Lecture 4

Lessons from the first tute

1. Algebra is about the manipulation of symbols not numbers.

Example As children we learn that $1+2=2+1$ or $5+4=4+5$ (perhaps by manipulating objects). But all of these are special cases of the axiom $x+y=y+x$.

Example The roots of $x^2+bx+c=0$ are

$$x_1 = \frac{-b+\sqrt{b^2-4c}}{2} \quad \text{and} \quad x_2 = \frac{-b-\sqrt{b^2-4c}}{2}$$

and so observe that $x_1 - x_2 = \sqrt{b^2-4c}$

$$\boxed{b^2-4c = (x_1-x_2)^2}$$

The discriminant of a monic quadratic is the square of the difference of the roots.

2. Algebra is governed by rules or axioms.

Example Solve the linear equation $ax+b=0$ where $a \neq 0$ stating the axioms used.

$$ax+b=0$$

$$(ax+b)+(-b) = 0+(-b) \qquad \text{Add } -b \text{ to both sides}$$

$$ax+(b+(-b)) = 0+(-b) \qquad \text{Associativity of } +$$

$$ax+0 = 0+(-b) \qquad \text{Additive inverses}$$

$$ax = -b \qquad 0 \text{ is the additive identity}$$

Keywords Associativity, Commutativity, distributivity.

$a^{-1}(ax) = a^{-1}(-b)$ $a \neq 0$ so a^{-1}, multiplicative inverse, exists.

$(a^{-1}a)x = a^{-1}(-b).$ Associativity of $\times$

$1x = a^{-1}(-b)$ Property of multiplicative inverse

$x = a^{-1}(-b) = -\frac{b}{a}.$ Notation.

$\boxed{\text{3. Always check algebra.}}$

__Example__ Writing $(a+b)^2 = a^2+b^2$ (which is WRONG BTW) is such a common mistake amongst first-year math(s) students, that in the US, it is often called the __Freshman Lemma__ (as a joke - though obviously not in the 'laugh out loud' variety)

__MISC__

(1) Let $x \in \mathbb{R}$, $x \neq 0$. Then $x^{-1} = \frac{1}{x}$ is always useful. [Good question about the __exponential function__ $x \longmapsto a^x$ $(a>0)$. This is different, but related]

(2) We shall use Σ-notation in a few places

$$\sum_{i=0}^{n} a_i \text{ means } a_0 + a_1 + \dots + a_n$$

A few loose ends

I need to prove the lemma used in the proof that $\sqrt{2}$ is irrational.

Lemma

(1) If a is even then a^2 is even.

(2) If a is odd then a^2 is odd.

(3) If a^2 is even then a is even.

Proof

(1) ~~[illegible]~~ Let a be even. Then by definition $a = 2x$ for some integer x. Squaring, $a^2 = (2x)^2 = 4x^2 = 2(2x^2)$. It follows that a^2 is even.

(2) Let a be odd. Then by definition, $a = 2x+1$ for some integer x. Squaring, $a^2 = 4x^2 + 4x + 1 = 2(2x^2 + 2x) + 1$. It follows that a^2 is odd.

(3) We are given that a^2 is even.

We know that either a is odd or a is even.

If a were odd, then by (2) above a^2 would be odd. Thus a cannot be odd. It follows that a must be even. $\blacksquare$ ← end of proof marker.

Patterns in real numbers

Example

(1) $0\cdot5$ is a finite decimal expansion.

$(= 0\cdot5000\ldots\ldots)$

(2) $0\cdot121212\ldots\ldots$

written $0\cdot\overline{12}$ is called a (purely) periodic decimal expansion.

(3) $0\cdot36\overline{12}$ is called an ultimately periodic decimal expansion.

[We regard periodic expansions as special cases of ultimately periodic ones.]

<u>Proposition</u> Every real no. that has an ultimately periodic decimal expansion is rational.

Rather than give you a formal proof, I shall given an example that demonstrates the idea.

<u>Example</u> Write $r = 0.1\overline{23}$ as a rational no.

$$r = 0.123\,23\,23\ldots$$

$$10r = 1.\underbrace{23\,23\,23}\ldots$$

$$100r = 12.32\,32\,32\ldots$$

$$1000r = 123.\underbrace{23\,23\,23}$$

Same after decimal point

Thus $1000r - 10r = 123 - 1$

It follows that $r = \dfrac{122}{990}$

(and so is a rational number).

[π is irrational and so cannot be ultimately periodic]