2} = \frac{z - a_3}{c_3}$$

Equivalently

$$\boxed{\begin{gathered} \frac{x-a_1}{c_1} = \frac{y-a_2}{c_2} \\ \frac{y-a_2}{c_2} = \frac{z-a_3}{c_3} \\ \text{Non-parametric equations} \\ \text{of the line.} \end{gathered}}$$

This is nothing other than a system of two linear equations in 3 unknowns

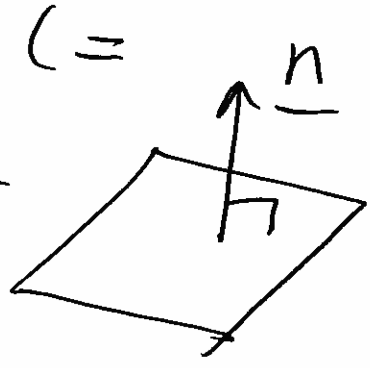
$$\boxed{\begin{aligned} c_2 x - c_1 y &= c_2 a_1 - c_1 a_2 \\ c_3 y - c_2 z &= c_3 a_2 - c_2 a_3 \\ \text{Solution set} &= \text{our line} \end{aligned}}$$

Equations of the plane

Planes are det'd by (amongst other things)

(1) Three distinct points not all in a straight line. Thus by 3 position vectors $\underline{a}, \underline{b}, \underline{c}$.

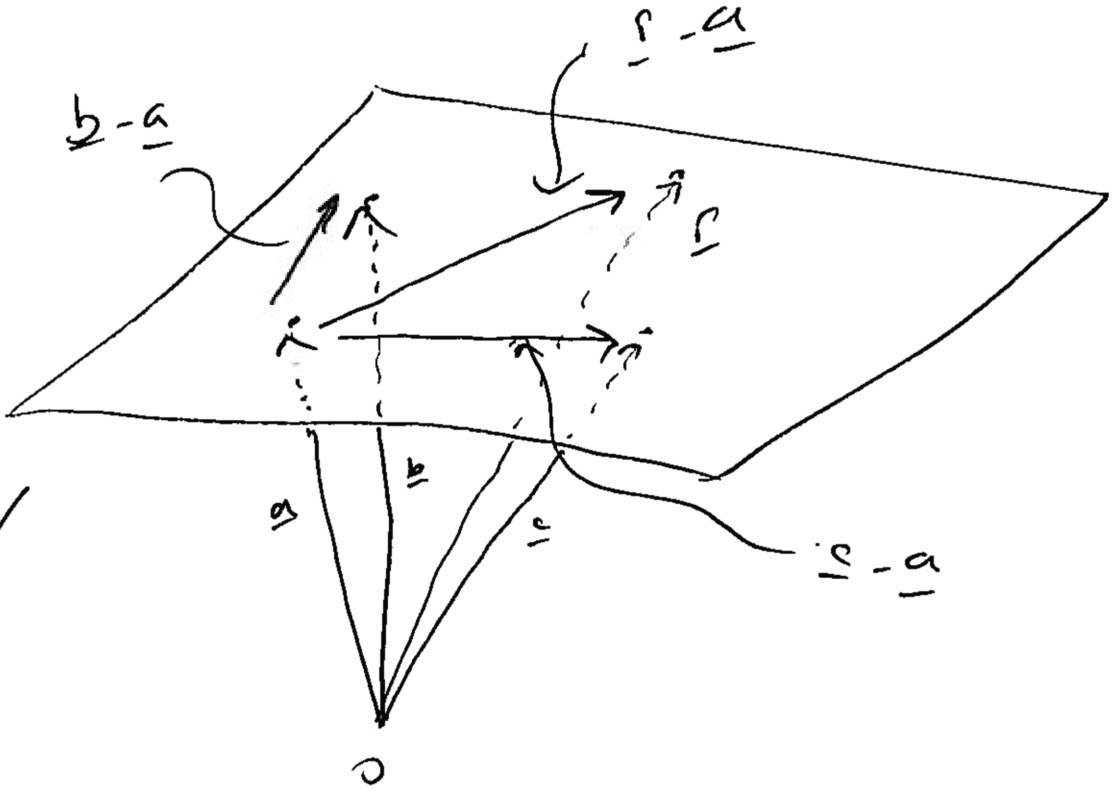
(2) One point and a direction perpendicular to the plane. Thus by one position vector $\underline{a}$ and a free vector $\underline{n}$ <u>normal</u> (= perpendicular) to the plane



[Is anyone reading this? Who knows!]

Three points

(1)



