Exercises 7

(1) Find the quotient and remainder when the first polynomial is divided by the second.
   (a) $2x^3 - 3x^2 + 1$ and x.
   (b) $x^3 - 7x - 1$ and $x - 2$.
   (c) $x^4 - 2x^2 - 1$ and $x^2 + 3x - 1$.

(2) Find all roots using the information given.
   (a) 4 is a root of $3x^3 - 20x^2 + 36x - 16$.
   (b) $-1, -2$ are both roots of $x^4 + 2x^3 + x + 2$.

(3) Find a cubic having roots $2, -3, 4$.

(4) Find a quartic having roots i, $-i$, $1 + i$ and $1 - i$.

(5) The cubic $x^3 + ax^2 + bx + c$ has roots x_1, x_2 and x_3. Show that a, b, c can each be written in terms of the roots.

(6) $3 + i\sqrt{2}$ is a root of $x^4 + x^3 - 25x^2 + 41x + 66$. Find the remaining roots.

(7) $1 - i\sqrt{5}$ is a root of $x^4 - 2x^3 + 4x^2 + 4x - 12$. Find the remaining roots.

(8) Find all the roots of the following polynomials.
   (a) $x^3 + x^2 + x + 1$.
   (b) $x^3 - x^2 - 3x + 6$.
   (c) $x^4 - x^3 + 5x^2 + x - 6$.

(9) Write each of the following polynomials as a product of real linear or real irreducible quadratic factors.
   (a) $x^3 - 1$.
   (b) $x^4 - 1$.
   (c) $x^4 + 1$.

(10) Find the 4th roots of unity as radical expressions.

(11) Find the 6th roots of unity as radical expressions.

(12) Find the 8th roots of unity as radical expressions.

(13) Find radical expresssions for the roots of $x^5 - 1$, and so show that

$$\cos 72^\circ = \frac{\sqrt{5} - 1}{4} \text{ and } \sin 72^\circ = \frac{\sqrt{10 + 2\sqrt{5}}}{4}.$$

Hint: Consider the equation

$$x^4 + x^3 + x^2 + x + 1 = 0.$$

Divide through by x^2 to get

$$x^2 + \frac{1}{x^2} + x + \frac{1}{x} + 1 = 0.$$

Put $y = x + \frac{1}{x}$. Show that y satisfies the quadratic

$$y^2 + y - 1 = 0.$$

You can now find all four values of x.

(14) In the following, express your answers in trigonometric form, and in radical form if possible.

  (a) Find the cube roots of $-8i$.
  (b) Find the fourth roots of 2.
  (c) Find the sixth roots of $1 + i$.