Lecture 6

Set notation does not record order or repetitions. We now introduce some notation that does.

An ordered pair (a,b) records order and repetitions. We have that $(a,b)=(c,d)$ iff $a=c$ and $b=d$. An ordered triple (a,b,c) is such that $(a,b,c)=(d,e,f)$ iff $a=d$, $b=e$, $c=f$. More generally we define an ordered n-tuple by $(a_1,\dots,a_n)$.

Example $(1,10,2015)$ is a date and is an ordered triple. (day, month, year).

Let $A_1,\dots,A_n$ be sets. Define the product set $A_1\times\dots\times A_n$ to be the set of all ordered n-tuples $(a_1,\dots,a_n)$ where $a_1\in A_1,\dots,a_n\in A_n$.

Obvious abbreviation $\underbrace{A\times\dots\times A}_{n\text{ times}}=A^n$.

Examples

(1) $A = \{a, b, c\}, \quad B = \{1, 2\}$

$$A \times B = \{(a,1), (a,2), (b,1), (b,2), (c,1), (c,2)\}$$

$$B \times A = \{(1,a), (2,a), (1,b), (2,b), (1,c), (2,c)\}$$

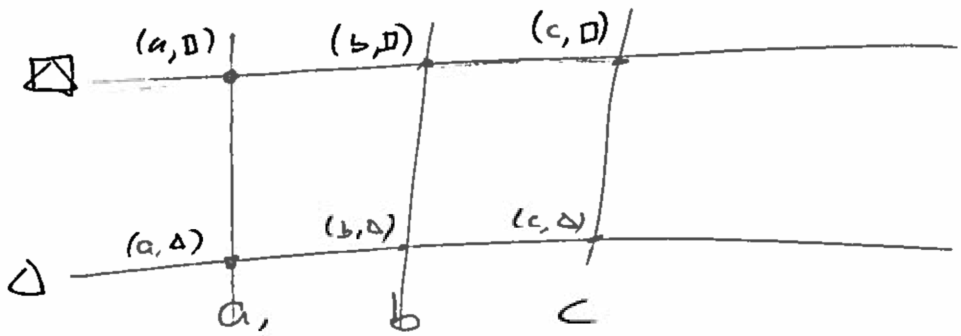
Observe that $A \times B \neq B \times A$.

(2) $\mathbb{R}^2 = \{(x,y) : x, y \in \mathbb{R}\}$ <u>real plane</u>

(3) $\mathbb{R}^3 = \{(x,y,z) : x, y, z \in \mathbb{R}\}$ <u>real space</u>

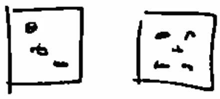
$$|A_1 \times \ldots \times A_n| = |A_1| \ldots |A_n|$$

<u>Example</u> $\{a, b, c\} \times \{\Delta, \square\}$ 6 elements.



__Example__ Two dice are throws. What are the possible outcomes?

$$X = \{1,..,6\} \times \{1,..,6\}$$

is called the __sample space__. An element of X is an ordered pair. Thus $(3, 5) \in X$ represents . A subset of X is called an __event__. Thus $A = \{2,4,6\} \times \{2,4,6\}$ is the event "both dice show an even number". The probability of this event is $\frac{|A|}{|X|} = \frac{9}{36} = \frac{3}{12}$

$$= \frac{1}{4}$$

__Strings__

Strings over a set

This is a notation/notion related to ordered n-tuples. Rather than write $(a_1, \ldots a_n)$ we write ~~E~~. $a_1 \ldots a_n$. This is a finite ordered sequence of symbols.

The __length__ of a string is the number of symbols that occur counting repetitions.

<u>Example</u> Let $A = \{0,1\}$. The set of all non-empty binary strings over A is the set

$$\{0, 1, 00, 01, 10, 11, \ldots\}$$

<u>Example</u> Let $B = \{x, y\}$. The set of all non-empty strings over B is the set

$$\{x, y, xx, xy, yx, xxx, \ldots\}$$

The number of binary strings of length n is 2^n. This is essentially the set $\{0,1\}^n$. By our previous results

$$|\{0,1\}^n| = 2^n.$$

We use this result.

Theorem $|P(X)| = 2^{|X|}$.

Proof I'll sketch the idea by looking again at the special case where $X = \{a, b, c\}$.

We code each subset by means of a binary string of length 3

a b c
0/1 0/1 0/1

Subset	Binary code
$\emptyset$	000
$\{a\}$	100
$\{b\}$	010
$\{c\}$	001
$\{a,b\}$	110
$\{b,c\}$	011
$\{a,c\}$	101
$\{a,b,c\}$	111

We now see why $|P(\{a,b,c\})| = 2^3$.

Example

$$2^n = \sum_{i=0}^{n} \binom{n}{i}$$

all possible subsets of an n-element set.

i-subsets.

This is a nice example of proving an algebraic result without using algebra — but by counting

There is an important consequence of our calculations.

$$\boxed{\text{The number of strings of length } n \text{ over } \{x,y\} \text{ with exactly } i \text{ } x\text{s is } \binom{n}{i}}$$

Example

All strings of xs and ys of length 4 with 2 xs.

$xxyy$, $xyyx$, $xyxy$,
$yxyx$, $yyxx$, $yxxy$,

$$\binom{4}{2} = \frac{4!}{2!\,2!} = \frac{4 \cdot 3}{2} = \underline{\underline{6}}$$