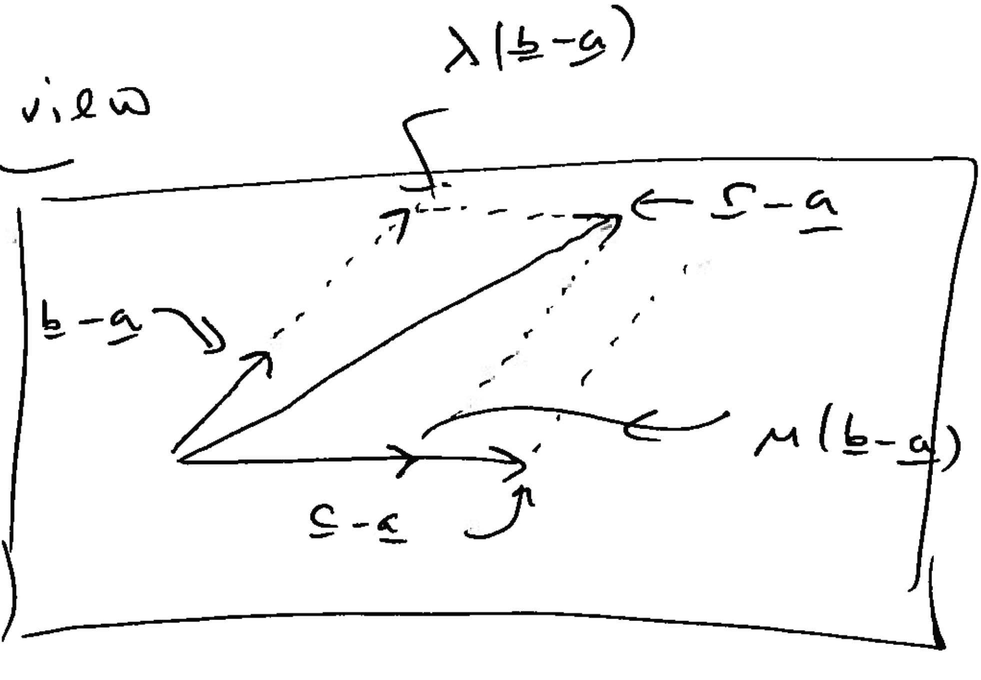
$$\underline{r} - \underline{a} = \lambda(\underline{b} - \underline{a}) + \mu(\underline{c} - \underline{a})$$

$\therefore$

$$\boxed{\underline{r} = \underline{a} + \lambda(\underline{b} - \underline{a}) + \mu(\underline{b} - \underline{a})}$$

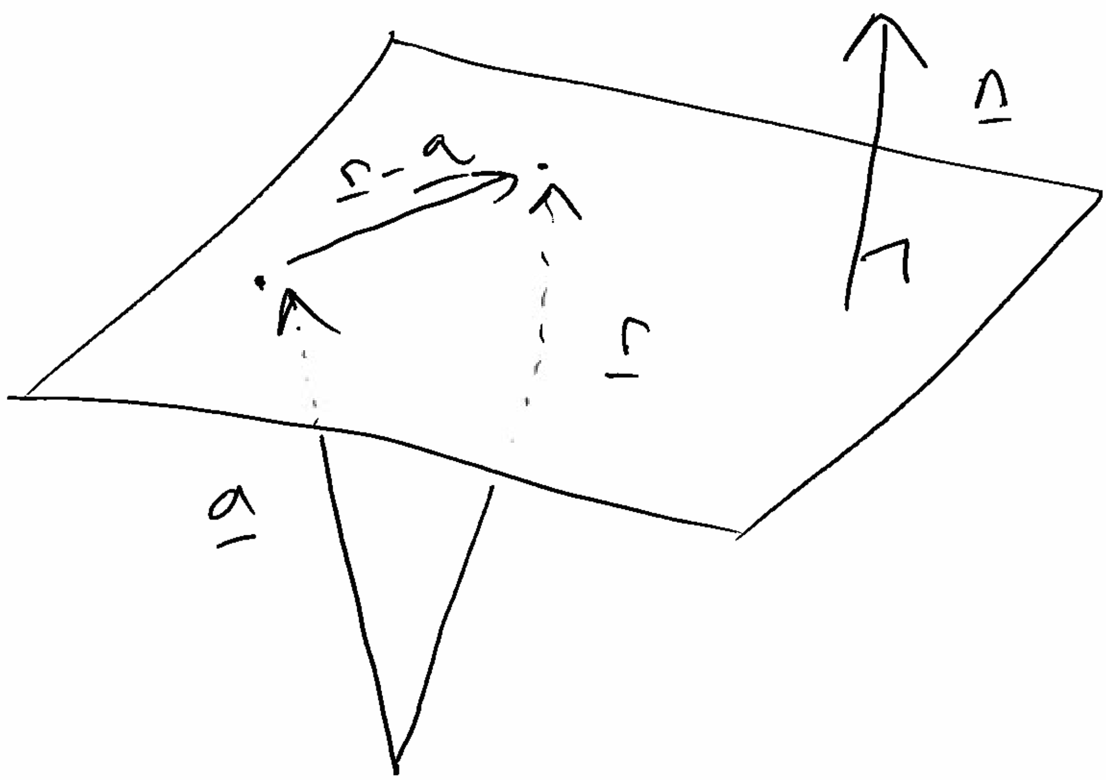
parametric vector equation of the plane.
then λ, μ are the parameters.

Top view



one point and normal

(2)



$$(\underline{r}-\underline{a})\cdot\underline{n} = 0$$

Now translate into scalar form

$$\underline{r}\cdot\underline{n} = \underline{a}\cdot\underline{n}$$

$$\cancel{(x-a_1)n_1 + (y-a_2)n_2 + (z-a_3)n_3 = 0}$$

$$n_1 x + n_2 y + n_3 z = a_1 n_1 + a_2 n_2 + a_3 n_3$$

Non-parametric equation of the plane. Has form

$$ax + by + cz = d$$

i.e. a <u>linear equation</u>.

$ax + by + cz = d$ always describes a plane except

- when $a = b = c = 0$, $d \neq 0$ when it describes $\emptyset$

- when $a = b = c = d = 0$ when it describes $\mathbb{R}^3$

To pass from (1) to (2)

use $\underline{n} = (\underline{b} - \underline{a}) \times (\underline{c} - \underline{a})$

(pourquoi?)

We now see that the non-parametric equation of the line describes said line as the intersection of two planes

line of intersection

planes

matrices + vectors = linear algebra

And a Merry Christmas to all our readers and a Happy New Year...