Exercises 1

(1) Please work your way through the exercies in Chapter 1 of reference [2].

(2) Let A be an alphabet. Prove that A^* is *cancellative* with respect to concatenation, meaning that if $x, y, z \in A^*$ then $xz = yz$ implies $x = y$, and $zx = zy$ implies $x = y$.

(3) Let $x, y, u, v \in A^*$. Suppose that $xy = uv$. Prove the following hold:

  (i): If $|x| > |u|$, then there exists a non-empty string w such that $x = uw$ and $v = wy$.

  (ii): If $|x| = |u|$, then $x = u$ and $y = v$.

  (iii): If $|x| < |u|$, then there exists a non-empty string w such that $u = xw$ and $y = wv$.

(4) In general, if $u, v \in A^+$, then the strings uv and vu are different as we have noted. This raises the question of finding conditions under which $uv = vu$. Prove that the following two conditions are equivalent:

  (i): $uv = vu$.

  (ii): There exists a string z such that $u = z^p$ and $v = z^q$ for some natural numbers $p, q > 0$.

(5) Prove that in a semigroup there is at most one identity.

(6) Determine which of the following are semigroups and which are not, and give reasons.

  (i): The set $T(X)$ of all functions defined from the set X to itself equipped with the binary operation of composition of functions.

  (ii): The set $M_n(\mathbb{R})$ of all $n \times n$ real matrices equipped with matrix multiplication.

  (iii): The set of all three dimensional vectors equipped with the vector product.