Solutions to Ex 1

2. In A^*, strings x and y are equal if (1) $|x| = |y|$ and
(2) $x_i = y_i$ for $1 \le i \le |x| = |y|$ where x_i denotes the ith letter of x.

We are given that $xz = yz$. Thus taking lengths, $|xz| = |yz|$ and so $|x| + |z| = |y| + |z|$. It follows that $|x| = |y|$.

By definition, $(xz)_i = (yz)_i$ for $1 \le i \le |xz| = |yz|$.

Thus, in particular $(xz)_i = (yz)_i$ for $1 \le i \le |x| = |y|$.

But $(xz)_i = x_i$ and $(yz)_i = y_i$ (for i in this range).

Thus $x_i = y_i$ for $1 \le i \le |x| = |y|$.

Hence $x = y$, as required.

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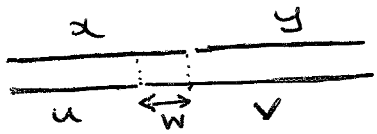
3. Let $xy = uv$. S'pose first, that $|x| > |u|$.
We may picture this situation as follows



We see that there is a string w s.t. $x = uw$.

From the picture, we see that $v = wy$.

Thus $x = uw$ and $v = wy$ as claimed.

The other two cases can be dealt with similarly.

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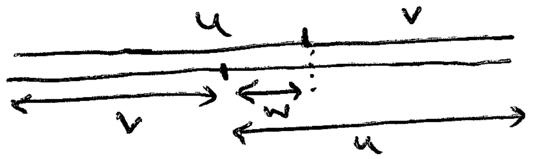
4. (ii) $\Rightarrow$ (i) is easy. We prove (i) $\Rightarrow$ (ii), the hard direction.

We are given that $uv = vu$. We use the result of Q3.
S'pose first that $|u| = |v|$. Then $u = v$ and there is nothing to prove.
We consider w.l.o.g the case where $|u| > |v|$.



We may write $u = vw$ and $u = wv$ where $|w| \neq 0$ and $|w| < |u|$.
Thus $vw = wv$. This suggests an induction argument.

$P_n :=$ " If $uv = vu$ and $|uv| = n$, where $n \geqslant 2$,
then $\exists z \in A^+$ and $p > 0$, $q > 0$ s.t. $u = z^p$ and $v = z^q$ ".

<u>Base step</u>: P_2 is true since u and v will be individual letters and will commute iff they are equal.

<u>(IH)</u>: P_n is true, P_{n-1} is true, .., P_2 is true

We prove P_{n+1} : So $uv = vu$ and $|uv| = n+1$.
We now use the argument above and the (IH) to get the result.

(5) Let S be a semigroup with identities e and f. Then by definition $ef = e$ (f is an identity) and $ef = f$ (e is an identity) and so $e = f$, as required.

(6)

(i) This is a semigp. Let $f, g, h \in T(X)$. Then for any $x \in X$, we have that

$$((f \circ g) \circ h)(x) = (f \circ g)(h(x)) = f(g(h(x)))$$

& similarly $(f \circ (g \circ h))(x) = f(g(h(x)))$.

(ii) This is a semigp. Let A, B and C be three $n \times n$ matrices.

$$\begin{aligned}
((AB)C)_{ij} &= \sum_k (AB)_{ik} C_{kj} \\
&= \sum_k \Big(\sum_\ell A_{i\ell} B_{\ell k} \Big) C_{kj} \\
&= \sum_k \sum_\ell A_{i\ell} B_{\ell k} C_{kj} \\
&= \sum_\ell \sum_k A_{i\ell} B_{\ell k} C_{kj} \qquad \text{order of sums may be interchanged} \\
&= \sum_\ell A_{i\ell} \Big(\sum_k B_{\ell k} C_{kj} \Big) \\
&= \sum_\ell A_{i\ell} (BC)_{\ell j} \\
&= (A(BC))_{ij}
\end{aligned}$$

It follows that $(AB)C = A(BC)$, as claimed.

(iii) Not a semigp. $(\underline{i} \times \underline{k}) \times \underline{k} = -\underline{i}$ but

$\underline{i} \times (\underline{k} \times \underline{k}) = \underline{0}$