Lecture 3

Algebra is about manipulating symbols. This implies that there must be <u>rules</u> (or <u>axioms</u>) to tell us <u>how</u> we may manipulate these symbols.

<u>But</u> there is not one algebra but many different <u>algebras</u> which assume different rules. What is called algebra in school is usually the algebra of real no's ($\mathbb{R}$). We shall study different algebras later, but first, we list the algebraic rules that govern $\mathbb{R}$.

> We take $+$ and $\times$ as primary operations and $-$ and $\div$ as derived operations. This will be explained later.

Algebraic axioms for $\mathbb{R}$

(F1) $(x+y)+z = x+(y+z)$. <u>associativity</u> of $+$

(F2) $x+0 = x = 0+x$. 0 is the <u>additive identity</u>

(F3) $x+(-x) = 0 = (-x)+x$. $-x$ is the <u>additive inverse</u> of x

(F4) $x+y = y+x$. <u>Commutativity</u> of addition

(F5) $(xy)z = x(yz)$.
<u>associativity</u> of $\times$

(F6) $1x = x = x1$.
1 <u>multiplicative</u> identity

A <u>field</u> is any system satisfying these axioms. (Advanced)

(F7) NB $\boxed{\text{If } x \neq 0}$ then $x^{-1}x = 1 = xx^{-1}$.
x^{-1} is the <u>multiplicative inverse</u> of x

(F8) $xy = yx$. <u>Commutativity</u> of $\times$

F9) $0 \neq 1$.

F10) $0x = 0 = x0$.

<u>distributivity</u>

F11) $x(y+z) = xy + xz$.

$(y+z)x = yx + zx$.

<u>Define</u> $a - b = a + (-b)$

<u>Define</u> If $a \neq 0$ then $b \div a$ or $\frac{b}{a}$ is equal to ba^{-1}.

<u>Abbreviations</u> $a^2 = aa$, $a^3 = a^2 a$ etc

$a + a = 2a$, $a + a + a = 3a$ etc.

<u>Axioms for equality</u>

(E1) $a = a$.

(E2) $a = b \Rightarrow b = a$.

(E3) $a = b$ and $b = c \Rightarrow a = c$.

(E4) $a = b$ and $c = d \Rightarrow a + c = b + d$.

(E5) $a = b$ and $c = d \Rightarrow ac = bd$.

<u>Rem</u> $(abc)^2 = abc \cdot abc$

$= aa\,bb\,cc$

$= a^2 b^2 c^2$

Sigma-notation

<u>Notation</u>

$$a_1 + \dots + a_n = \sum_{i=1}^{n} a_i$$

$$a_0 + \dots + a_m = \sum_{j=0}^{m} a_j$$

<u>Example</u> We prove from the axioms of $\mathbb{R}$ that $(-1) \times (-1) = 1$

$$(-1) \times (-1) - 1 \overset{\text{def}}{=} (-1)(-1) + (-1)$$

$$= \underline{(-1)}(-1) + \underline{(-1)}\,1$$

$$= (-1)\left[(-1) + 1\right]$$

$$= (-1)\,0$$

$$= 0$$

$$\therefore (-1) \times (-1) = 1$$

<u>Exercise</u> Solve $ax + b = 0$ where $a \neq 0$ using the axioms for $\mathbb{R}$.

Completing the square

$$a x^{2} + b x + c$$

$\leftarrow$ degree

$\leftarrow$ constant term

$(a \neq 0)$ a, b, c called coefficients

if $a = 1$ called monic

polynomial of degree 2 or

quadratic polynomial

$$a x^2 + b x + c = 0$$

$\surd$ called a quadratic equation

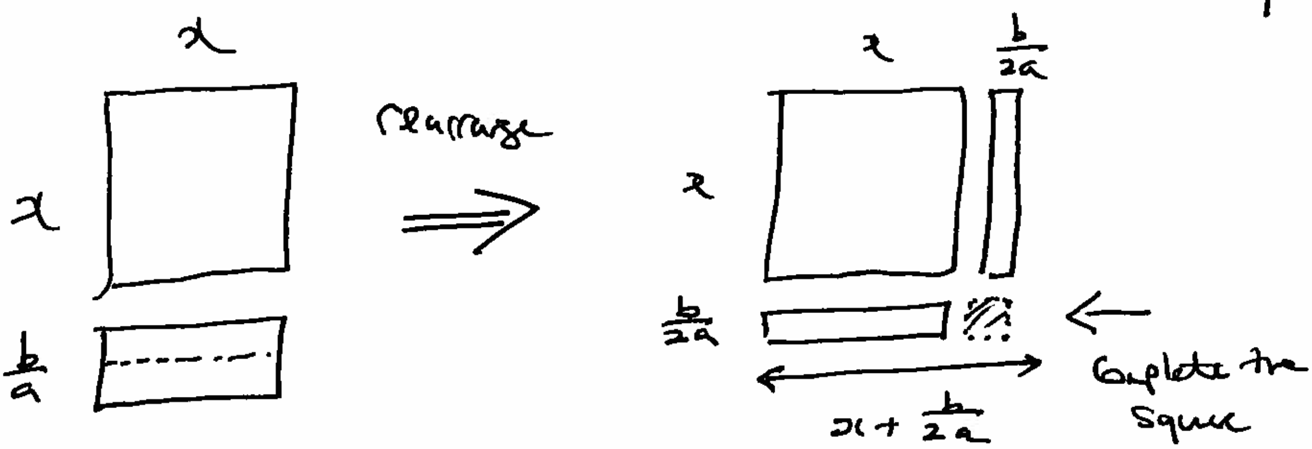
Solutions called roots.

Goal Find all solutions to the above quadratic equation. Divide through by a (since $a \neq 0$) to get a monic equation

number

$$\boxed{x^2 + \frac{b}{a} x} + \boxed{\frac{c}{a}} = 0$$

we shall draw a picture of this



area = $x^2 + \frac{b}{a}x$

$$\boxed{x^2 + \frac{b}{a}x = \left(x + \frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2}$$

our equation

$$\boxed{x^2 + \frac{b}{a}x} + \frac{c}{a} = 0$$

Completing the square

therefore becomes

$$\underbrace{\left(x + \frac{b}{2a}\right)^2}_{\text{perfect square}} \underbrace{- \left(\frac{b}{2a}\right)^2 + \frac{c}{a}}_{\text{number}} = 0$$

$$\therefore \left(x + \frac{b}{2a}\right)^2 = \left(\frac{b}{2a}\right)^2 - \frac{c}{a}$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

$$\therefore \quad x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}}$$

$$= \frac{\pm\sqrt{b^2 - 4ac}}{2a}$$

$$\therefore \quad \boxed{x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}}$$

Check that both these solutions really do work.

Defn $D = b^2 - 4ac$ the discriminant

$D > 0$ two real solutions

$D = 0$ one real solution (twice)

$D < 0$ no real solutions. The irreducible case.