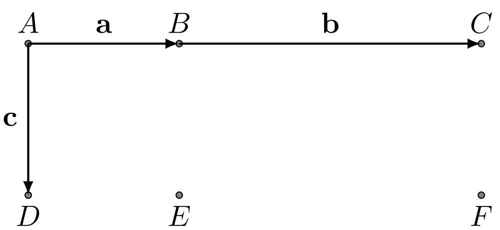
Exercises 10

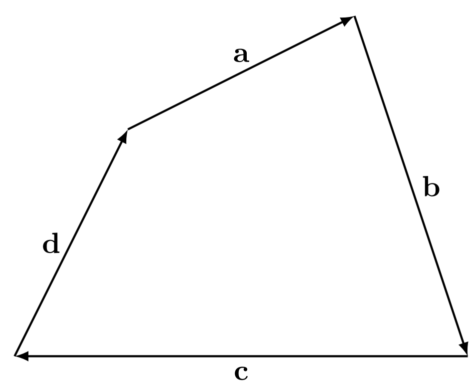
(1) Consider the following diagram.



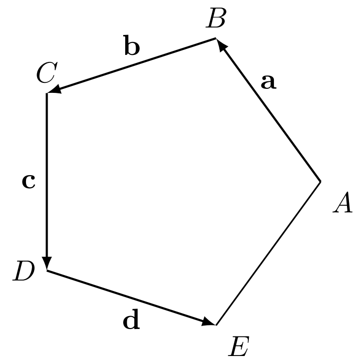
Now answer the following questions.

(a) Write the vector $\overrightarrow{BD}$ in terms of **a** and **c**.
(b) Write the vector $\overrightarrow{AE}$ in terms of **a** and **c**.
(c) What is the vector $\overrightarrow{DE}$?
(d) What is the vector $\overrightarrow{CF}$?
(e) What is the vector $\overrightarrow{AC}$?
(f) What is the vector $\overrightarrow{BF}$?

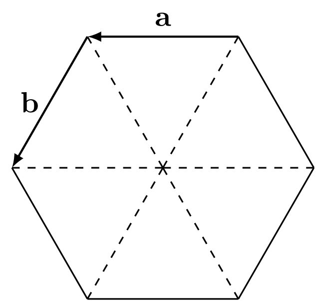
(2) If **a**, **b**, **c** and **d** represent the consecutive sides of a quadrilateral, show that the quadrilateral is a parallelogram if and only if $\mathbf{a}+\mathbf{c} = \mathbf{0}$.



(3) In the regular pentagon $ABCDE$, let $\overrightarrow{AB} = \mathbf{a}$, $\overrightarrow{BC} = \mathbf{b}$, $\overrightarrow{CD} = \mathbf{c}$ and $\overrightarrow{DE} = \mathbf{d}$. Express $\overrightarrow{EA}$, $\overrightarrow{DA}$, $\overrightarrow{DB}$, $\overrightarrow{CA}$, $\overrightarrow{EC}$, $\overrightarrow{BE}$ in terms of **a**, **b**, **c** and **d**.



(4) Let $\mathbf{a}$ and $\mathbf{b}$ represent adjacent sides of a regular hexagon so that the initial point of $\mathbf{b}$ is the terminal point of $\mathbf{a}$. Represent the remaining sides by means of vectors expressed in terms of $\mathbf{a}$ and $\mathbf{b}$.



(5) Prove that $\|\mathbf{a}\|\,\mathbf{b}+\|\mathbf{b}\|\,\mathbf{a}$ is orthogonal to $\|\mathbf{a}\|\,\mathbf{b}-\|\mathbf{b}\|\,\mathbf{a}$ for all vectors $\mathbf{a}$ and $\mathbf{b}$.

(6) Let $\mathbf{a}$ and $\mathbf{b}$ be two non-zero vectors. Define

$$\mathbf{u} = \left(\frac{\mathbf{a}\cdot\mathbf{b}}{\mathbf{a}\cdot\mathbf{a}}\right)\mathbf{a}.$$

Show that $\mathbf{b}-\mathbf{u}$ is orthogonal to $\mathbf{a}$.

(7) (a) Simplify $(\mathbf{u}+\mathbf{v})\times(\mathbf{u}+\mathbf{v})$.
(b) Simplify $(\mathbf{u}+\mathbf{v})\times(\mathbf{u}-\mathbf{v})$.

(8) Let $\mathbf{a}$ and $\mathbf{b}$ be two unit vectors, the angle between them being $\frac{\pi}{3}$. Show that $2\mathbf{b}-\mathbf{a}$ and $\mathbf{a}$ are orthogonal.

(9) Prove that

$$\|\mathbf{u}-\mathbf{v}\|^2 + \|\mathbf{u}+\mathbf{v}\|^2 = 2(\|\mathbf{u}\|^2 + \|\mathbf{v}\|^2).$$

Deduce that the sum of the squares of the diagonals of a parallelogram is equal to the sum of the squares of all four sides.

(10) Let $\mathbf{a} = 2\mathbf{i} + \mathbf{j} + \mathbf{k}$ and $\mathbf{b} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$.
  (a) Calculate $\|\mathbf{a}\|$ and $\|\mathbf{b}\|$.
  (b) Calculate $\mathbf{a} + \mathbf{b}$.
  (c) Calculate $\mathbf{a} - \mathbf{b}$.
  (d) Calculate $\mathbf{a} \cdot \mathbf{b}$.
  (e) Calculate the angle between $\mathbf{a}$ and $\mathbf{b}$ to the nearest degree.
  (f) Calculate $\mathbf{a} \times \mathbf{b}$.
  (g) Find a unit vector orthogonal to both $\mathbf{a}$ and $\mathbf{b}$.

(11) Calculate $\mathbf{i} \times (\mathbf{i} \times \mathbf{k})$ and $(\mathbf{i} \times \mathbf{i}) \times \mathbf{k}$. What do you deduce as a result of this?

(12) Calculate $\mathbf{a} \cdot (\mathbf{a} \times \mathbf{b})$ and $\mathbf{b} \cdot (\mathbf{a} \times \mathbf{b})$.

(13) The unit cube is determined by the three vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$. Find the angle between the long diagonal of the unit cube and one of its edges.

(14) Calculate $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$ where $\mathbf{u} = 3\mathbf{i} - 2\mathbf{j} - 5\mathbf{k}$, $\mathbf{v} = \mathbf{i} + 4\mathbf{j} - 4\mathbf{k}$ and $\mathbf{w} = 3\mathbf{j} + 2\mathbf{k}$.