# Ex 3: Solutions part 1

(1)
(a) I'll show one calculation of Blankinship's algorithm.

$$\left(\begin{array}{cc|c} 1 & 0 & 741 \\ 0 & 1 & 715 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & -1 & 26 \\ 0 & 1 & 715 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & -1 & 26 \\ -27 & 28 & 13 \end{array}\right) \to \left(\begin{array}{cc|c} 55 & -57 & 0 \\ -27 & 28 & 13 \end{array}\right) \leftarrow \text{gcd}.$$

$$\therefore \begin{pmatrix} 55 & -57 \\ -27 & 28 \end{pmatrix} \begin{pmatrix} 741 \\ 715 \end{pmatrix} = \begin{pmatrix} 0 \\ 13 \end{pmatrix}$$

$$\therefore \quad 13 = (-27) \times 74 + 28 \times 715$$

(b) $259 = 1 \times 2072 - 1 \times 1813$.

(c) $5 = (-47) \times 1485 + 40 \times 1745$

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(2) We need to prove that $f_n = \frac{1}{\sqrt{5}}(\varphi^n - \bar{\varphi}^n)$ where $\varphi$ and $\bar{\varphi}$ are roots of $x^2 - x - 1 = 0$.

<u>Base cases:</u> Check formula works when $n=0$ and $n=1$.

<u>IH</u>: Assume formula works for $n-1$ and $n$.

<u>Prove</u> that formula works for $n+1$.

$$f_{n+1} = f_n + f_{n-1} = \frac{1}{\sqrt{5}}\left(\varphi^{n-1}(\varphi+1) - \bar{\varphi}^{n-1}(\bar{\varphi}+1)\right) \text{ by IH}$$

But $\varphi^2 = \varphi + 1$ and $\bar{\varphi}^2 = \bar{\varphi} + 1$. $\therefore$

$$= \frac{1}{\sqrt{5}}\left(\varphi^{n+1} - \bar{\varphi}^{n+1}\right), \text{ as required.}$$

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(3) Similar to the above. Base cases here are $n=3$ and $n=4$.

<u>IH</u> Assume inequality holds for $n-1$ and $n$.

<u>Prove the</u> formula holds for $n+1$:

$$f_{n+1} = f_n + f_{n-1} \underset{\text{by IH}}{>} \varphi^{n-2} + \varphi^{n-3} = \varphi^{n-3}(\varphi+1) = \varphi^{n-1}, \text{ as required.}$$

4. $$\lim_{x\to\infty} \frac{li(x)}{x/\ln(x)} = \frac{1/\ln(x)}{(\ln(x) - x\frac{1}{x})/(\ln(x))^2}$$

$$= \frac{1}{1 - \frac{1}{\ln(x)}}$$

But $\lim_{x\to\infty} \frac{1}{\ln(x)} = 0$ $\therefore \lim_{x\to\infty} \frac{1}{1-\frac{1}{\ln(x)}} = 1$

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5. The primes $p$ we need to count are $10^{299} < p < 10^{300}$.

By the PNT, $\pi(10^{300}) = \frac{10^{300}}{\ln(10^{300})}$ and $\pi(10^{299}) = \frac{10^{299}}{\ln(10^{299})}$.

$\therefore \pi(10^{300}) - \pi(10^{299}) \approx 1.3 \times 10^{297}$

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6. $\sqrt{13,837} = 117.6$

$\therefore$ Initial value of $t = 118$

| $t$ | $t^2 - n$ |
|---|---|
| 118 | 87 |
| 119 | $324 = 18^2$ ✗ |

$\therefore 119^2 - 13,837 = 18^2 \Rightarrow 13,837 = 137 \times 101$

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7. $n = 809,009$. $\lfloor\sqrt{n}\rfloor + 1 = 900$

| $t$ | $t^2 - n$ |
|---|---|
| 900 | 991 |
| 901 | 2792 |
| 902 | 4595 |
| 903 | $6400 = 80^2$ |

$\Rightarrow n = 903^2 - 80^2$

$= 983 \times 823$

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8. $10^9 - 3^9 = (10-3)(\text{—})$ $\therefore 7$ is a factor

$(10^3)^3 - (3^3)^3 = (10^3 - 3^3)(\text{—})$ $\therefore 973$ is a factor, & $7 \mid 973$

$= 7 \cdot 19 \cdot 139 \cdot 54,091$.