Exercises 7

(1) Find the Hamming distance between 11000 and 11000 and between 1100101 and 0111011.

(2) Below you will find listed four codes. Show that each code is linear, find generator and parity check matrices, and determine the parameters of each code (n, k, d).

Code 1

$$
\begin{matrix}
0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 \\
0 & 1 & 0 & 1 \\
0 & 1 & 1 & 0 \\
1 & 0 & 0 & 1 \\
1 & 0 & 1 & 0 \\
1 & 1 & 0 & 0 \\
1 & 1 & 1 & 1
\end{matrix}
$$

Code 2

$$
\begin{matrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 1 & 0 \\
0 & 1 & 0 & 1 & 0 & 1 \\
0 & 1 & 1 & 0 & 1 & 1 \\
1 & 0 & 0 & 0 & 1 & 1 \\
1 & 0 & 1 & 1 & 0 & 1 \\
1 & 1 & 0 & 1 & 1 & 0 \\
1 & 1 & 1 & 0 & 0 & 0
\end{matrix}
$$

Code 3

$$
\begin{matrix}
0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 \\
1 & 0 & 1 & 0 \\
1 & 1 & 1 & 1
\end{matrix}
$$

Code 4

0	0	0	0
1	1	0	0
0	0	1	1
1	1	1	1

(3) Let $\mathbf{v} \in \mathbb{Z}_2^n$. Define $S(\mathbf{v}, r)$ to be the set of all vectors $\mathbf{u}$ such that $d(\mathbf{u}, \mathbf{v}) \leq r$. This is called a *sphere of radius r centered on* $\mathbf{v}$.

(a) How many elements does the set $S(\mathbf{v}, r)$ contain?

(b) Let $C \subseteq \mathbf{Z}_2^n$ be a code containing M elements. Suppose that C can correct upto t errors. Prove the *Hamming bound*

$$M(1 + \binom{n}{1} + \binom{n}{2} + \ldots + \binom{n}{t}) \leq 2^n$$

(c) Show that for the Hamming codes the above inequality is in fact an equality. Codes such as this are said to be *perfect*.

(d) Is there a linear code with the parameters $(21, 14, 5)$?

(e) A college wants to issue ID numbers in the form of binary strings n bits long to 100 students. Students are likely to make errors when typing these ID's and so it has been decided that a 2-error correcting code should be used. What is the smallest value of n for which this is possible?