Exercises 6

(1) Solve the following problems in complex number arithmetic. In each case, the answer should be in the form $a+bi$ where a and b are real.
   (a) $(2+3i)+(4+i)$.
   (b) $(2+3i)(4+i)$.
   (c) $(8+6i)^2$.
   (d) $\frac{2+3i}{4+i}$.
   (e) $\frac{1}{i}+\frac{3}{1+i}$.
   (f) $\frac{3+4i}{3-4i}-\frac{3-4i}{4+4i}$.

(2) Find the square roots of each of the following complex numbers.
   (a) $-i$.
   (b) $-1+\sqrt{24}i$.
   (c) $-13-84i$.

(3) Solve the following quadratic equations.
   (a) $x^2+x+1=0$.
   (b) $2x^2-3x+2=0$.
   (c) $x^2-(2+3i)x-1+3i=0$.

(4) Define the *Gaussian integers*, denoted by $\mathbb{Z}[i]$, to be all complex numbers of the form $m+ni$ where m and n are integers. A *perfect square* in $\mathbb{Z}[i]$ is a Gaussian integer that can be written $(a+bi)^2$ for some Gaussian integer $a+bi$. Prove that if $x+yi$ is a perfect square then x and y form part of a Pythagorean triple (see Section 1.1).

(5) Express $\cos 5x$ and $\sin 5x$ in terms of $\cos x$ and $\sin x$.

(6) Prove the following where x is real.[1]
   (a) $\sin x = \frac{1}{2i}(e^{ix}-e^{-ix})$.
   (b) $\cos x = \frac{1}{2}(e^{ix}+e^{-ix})$.
   Hence show that $\cos^4 x = \frac{1}{8}[\cos 4x + 4\cos 2x + 3]$.

(7) **Thought provoking.** Determine all the values of i^i. What do you notice?

[1]Compare (a) and (b) with $\sinh x = \frac{1}{2}(e^x - e^{-x})$ and $\cosh x = \frac{1}{2}(e^x+e^{-x})$.